Displacement Convexity – A Useful Framework for the Study of Spatially Coupled Codes

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Abstract—Spatially coupled (SC) codes have been shown to exhibit threshold saturation under low-complexity message-passing algorithms; that is, the belief-propagation (BP) threshold of a single system increases up to the maximum-a-posteriori (MAP) threshold upon coupling. We review a simple proof technique of threshold saturation in [1].

Displacement convexity was introduced in [2] and studies the convexity of functionals with respect to an alternative structure of probability measures. It can be used in coding systems as another proof technique of threshold saturation: if the “potential” functional associated to the system is strictly convex, then it admits a unique minimizing “decoding wave” along the coupling chain. The potential needs to be studied in the continuum limit so that this analysis is applicable. In such settings, we find it useful to understand a general rearrangement inequality that was shown in [3].

Index Terms—spatial coupling, threshold saturation, displacement convexity, fixed-point equation, rearrangement inequalities.

I. INTRODUCTION

Spatial coupling of low-density parity-check (LDPC) codes was first introduced by Felstrom and Zigangirov in [4]. SC systems have shown to work very well under low-complexity BP algorithms [5], [6], [1], [7], [8]. In fact, this combination achieves essentially optimal performance. The principle behind this is called threshold saturation via spatial coupling and was coined by Kudekar, Richardson and Urbanke in [5]. So far, threshold saturation has been proved for SC regular and irregular LDPC codes over the class of Binary Memoryless Symmetric (BMS) channels. Spatial coupling has been proving useful in several areas, such as statistical physics, satisfiability, and compressive sensing, so it is worth investigating such systems further.

Consider for example the ($\ell$, $r$)-regular Gallager ensemble on the Binary Erasure Channel (BEC) with erasure probability $\epsilon$. That is, the channel transmits the true value of the bit with probability $1 - \epsilon$ and an erased bit with probability $\epsilon$. The ensemble is represented by a bipartite graph, consisting of variable nodes of degree $\ell$ and check nodes of degree $r$. Messages are exchanged between nodes in BP decoding and at any iteration of decoding, the outgoing message emitted by a node is either the true value of the message or an erased message. Denote by $x^{(l)}$ the erasure probability at iteration $l$. A check node sends an erasure message on an edge (to a variable node) if it receives an erasure message on any other edge; that is, with probability $1 - (1 - x^{(l)})^{r-1}$. A variable node at iteration $l + 1$ sends an erasure message on an edge (to a check node) if its initial message is an erasure and it receives erasure messages on all its other edges. Hence, BP decoding can be tracked by the following recursion on the erasure probability

$$x^{(l+1)} = \epsilon(1 - (1 - x^{(l)})^{r-1})^{\ell-1}.$$ 

We consider a more general setting, where BP is run on any bipartite graph. The channel noise parameter is denoted by $\epsilon$ and the message–passing rules on the nodes are denoted by the functions $f$ and $g$. The quantity $x^{(l)}$ now denotes the output distribution of a variable node at iteration $l$. For the example above, $f(x; \epsilon) = \epsilon(1 - (1 - x^{(l)})^{r-1})^{\ell-1}$ and $g(x) = 1 - (1 - x^{(l)})^{r-1}$. The generic single-system recursion can thus be expressed as

$$x^{(l+1)} = f(g(x^{(l)}); \epsilon).$$  

(1)

This equation is also known as the density evolution equation, as it tracks the state of the system as decoding occurs.

To analyze the BP algorithm, we consider the equation (1) that characterizes a system. We are interested in finding solutions of the fixed-point (FP) equation $x = f(g(x); \epsilon)$. More specifically, we are interested in checking whether the
reduction converges to the trivial (zero) solution, which means that decoding has been successful.

We represent a single system as a protograph consisting of one \( f \)-system and one \( g \)-system. In reality, the \( f \)-system (\( g \)-system) still consists of all the original \( f \)-nodes (\( g \)-nodes) of the bipartite graph. We will refer to these nodes as “underlying” \( f \)-nodes (\( g \)-nodes).

An \((L, w)\) SC ensemble is obtained from the single system by replicating the single-system protograph \( 2L + 1 \) times on the positions in the interval \([−L, \ldots, L]\). The replicas are then connected over a window size \( w \) uniformly at random, as will be explained in more detail in Section II. A generic SC ensemble that was used in [6] is reproduced in Figure 1.

In the parameter space, the algorithmic threshold of the channel noise parameter \( \epsilon \) for a specific decoding algorithm is defined to be the maximum value of \( \epsilon \) at which it is possible to decode; that is, for which the recursion converges to the trivial solution. The optimal value for \( \epsilon \) is that obtained by the MAP decoder, and is denoted by \( \epsilon_{\text{MAP}} \). BP algorithms are desirable because they are of low complexity. However, under BP decoding, single system ensembles can only decode up to a threshold \( \epsilon_{\text{BP}} < \epsilon_{\text{MAP}} \). The goal is to increase \( \epsilon_{\text{BP}} \) to \( \epsilon_{\text{MAP}} \); that is, to let \( \epsilon_{\text{BP}} \) \textit{saturate} to \( \epsilon_{\text{MAP}} \). Spatial coupling has been shown to exhibit this phenomenon, called \textit{threshold saturation}.

In [1], Yedla et al. express this problem in variational form. Specifically, they define a “potential” functional associated to a coupled system, whose minimization yields the FP equation. Yedla et al. show that for all \( \epsilon < \epsilon_{\text{MAP}} \), the only FP solution of the recursion of the coupled system is the trivial solution. This shows that, indeed, the \( \epsilon \)-threshold has saturated to the maximum value upon coupling.

The analysis done by Yedla et al. assumes a specific initial state on the single systems along the coupling chain. The natural question then arises: is this FP solution unique or is it specific to the initial conditions? To answer this question, we propose, in Section V, the study of displacement convexity as a tool to find non-trivial FP solutions for general channels. The analysis for a particular setting has already been carried out in [9].

II. THRESHOLD SATURATION FOR COUPLED VECTOR ENSEMBLES

Yedla et al. [6] consider a system governed by a vector-valued recursion of dimension \( d \). They define, accordingly, the vector-valued message passing rules \( f \) and \( g \) of the (uncoupled) system, and call the system \((f, g)\) \textit{vector admissible} if it satisfies certain conditions (see Section I). They consider two techniques of spatial coupling and prove that, in both cases, the (uncoupled) vector admissible system \((f, g)\) exhibits threshold saturation upon spatial coupling.

We start by some notation that will allow us to carry out the analysis using vector quantities.

A. Notation

Consider a vector recursion of dimension \( d \in \mathbb{N} \). The space on which the \( d \)-dimensional recursion is defined is denoted by \( \mathcal{X} = [0,1]^d \), and the parameter space of the recursive system by \( \mathcal{E} = [0,1] \). For convenience, \( \mathcal{X}_0 \triangleq \mathcal{X} \setminus \{0\} \) is also defined. Vectors are denoted in boldface lowercase and matrices in boldface uppercase. Vector-valued functions are denoted in boldface (e.g. \( f(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_d(\mathbf{x})] \)) and scalar-valued functions in standard-weight typeface. A matrix \( \mathbf{X} \in \mathbb{R}^{n \times d} \) is an \( n \times d \) matrix such that \( \mathbf{X}_{i} = x_i \) and \( \mathbf{X}_{i,j} = x_{i,j} \). The gradient of a scalar function of a vector \( \mathbf{x} \) is defined by \( F'(\mathbf{x}) \triangleq [\partial F(\mathbf{x})/\partial x_1, \ldots, \partial F(\mathbf{x})/\partial x_d] \) and that of a scalar function of a matrix \( \mathbf{X} \) is defined by \( F'(\mathbf{X}) \triangleq [\partial F(\mathbf{X})/\partial x_{i,j}] \). The matrix operator “vec” constructs a vector out of a matrix \( \mathbf{X} = [x_1, \ldots, x_d] \) such that \( \text{vec}(\mathbf{X}) = [x_1, \ldots, x_d] \). Then the Hessian of a scalar function of a matrix \( \mathbf{X} \) is defined by \( F''(\mathbf{X}) \triangleq \partial \text{vec}(F'(\mathbf{X}))/\partial \mathbf{x} \).
A natural partial order is defined between two vectors \( x, y \in \mathcal{X} \). Specifically, if \( x = [x_1, \ldots, x_d] \) and \( y = [y_1, \ldots, y_d] \), then \( x \succeq y \) if \( x_i \leq y_i \) for all \( i = 1 \ldots d \) and \( x \succeq y \) if \( x_i \geq y_i \) for all \( i = 1 \ldots d \).

### B. Definitions and Preliminaries

Consider the \( d \)-dimensional vector recursion

\[
x^{(l+1)} = f(g(x^{(l)}); \epsilon),
\]

where \( f : \mathcal{X} \times \mathcal{E} \to \mathcal{X} \) and \( g : \mathcal{X} \to \mathcal{X} \) are mappings on the spaces \( \mathcal{X} = [0,1]^d \) and \( \mathcal{E} = [0,1] \). As explained in Section I, this equation describes the density evolution of the decoding process. Specifically, the vector \( x^{(l)} \) in the recursion denotes the output distribution of a variable node (an \( f \)-system) at iteration \( l \) of decoding.

Define the functionals \( F : \mathcal{X} \times \mathcal{E} \to \mathcal{R} \) and \( G : \mathcal{X} \to \mathcal{R} \) by \( F(x; \epsilon) = f(x; \epsilon)D \) and \( G(x) = g(x)D \), where \( D \) is a positive \( d \times d \) diagonal matrix.

**Definition 1.** A system \((f, g)\) is called **vector admissible** if

1. \( f, g \) are twice continuously differentiable,
2. \( f(x; \epsilon), g(x) \) are non-decreasing in \( x \),
3. \( f(x; \epsilon) \) is strictly increasing in \( \epsilon \) for \( x \in \mathcal{X}_0 \),
4. \( f(0; \epsilon) = f(\epsilon) = g(0) = 0 \) and \( F(x; 0) = 0 \).

The goal of the paper is to show that threshold saturation occurs for a SC system whose underlying ensemble is a vector admissible system that is governed by the recursion in (2).

**Definition 2.** The potential functional \( U(x; \epsilon) \) of a vector admissible system \((f, g)\) is defined by

\[
U(x; \epsilon) = \int_0^x \left[ (z - f(g(x)); \epsilon)Dz \right] \cdot dz
\]

\[
= g(x)Dx^\top - G(x) - F(g(x); \epsilon).
\]

The potential functional is defined such that it is the integral of the gradient of a function, and so it is path independent. It is important to note that the potential is strictly decreasing in \( \epsilon \) for \( x \in \mathcal{X}_0, \epsilon \in \mathcal{E} \) since \( f \) is strictly increasing in \( \epsilon \) on this domain.

**Definition 3.** Consider a pair \((x, \epsilon) \in \mathcal{X} \times \mathcal{E}\) and the initial value \( x_0 = x \) of the recursion in (2). The **limiting value** is defined as \( x^\infty(x; \epsilon) = \lim_{l \to \infty} f(g(x^{(l)}); \epsilon) \) if the limit exists.

**Lemma 1 ([11]).** Consider a pair \((x, \epsilon) \in \mathcal{X} \times \mathcal{E}\). If \( x \succeq f(g(x); \epsilon) \) or \( x \succeq f(g(x); \epsilon) \), then \( U(x; \epsilon) \geq U(f(g(x); \epsilon)) \).

**Remark.** Using Lemma 1 and since \( x \) is bounded in \([0,1]\), it can be shown that, if \( x \succeq f(g(x); \epsilon) \) or \( x \succeq f(g(x); \epsilon) \), then \( x^\infty(x, \epsilon) \) exists. Specifically, \( x^\infty(1, \epsilon) \) exists.

**Definition 4.** The FP set \( \mathcal{F}(\epsilon) \), its \( x \)-support \( \mathcal{X}_f \), and the epsilon set \( \mathcal{E}(\epsilon) \) are defined by

\[
\mathcal{F}(\epsilon) = \{ x \in \mathcal{X}_0 \text{ s.t. } x = f(g(x); \epsilon) \},
\]

\[
\mathcal{X}_f = \bigcup_{\epsilon \in \mathcal{F}(\epsilon)} g(x),
\]

\[
\mathcal{E}(\epsilon) = \{ \epsilon \in \mathcal{E} \text{ s.t. } x \in \mathcal{F}(\epsilon) \}.
\]

**Definition 5.** The single-system threshold (also called the BP threshold) is defined by

\[
\epsilon^* = \sup \{ \epsilon \in \mathcal{E} | x^\infty(1; \epsilon) = 0 \},
\]

and the potential threshold (also called the MAP threshold) is defined by

\[
\epsilon^* = \sup \{ \epsilon \in \mathcal{E} | \min_{x \in \mathcal{F}(\epsilon)} U(x; \epsilon) \geq 0 \}.
\]

For \( \epsilon > \epsilon^* \), the **energy gap** is defined by

\[
\Delta(E) = \min_{x \in \mathcal{F}(\epsilon)} U(x; \epsilon).
\]

Yedla et al. then introduce two types of spatial coupling. The system we aim to analyze is called the basic SC vector system and was introduced briefly in Section I. The second type of coupling introduced in [1] is called the one-sided SC vector system; it is closely related to the basic system and is easier to analyze. As will be shown below, it is possible to deduce properties about the basic system once they are shown to hold for the one-sided system. We recall the notion of a protograph from Section I and start with a description of the two setups.

An \((L, w)\) basic SC vector system is obtained from the single vector system as follows: the single-system protograph is replicated \( 2L + 1 \) times on the positions in the interval \( \mathcal{L}_f = [-L, \ldots, L] \). We define \( w \) as the spatial coupling window size. The underlying nodes in the \( f \)-system at position \( i \) are connected to underlying nodes in the \( g \)-systems at positions in \([i, \ldots, i + w - 1]\). Conversely, the underlying nodes in the \( g \)-system at position \( i \) will be connected to underlying nodes in the \( f \)-system at positions in \([i, \ldots, i - w + 1]\).

The connections are chosen independently and uniformly at random. The edges between underlying nodes are then chosen by a random permutation.

We remark that the output density is different at every position of the ensemble. Thus, the analysis changes from that of a vector \( x \) (for the uncoupled recursion in (2)) to a matrix \( X \) along the length of the coupling chain. For \( i \in \mathcal{L}_w \), \( [X]_i = x_i \) is the vector consisting of the output density at position \( i \) and the density evolution equation associated with the coupled system is

\[
x^{(l+1)} = \frac{1}{w} \sum_{k=0}^{w-1} f \left( \frac{1}{w} \sum_{j=0}^{w-1} g(x^{(l)}_{i+j-k}); \epsilon_{i-k} \right),
\]

where \( \epsilon_i = \epsilon \) for \( i \in \mathcal{L}_f \) and \( \epsilon_i = 0 \) for \( i \notin \mathcal{L}_f \). \( x_0^{(l)} = 1 \) for \( i \in \mathcal{L}_g \) and \( x^{(l)} = 0 \) for \( i \notin \mathcal{L}_g \) and all \( l \). This recursion can also be expressed in matrix form as

\[
X^{(l+1)} = A^\top f(Ag(X^{(l)}); \epsilon),
\]

where \( A = \frac{1}{w} \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 1 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 1 & \ldots & 1 \end{bmatrix} \).

We can see from the recursion above that the columns of \( X \in \mathcal{X}^{2L+w} \) are non-decreasing and are constructed at each iteration by averaging the rows of a matrix over a sliding window of size \( w \).
The one-sided SC vector system is a variant of the basic system. The curve of $x$ attains its maximum at position $i_0 \in \left[\frac{w-1}{2}\right]$ [5]. The one-sided system is obtained from the basic system by fixing the values $x_i^{(l)} = x_i^{(l)}$ for all $i_0 < i \leq 2L + w$ and for all $l$.

**Lemma 2.** The recursions corresponding to both the basic and one-sided SC vector systems are component-wise decreasing with iteration and converge to well-defined FPs. Also, for $i \in \mathcal{L}_g$, the one-sided recursion is a component-wise upper-bound on the basic recursion and converges to a non-decreasing FP.

**Sketch of proof:** Consider the recursion in (7). Starting with the initial conditions described, and due to the component-wise monotonicity of the functions $f$ and $g$ in $x$, the first part of the lemma follows. As for the second part, it is enough to remark that the one-sided system is obtained from the basic system by fixing the values $x_i^{(l)} = x_i^{(l)}$ for all $i_0 < i \leq 2L + w$, where $x_{i_0}$ is the maximum value of the matrix $X$ of the basic system. Thus, the one-sided recursion will always yield values that upper-bound those of the basic recursion. Finally, the recursions in both cases are bounded in $[0, 1]^d$, which ensures convergence.

**Definition 6.** The coupled-system potential for admissible vector recursions is defined as

$$U(X; \epsilon) \triangleq Tr(g(X)DX^\top) - G(X) - F(Ag(X); \epsilon),$$

where $G(X) = \sum_i G(x_i)$ and $F(X; \epsilon) = \sum_i F(x_i; \epsilon)$.

By direct computation, it can be shown that the derivative of $U(X; \epsilon)$ with respect to $|X|_i = x_i$ satisfies

$$[U'(X; \epsilon)]_i = \left(x_i - [A^\top f(Ag(x); \epsilon)]_i\right)Dg'(x_i),$$

and the Hessian of $U(X; \epsilon)$ satisfies

$$\|U''(X; \epsilon)\|_\infty \leq \|D\|_{\infty}(g'_m + g''_m + 2(g''_m)^2 f'_m) \triangleq K_{f,g}.$$ 

where $g'_m = \sup_{x \in [0,1]} \|g(x)\|_{\infty}$, $g''_m = \sup_{x \in [0,1]} \|g''(x)\|_{\infty}$, and $f'_m = \sup_{x \in [0,1]} \|f'(x; \epsilon)\|_{\infty}$.

Yedla et al. then define the “shifting” operator $S : \mathcal{X}^{2L+w} \to \mathcal{X}^{2L+w}$ such that $[S]_i = 0$ and $[S]_i = x_{i-1}$ for $2 \leq i \leq 2L + w$. The following properties can thus be shown.

**Lemma 3 ([11]).** Let $X \in \mathcal{X}^{2L+w}$ denote a matrix in the one-sided SC system and define $SX$ as above. Then, $\|\text{vec}(SX - X)\|_{\infty} \leq \frac{K_{f,g}}{2}$ and $\|\text{vec}(SX - X)\|_1 = \|X\|_{2L+w}$. Also, $U(SX; \epsilon) - U(X; \epsilon) \leq -U(x_{i_0}; \epsilon)$.

**Lemma 4 ([11]).** For a FP $X$ of the one-sided SC system, $\text{vec}(U'(X; \epsilon)) \cdot \text{vec}(SX - X) = 0$.

**Lemma 5 ([11]).** For a FP $X$ of the one-sided SC system, $x^\infty(x_{i_0}; \epsilon)$ exists, $x_{i_0} \leq x^\infty(x_{i_0}; \epsilon)$, and $U(x_{i_0}; \epsilon) \geq U(x^\infty(x_{i_0}; \epsilon); \epsilon)$.

Using the notions and lemmas above, Yedla et al. proceed to state and prove the main result of the paper.

**Theorem 6.** For a vector admissible system $(f, g)$ with $\epsilon < \epsilon^*$ and $w > C(\epsilon) = dK_{f,g}/(2\Delta E(\epsilon))$, the only fixed point of the (basic) SC system is $0$.

**Proof:** The lemma is first proved by contradiction on the one-sided system and then extended to the basic system. Consider the one-sided recursion with $\epsilon < \epsilon^*$ and $w > C(\epsilon) = dK_{f,g}/(2\Delta E(\epsilon))$, and assume that it admits a FP $X \neq 0$. The Taylor expansion of $U(SX; \epsilon)$ around $U(X; \epsilon)$ yields

$$\frac{1}{2} \text{vec}(SX - X)^\top U''(Z(t); \epsilon) \text{vec}(SX - X) = U(SX; \epsilon) - U(X; \epsilon) - \text{vec}(U'(X; \epsilon)) \cdot \text{vec}(SX - X) \leq -U(x_{i_0}; \epsilon) \leq -\Delta E(\epsilon)$$

such that $Z(t) = X + t(SX - X)$ for some $t \in [0, 1]$. By taking the absolute value and using again Lemma 3,

$$\Delta E(\epsilon) \leq \frac{1}{2} \text{vec}(SX - X)^\top U''(Z(t); \epsilon) \text{vec}(SX - X) \leq \frac{1}{2} \|\text{vec}(SX - X)\|_1 \|U''(Z(t); \epsilon)\|_\infty \|\text{vec}(SX - X)\|_\infty \leq \|X\|_{\infty} K_{f,g} \frac{1}{w}.$$ 

Since $\|X\|_{\infty} \leq d$, and noting that $\Delta E > 0$ if $\epsilon^* \leq \epsilon < \epsilon^*$, the inequality above implies that $w \leq dK_{f,g}/(2\Delta E(\epsilon))$, which contradicts the hypothesis. Thus, the one-sided recursion does not have a non-trivial FP. Since the FP of the basic recursion is upper bounded by that of the one-sided recursion (by Lemma 2), then the only FP of the basic SC system is also $0$.

III. DISPLACEMENT CONVEXITY

In his paper, McCann considers a $d$-dimensional gas of particles represented by its mass density $\rho(x)$ on $\mathbb{R}^d$. He expresses the energy of the gas in the form of a functional and his goal is to check whether the gas reaches a state of equilibrium, in which case the energy is minimized, and if so, to check whether this minimizing state is unique.

To do that, he develops the theory of a new convexity principle called displacement convexity. He introduces an interpolation on the space of probability measures on $\mathbb{R}^d$ and defines displacement convexity of a functional as its convexity with respect to this interpolation parameter. He considers functionals of a specific form and proves that they are displacement convex and admit a unique minimizer under certain conditions.

Although the problem McCann addresses originates from fluid mechanics, it is applicable to a more general class of problems that can be formulated as a similar variational problem. In the remainder of this section, we will introduce the tool of displacement convexity in the general setting in which it is applicable.
A. Definitions and Setup

We now introduce the convex structure on which McCann defines displacement convexity. We first recall the definition of convexity in the conventional sense of a generic functional $E$ on the space of probability measures $\mathcal{P}(\mathbb{R}^d)$. Given two probability measures $\rho, \rho' \in \mathcal{P}(\mathbb{R}^d)$, the linear interpolation $\rho_t$ is defined as $\rho_t = (1-t)\rho + t\rho'$, and $E$ is convex if

$$E(\rho_t) \leq (1-t)E(\rho) + tE(\rho').$$

(12)

McCann introduces an alternative definition of the interpolated measure $\rho_t$ and calls $E$ displacement convex if it satisfies the inequality in (12) under this interpolation. In his paper, he considers functionals of the form $E(\rho) := U(\rho) + G(\rho)/2$, where

$$U(\rho) := \int_{\mathbb{R}^d} A(\rho(x))dx,$$

(13)

$$G(\rho) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\rho(x)V(x-y)d\rho(y),$$

(14)

and proves that $E$ is displacement convex if $A$ and $V$ satisfy certain conditions that we will discuss.

The alternative structure on which displacement convexity is defined is the $d$-dimensional space of absolutely continuous probability measures $\mathcal{P}_{ac}(\mathbb{R}^d)$ and will be used as such throughout this section. We first give the definition of the interpolant for $d = 1$ for simplicity.

Denote by $\rho[M] \mathcal{M}$ the measure of the set $M$ on $\mathbb{R}$ by $\rho$. Then, given two measures $\rho, \rho' \in \mathcal{P}_{ac}(\mathbb{R})$ and a point $x \in \mathbb{R}$, there exists a value $y(x) \in \mathbb{R} \cup \{\pm \infty\}$ such that

$$\rho([-\infty, x]) = \rho'([-\infty, y(x)]).$$

(15)

According to a theorem by Brenier [10], [11], the transformation $y$ may be taken to be non-decreasing so that $y(x)$ is uniquely determined for a given $x \rho$-a.e. The interpolated measure $\rho_t$ acts on $\rho, \rho'$ by linearly displacing the mass lying under $\rho$ at $x$ towards the corresponding point $y(x)$ for $\rho'$ as a function of the time $t \in [0, 1]$. That is, the interpolant $\rho_t$ satisfies

$$\rho([-\infty, x]) = \rho_t([-\infty, (1-t)x + ty(x)]).$$

(16)

In order to give a generalized definition of the interpolant on the space of measures on $\mathbb{R}^d$, we recall the definition of the pushforward. Given a measure $\rho \in \mathcal{P}(\mathbb{R}^d)$ and a Borel transformation $y: \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined $\rho$-a.e., the pushforward of $\rho$ through $y$ is denoted as $y_{\#}\rho$ and defined for any Borel set $M \in \mathbb{R}^d$ by

$$y_{\#}\rho[M] := \rho[y^{-1}(M)].$$

(17)

McCann imposes two “smoothness” conditions on the transformation $y$: that it be locally irrotational\(^2\) and involve no crossings\(^3\). According to Brenier’s theorem [10], [11], as applied to the general dimension case, a pushforward map $y$ that satisfies these conditions may be taken to be the gradient of a convex function $\psi: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, where the map $\nabla\psi$ is unique $\rho$-almost everywhere (a.e.) (although $\psi$ may not be unique). This allows us to define the displacement interpolation.

**Definition 7.** [2] Given probability measures $\rho, \rho' \in \mathcal{P}_{ac}(\mathbb{R}^d)$, there exists $\psi$ convex on $\mathbb{R}^d$ such that $\nabla\psi_{\#}\rho = \rho'$. If $id$ denotes the identity mapping on $\mathbb{R}^d$, then the displacement interpolant $\rho_t \in \mathcal{P}(\mathbb{R}^d)$ between $\rho$ and $\rho'$ at time $t \in [0, 1]$ is defined by

$$\rho_t := [(1-t)id + t\nabla\psi]_{\#}\rho.$$  

(18)

We assume that the function $\psi$ is twice differentiable, and denote by $\nabla^2\psi$ its gradient and by $\nabla^2\psi$ its Hessian.

B. Displacement Convexity of $G(\rho)$

In this section, we reproduce McCann’s proof of the displacement convexity of $G(\rho)$, under certain conditions on $V$. The proof is short and gives insight into the convex structure on which the interpolation is defined.

We first remark that the change of variables theorem and the definition of the pushforward (17) imply

$$\int_{\mathbb{R}^d} f(x)d(\nabla\psi_{\#}\rho)(x) = \int_{\mathbb{R}^d} f(\nabla\psi(x))d\rho(x).$$

(19)

**Proposition 7.** [2] The functional $G(\rho_t)$ is a convex function of $t$ on $[0, 1]$. It is strictly convex when $V(x)$ is strictly convex, unless $\rho'$ is a translate of $\rho$.

**Proof:** The property (19) allows us to write

$$G(\rho_t) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\rho_t(x)V(x-y)d\rho_t(y)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} d\rho(x)V((1-t)(x-y) + t(\nabla\psi(x) - \nabla\psi(y)))d\rho(y).$$

The displacement convexity of $G(\rho)$ – the convexity of $G(\rho_t)$ along the lines of the displacement interpolation – follows directly from the convexity of $V(x)$. Further, strict convexity of $V(x)$ implies strict displacement convexity of $G(\rho)$ unless $x - y = \nabla\psi(x) - \nabla\psi(y)$ holds $\rho \times \rho$-a.e., that is, unless $\rho'$ is a translation of $\rho$ by $\nabla\psi(x) - x$.

C. Displacement Convexity of $U(\rho)$

In this section, we give an outline of the proof of the displacement convexity of $U(\rho)$, under certain conditions on $A$ (see (13)). The proof relies on analytic derivations as well as a lemma that we state below.

**Lemma 8.** [2] Let $\Lambda$ be a non-negative $d \times d$ matrix and $v(t) := \det([1-t]I + t\Lambda)$, where $I$ is the identity matrix. Then $v^{1/d}(t)$ is concave on $t \in [0, 1]$, and the concavity is strict unless $\Lambda = \Lambda I$.

\(^{2}\)A function is locally irrotational if its circulation around every closed curve vanishes. In McCann’s setting, it ensures that mass is transferred from one set to another in a well-defined direction, so that the transfer is “invertible”.

\(^{3}\)A function $y$ involves no crossings if $(1-t)x + ty(x) = (1-t)x' + ty(x')$ implies that $x = x'$ for $t \in [0, 1]$. This also ensures that the transfer is mass-preserving.
Theorem 9. If $\lambda^d A(\lambda^{-d})$ is convex non-increasing on $\lambda \in (0, \infty)$ with $A(0) = 0$, then $U(\rho)$ is a convex function of $t$ on $[0, 1]$. Strict convexity follows from that of $\lambda^d A(\lambda^{-d})$ unless $\nabla^2 \psi(x) = I \rho$-a.e., where $I$ is the identity matrix.

Proof: It can be shown [2] that

$$U(\rho) = \int_{R^d} A \left( \frac{\rho(x)}{\det(1-tI+t\Lambda(x))} \right) \det((1-t)I+t\Lambda(x)) dx, \quad (20)$$

where $\Lambda = \nabla^2 \psi$ is the Hessian of $\psi$.

The integrand in (20) can be written as the composition of the function $h(\lambda) = \lambda^d A(\lambda^{-d}) \rho(x)$ with the function $g(t) = \lambda^{1/\delta}(t)$. Since $h(\lambda)$ is convex non-increasing (by the hypothesis) and $g(t)$ is concave (due to Lemma 8), their composition is convex. Strict convexity of the integrand follows from that of $h$ unless $g(t)$ is a constant, which follows from Lemma 8 in the case when $\nabla^2 \psi(x) = I \rho$-a.e.

Since each of $G(\rho)$ and $U(\rho)$ are displacement convex, then $E(\rho) = U(\rho) + G(\rho)/2$ is displacement convex.

D. Existence and Uniqueness of Minimizer

After proving that the functional $E$ of interest is displacement convex, McCann proceeds to the question motivating his new theory. Specifically, he proves that the functional admits a minimizer that it is unique (up to translation) given certain conditions. In this section, we recall the definition in (13) and give the main ideas in his analysis.

Theorem 10. Assume that $\lambda^d A(\lambda^{-d})$ is convex non-increasing on $\lambda \in (0, \infty)$ with $A(0) = 0$ and that the energy minimizing measure is in $\mathcal{P}_{ac}(R^d)$. Assume that $V : R^d \to R \cup \{+\infty\}$ is strictly convex. Then, if $E_\gamma \equiv \inf_{\rho \in \mathcal{P}_{ac}(R^d)} E(\rho) < \infty$, the infimum of $E(\rho)$ is uniquely attained up to translation.

Sketch of Proof: We replace $V(x)$ by $(V(x)+V(-x))/2$ in $G(\rho)$ since that will leave the functional unchanged. We add a constant so that $V(0) = 0$; this will shift $E(\rho)$ by the same value. Since $U(\rho) \geq 0$ and $V(0) \geq 0$, then $E(\rho) \geq 0$. This allows us to extract a sequence $\rho_n \in \mathcal{P}_{ac}(R^d)$ that satisfies

$$\lim_{n \to +\infty} E(\rho_n) = \inf_{\rho \in \mathcal{P}_{ac}(R^d)} E(\rho), \quad (21)$$

$\rho_n$ is thus called a minimizing sequence of $E(\rho)$. Since $U(\rho)$ and $G(\rho)$ are weak-* lower semi-continuous (see [2]), then any weak-* limit point $\rho_0 \in \mathcal{P}_{ac}(R^d)$ of the sequence $\rho_n$ would satisfy

$$E(\rho_0) \leq \liminf_{n \to +\infty} E(\rho_n). \quad (22)$$

Since $\inf_{\rho} E(\rho) \leq E(\rho_0)$ and $\liminf_{n \to +\infty} E(\rho_n) \leq \liminf_{n \to +\infty} E(\rho_n)$ by definition, then by combining (21) and (22), we obtain $E(\rho_0) = \inf_{\rho} E(\rho)$. A compactness argument is then used to prove that the limit $\rho_0$ is indeed attained on the space $\mathcal{P}_{ac}(R^d)$. This concludes the existence proof.

The uniqueness proof is as follows: assume there exist two minimizers $\rho_0, \rho'_0 \in \mathcal{P}_{ac}(R^d)$ and define their displacement interpolant $\rho_t$ as before, for $t \in (0, 1)$. Since $U(\rho)$ and $G(\rho)$ are displacement convex (by Theorem 9 and Proposition 7, respectively), then $E(\rho_t) \leq (1-t)E(\rho_0)+tE(\rho'_0)$. According to Proposition 7, strict inequality holds unless $\rho_t$ is a translate of $\rho_0$. Due to the definition of $E_\gamma$, uniqueness (up to translation) is established.

IV. REARRANGEMENT INEQUALITIES

Rearrangement inequalities have shown to serve as useful tools in several mathematical problems. A rearrangement of a function is a mass-preserving transformation. The inequality presented by Brascamp, Lieb, and Luttinger [3] generalizes inequalities that have been proven by Riesz [12], Hardy, Littlewood, and Pólya [13], and Luttinger and Friedberg [14]. We present, in this section, the general rearrangement inequality proved by Brascamp et al.

A. Definitions and Preliminaries

We start by defining a symmetric decreasing rearrangement and stating some useful preliminaries.

Definition 8. For a non-negative function $f$, we define the measure $M^r_f = \mu \{ x \mid f(x) \geq y \}$, where $\mu$ denotes the Lebesgue measure. The symmetric decreasing rearrangement of $f$ is denoted as $f^*: it has the same properties as $f$ and additionally satisfies $f^*(x) = f^*(-x) \forall x$, $f^*(x_2) \leq f^*(x_1)$ $\forall 0 < x_1 < x_2$, and $M_f^r = M_y^r \forall y > 0$.

Lemma 11 ([15]). Let $C$ be a convex set and $F$ be a function in $R^{d+1}$, and let $F(t)$ be the family of planes $\{ F(x, t) \}$, for $t \in R$. Let $v(t)$ be the volume of the convex set $F(t) \cap C$ in $R^d$, then $v(t)^{1/d}$ is a concave function of $t$ in the interval where $v(t) > 0$.

Corollary 12. Let $C, F$, and $v(t)$ be as in Lemma 11. If $C$ is balanced (i.e. $x \in C \Rightarrow -x \in C$), then $v(t) = v(-t)$ and $v(t_2) = v(t_1)$ for $0 \leq t_1 \leq t_2$.

B. Rearrangement for Several Cases

In this section, we present the following rearrangement inequality that has been proved by Brascamp et al. [3].

Theorem 13. Let $f_j, 1 \leq j \leq k$, be non-negative measurable functions on $R$, and $a_{jm}, 1 \leq j \leq k, 1 \leq m \leq d$, be real numbers. Then,

$$\int_{R^d} \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m \right) \leq \int_{R^d} \prod_{j=1}^k f_j^* \left( \sum_{m=1}^d a_{jm} x_m \right) \quad (23)$$

Brascamp et al. start by proving the theorem in the case when each function $f_j, 1 \leq j \leq k$ is the characteristic function of a single interval, then when each function is the characteristic function of a finite union of disjoint intervals. They
Since the partial ordering is different than the one used in Section 2, the claim.

Lemma 14. Let \( f_j(x) \), \( 1 \leq j \leq k \) be the characteristic functions of the corresponding intervals \([b_j - c_j, b_j + c_j]\). Define the function \( f_j(x|t) = f_j(x + b_j t) \). Then \( I(t) \) is a non-decreasing function of \( t \in [0, 1] \), where
\[
I(t) = \int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m | t \right).
\]

Sketch of proof: \( I(t) \) is the volume of the intersection of the set \( C \) and the plane \( x_d + t = 1 \), where
\[
C = \bigcap_{1 \leq j \leq k} \{ x \in R^{d+1} | -c_j \leq \sum_{m=1}^d a_{jm} x_m - b_j x_d + 1 \leq c_j \}.
\]

Since \( C \) is convex and balanced, Corollary 12 implies that \( I(t) \) is non-decreasing for \( t \in [0, 1] \).

Corollary 15. The inequality (23) holds if each \( f_j \), \( 1 \leq j \leq k \) is the characteristic function of a single interval.

Proof: The proof follows from Lemma 14 by remarking that \( f_j(x|0) = f_j(x) \) and \( f_j(x|1) = f_j^*(x) \).

Lemma 16. The inequality (23) holds if each \( f_j \), \( 1 \leq j \leq k \) is the characteristic function of a corresponding union of disjoint compact intervals.

Proof: Denote by \( n_j \) the number of intervals of the function \( f_j \), for \( 1 \leq j \leq k \) and define \( N = \{ n_1, \ldots, n_k \} \). The partial order \( \preceq \) is defined such that \( M \preceq N \) if \( m_j \leq n_j \) for \( 1 \leq j \leq k \) and \( n_i < n_j \) for some \( i \). (Remark that this partial ordering is different than the one used in Section II.) The lemma is proved by induction on \( N \). The base case \( N = \{1, \ldots, 1 \} \) is the result of Corollary 15. Assume that the lemma holds for all \( M \preceq N \). By construction, each \( f_j(x) \) is the characteristic function of the set
\[
\bigcup_{1 \leq p \leq n_j} \{ x \in R | b_{jp} - c_{jp} \leq x \leq b_{jp} + c_{jp} \}
\]
for \( 1 \leq j \leq k \), such that \( b_{jp} + c_{jp} < b_{jp+1} - c_{jp+1} \) (for each function, the intervals are disjoint). The function \( f_j(x|t) \) is defined to be the characteristic function of the set
\[
\bigcup_{1 \leq p \leq n_j} \{ x \in R | b_{jp}(1-t) - c_{jp} \leq x \leq b_{jp}(1-t) + c_{jp} \}
\]
for \( 1 \leq j \leq k \). Since the intervals must remain disjoint throughout the proof, the definition in (26) is only valid when \( 0 \leq t \leq \tau \), where \( \tau = \min[1 - (c_{jp+1} + c_{jp})/(b_{jp+1} - b_{jp})] \) is the smallest value at which at least two intervals coalesce for some \( j \). Since each \( f_j \) can be expressed as the sum of \( n_j \) characteristic functions \( g_{jp} \), \( 1 \leq p \leq n_j \), of single intervals, we rewrite the left-hand side of (23) as
\[
\int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m \right) = \int_{R^d} d^d x \prod_{h=1}^{\sum_{j=1}^k n_j} \hat{g}_h \left( \sum_{m=1}^d a_{jm} x_m \right),
\]
where the functions \( \hat{g} \) are the characteristic functions of all the single intervals that are characterized by the functions \( f_j \), \( 1 \leq j \leq k \). We thus apply Lemma 14 on the integral in (27) and find that
\[
\int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m \right) \leq \int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m | \tau \right).
\]

Since at \( t = \tau \), there is at least one coalescence of intervals, the total number of intervals is smaller and we have by the inductive hypothesis that
\[
\int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m | \tau \right) \leq \int_{R^d} d^d x \prod_{j=1}^k f_j \left( \sum_{m=1}^d a_{jm} x_m \right).
\]

Combining (28) and (29) completes the inductive argument.

Brascamp et. al refer to [13] for a generalization of this result to functions that take any non-negative values as well as those that have infinite support. We present here an argument that allows us to generalize the rearrangement inequality for functions that take any non-negative values and have finite support. We start by recalling the “layer-cake” formulation of a non-negative function.

Definition 9. For a non-negative function \( f \), define the function \( \chi_y(x) = 1 \) if \( f(x) \geq y \) and \( \chi_y(x) = 0 \) otherwise. Then \( f \) and its symmetric decreasing rearrangement \( f^* \) can be expressed in layer-cake form as
\[
f(x) = \int_0^\infty dy \chi_y(x), \quad f^*(x) = \int_0^\infty dy \chi_y^*(x).
\]

Lemma 17. The inequality in (23) holds if each \( f_j \), \( 1 \leq j \leq k \), is a function that takes any finite non-negative values and has finite support.

Proof: Using the definition of the layer-cake formula, we express the left-hand side of (23) as
\[
\int_{R^d} d^d x \int_{[0, \infty]^k} \prod_{j=1}^k dy_j \chi_{y_j} \left( \sum_{m=1}^d a_{jm} x_m \right)
\]
\[
= \int_{[0, \infty]^k} \prod_{j=1}^k dy_j \int_{R^d} d^d x \chi_{y_j} \left( \sum_{m=1}^d a_{jm} x_m \right)
\]
\[
\leq \int_{[0, \infty]^k} \prod_{j=1}^k dy_j \int_{R^d} d^d x \chi_{y_j}^* \left( \sum_{m=1}^d a_{jm} x_m \right)
\]
\[
= \int_{R^d} d^d x \int_{[0, \infty]^k} \prod_{j=1}^k dy_j \chi_{y_j}^* \left( \sum_{m=1}^d a_{jm} x_m \right),
\]
where the equalities follow from Fubini’s theorem since \( f_j \), \( 1 \leq j \leq k \) have finite support and take finite non-negative values, and the inequality is due to Lemma 16. This proves the claim.

V. DISCUSSION AND RESEARCH PLAN

Spatial coupling has shown to perform very well under low-complexity algorithms. Indeed, as shown in [1], the phenomenon of threshold saturation occurs for codes governed
by coupled vector recursions. This result had previously been proven for codes governed by coupled scalar recursions [6] and was later extended to codes on BMS channels [8]. It has been shown that spatial coupling is a powerful tool in approaching capacity regions of interest. Hence, it is clearly worth investigating further into the properties of SC systems.

Displacement convexity serves as a tool to analyze SC codes. In effect, it can be used as another proof technique for threshold saturation via spatial coupling. Consider for example the case of the $(ℓ, r)$-regular Gallager ensemble over the BEC, with the noise parameter fixed to $ε_{\text{MAP}}$. In that case, it has been shown in [9] that there exists a non-trivial FP solution to the density evolution equation, and that it is unique up to translation. The work in [9] only served as a proof of concept; that is, to show that displacement convexity is applicable for such analyses.

The proof technique employed in [9] consists of expressing the potential functional of the coupled ensemble in continuous form, and proving that it is strictly convex with respect to an alternative structure of probability measures. Since the minimization of the potential yields the FP solution, strict convexity proves the uniqueness of the minimizer. It is unique only up to translations due to the translation invariance of the potential. This result, along with that in [1], proves that the potential-minimizing decoding profile is unique regardless of the initial conditions on the error probabilities of the single systems along the coupling chain. In fact, the work in [9] shows that this FP profile increases from 0 to $ε$, which means that it is "stuck" and perfect decoding does not occur for the whole chain. The work in [3] allowed the authors to see that indeed, it is an increasing profile that minimizes the potential, and thus rearrangement inequalities played a crucial role in this proof.

Another important question not addressed in [9] concerns what happens when the noise parameter is not fixed at $ε_{\text{MAP}}$, but is slightly below it. We expect to reach a result similar to that explained in Section II; that is, that the FP solution is uniquely the trivial solution, and perfect decoding occurs. For such values of $ε$, the single system has a strictly positive energy gap (see Section II-B). It has been shown in [16] that such systems exhibit the phenomenon of "moving waves". We propose to use displacement convexity to characterize this travelling wave, in terms of existence and uniqueness, and perhaps also the velocity with which it travels.

As future research directions, we propose to generalize the analysis in [9] with regard to several aspects, as well as to answer the question above in those generalized settings. First, it is interesting to explore the conditions that a system must satisfy so that it is possible to employ displacement convexity as a proof technique. That is, we consider a more general system under BP decoding, governed by a generic recursion, in the scalar (1) and vector (2) cases, and we define the corresponding potentials, as given in [6] and [1], respectively. We are then interested in taking the continuum limit of the potential, so that we can investigate its displacement convexity. As such, we investigate the conditions on the message–passing rules $f$ and $g$, if they exist, so that the limit exists. We are also interested in coining the properties of $f$ and $g$ that are sufficient for the displacement convexity of the potential.

As shown in Section II, the minimizer of the potential of a coupled ensemble is indexed by the discrete values that refer to the spatial dimension on which the single system is replicated. However, in order to employ displacement convexity on such potentials, one has to take their continuum limit. Although this serves as a good approximation in the large chain-length limit, it is evidently more accurate if the tool of displacement convexity were adjusted to study discrete functionals. In that respect, we propose to analyze the error made in such approximations, or perhaps to find a discrete equivalent of displacement convexity to assess coding schemes of interest.

Finally, displacement convexity can prove to be universally useful as a proof technique if it can prove threshold saturation on any BMS channel. We thus propose the investigation of displacement convexity on potentials associated to BMS channels, where message–passing decoding involves the exchange of general probability densities. In that case, one has to check whether there are equivalent notions in such settings for tools that were used in the proof techniques of the above cases, such as rearrangement inequalities.

**REFERENCES**


