Threshold Saturation for LDPC Convolutional Code Ensembles

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Abstract—Low-density parity-check (LDPC) codes, originally invented by Gallager in 1963, are now well-known for their good performances. Their convolutional counterpart was introduced by Felström and Zigangirov in 1999 and have already been shown to be able to achieve the Shannon capacity under low-complexity iterative decoding algorithms. In principle, the belief-propagation (BP) threshold of the convolutional or spatially-coupled LDPC code ensemble can be increased up to the maximum a posteriori (MAP) threshold of the underlying uncoupled ensemble. In this write-up, we will focus on explaining this “threshold saturation” phenomenon for the binary erasure channel (BEC) in detail, and propose some future research plans.

Index Terms—Belief propagation decoding, channel capacity, density evolution, extrinsic information transfer functions, low-density parity-check convolutional code ensembles, maximum a posteriori decoding, potential functions, spatial coupling, and threshold saturation.

I. INTRODUCTION

Low-density parity-check (LDPC) codes [1] are powerful error-correcting codes which have been well-known for their excellent performance under low-complexity iterative BP decoding. Particularly, regular LDPC codes can asymptotically and universally achieve the capacity under MAP decoding [2]; also see [3] for a more general treatment on iterative decoding analysis for LDPC codes. The convolutional counterpart of LDPC codes was first proposed by Felström and Zigangirov in 1999 [4]. These codes, when properly terminated, are able to exhibit better performance, e.g., better thresholds, than the corresponding LDPC block codes [5]. More precisely, [5] analyzed the convergence behavior of the BP thresholds of a special class of terminated regular LDPC convolutional codes with rate one-half via density evolution (DE) techniques, over the binary erasure channel (BEC) and the binary-input additive white Gaussian noise channel (BAWGNC). Most importantly, [5] reported that the BP thresholds of the terminated LDPC convolutional codes are very close to the MAP thresholds of the corresponding regular LDPC block codes, via numerical simulations.

Later, Kudekar et al. [6] considered a “convolutional-like” or “spatially-coupled” regular LDPC code ensemble, and then formally showed that, over the BEC, the spatial-coupling of individual codes increases the iterative BP threshold of the coupled ensemble to its maximum possible value, which is equal to the MAP threshold of the underlying code ensemble. They called this phenomenon “threshold saturation via spatial coupling”. The proof technique relies on the so-called extrinsic information transfer (EXIT) function and the area theorem of the code ensemble [7], and the key to the proof is to show the existence of a unimodal fixed point (FP) of DE. The channel parameter associated with such an FP is necessarily very close to the MAP threshold of the underlying ensemble.

The discovery of this fundamental phenomenon, threshold saturation via spatial coupling, is rather important, in that it gives us a new way of achieving the channel capacity under low-complexity decoding algorithms. However, the proof of [6] is technical and it does not easily generalize to general binary memoryless symmetric (BMS) channels. A simpler proof of the threshold saturation is given by [8] by using the potential theory in statistical physics. Rather than focusing on spatially-coupled LDPC codes, [8] considered more general one-dimensional scalar recursions and their corresponding coupled scalar recursions, and developed a general theory for proving the threshold saturation results.

The organization of this write-up is as follows. Section II discusses an early analysis on decoding thresholds for LDPC convolutional codes by Lentmaier et al. [5]. In Section III, we will discuss in detail the work by Kudekar et al. [6]. Section IV analyzes the work by Yedla et al. [8]. In Section V, we will propose some future research plans.
II. AN EARLY BREAKTHROUGH [5]

Consider a rate $R = b/c$ time-varying binary convolutional code which can be defined as the set of sequences

$$v \triangleq (\cdots, v_1, v_2, \cdots, v_t, \cdots),$$

where $v_t \in \mathbb{F}_2$, satisfying the constraint $vH^\top = 0$, and $H^\top$ is the infinite syndrome former matrix [5].

$$H^\top \triangleq \begin{pmatrix}
H_0^\top(1) & \cdots & H_m^\top(1 + m_s)
\end{pmatrix}$$

where

$$H_i^\top(t) \triangleq \begin{pmatrix}
H_0^\top(t) & \cdots & H_m^\top(t + m_s)
\end{pmatrix},$$

satisfying, for $0 \leq i \leq m_s$, $H_i^\top(t + i)$ is a $c \times (c - b)$ binary matrix such that $H_0^\top(t)$ is full rank for all time instant $t$. Here $m_s$ is called the syndrome former memory, indicating the largest index $i$ such that $H_i^\top(t + i)$ is a non-zero matrix for some $t$.

Lentmaier et al. [5] analysed an ensemble $C_P(J,2J,M)$ of $(J,J)$ regular time-varying LDPC convolutional codes which is described as follows. For $J > 2$, let the syndrome former memory $m_s = J - 1$. For $0 \leq i \leq J - 1$, let

$$H_i^\top(t + i) \triangleq \begin{pmatrix}
P_i^{(0)}(t + i) & P_i^{(1)}(t + i)
\end{pmatrix}^\top,$$

where $P_i^{(h)}(t + i)$ are $M \times M$ permutation matrices for $h \in \{0,1\}$. All other entries of the syndrome former are zero matrices. Equivalently, each $H_i^\top(t + i)$ is a $2M \times M$ binary matrix, and by construction, each row of the syndrome former $H^\top$ has $J$ ones in each row and $2J$ ones in each column. Let the permutation matrices $P_i(t + i)$, $t \geq 1, 0 \leq i \leq J - 1$, and $h \in \{0,1\}$, be chosen uniformly at random and independently over the set of $M!$ permutation matrices of size $M \times M$.

The ensemble of $H^\top$ defines the corresponding ensemble $C_P(J,2J,M)$ of $(J,J)$ regular LDPC convolutional codes. As has been discussed in [5], almost all codes in $C_P(J,2J,M)$ can be terminated with a tail of length $m_s + 1$ blocks. By terminating LDPC convolutional codes, the investigation of their thresholds are reduced to the estimation of thresholds of LDPC block codes, in such a way that existing techniques for the latter become handy.

<table>
<thead>
<tr>
<th>${(J,2J)}$</th>
<th>$\varepsilon_{conv}$</th>
<th>$\varepsilon_{blk,map}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(3,6)}$</td>
<td>0.488</td>
<td>0.49134</td>
</tr>
<tr>
<td>${(4,8)}$</td>
<td>0.497</td>
<td>0.49798</td>
</tr>
<tr>
<td>${(5,10)}$</td>
<td>0.499</td>
<td>0.49951</td>
</tr>
</tbody>
</table>

Threshold comparison were reported in Table I [5]. As we can see, remarkably, terminated LDPC convolutional codes almost achieve the upper bounds on the MAP thresholds of the underlying regular LDPC codes, even under suboptimal BP decoding. This is our focus in subsequent sections.

III. THRESHOLD SATURATION FOR LDPC CONVOLUTIONAL CODE ENSEMBLES [6]

A. Definition and Preliminaries

We start with the specific construction described in [6]. This gives the following definition of the $(d_1,d_r,L,w)$ ensemble.

Definition 1 (The $(d_1,d_r,L,w)$ Ensemble): Let the variable nodes be at integer positions $[-L,L], L \in \mathbb{N}$. At each position there are $M$ variable nodes and $\sum_{i=0}^{L} M$ check nodes, $M \in \mathbb{N}$. Assume that each of $d_i$ connections of a variable node at position $i$ is uniformly and independently chosen from the range $[i, \cdots, i + w - 1]$, where $w$ is a “smoothing” parameter. Similarly, assume that each of the $d_r$ connections of a check node at position $i$ is uniformly and independently chosen from the range $[i - w + 1, \cdots, i]$.

Definition 2 (Design Rate): The design rate of the ensemble $(d_1,d_r,L,w)$ with $w \leq 2L$ is

$$R(d_1,d_r,L,w) = \left(1 - \frac{d_1}{{d_r}}\right) - \frac{d_1}{d_r} + 1 - 2 \sum_{i=0}^{L} \left(\frac{d_r}{2L+1}\right).$$

Discussion: Recall that we want to show the existence of a unimodal FP and such existence implies that the associated channel parameter, call it $\varepsilon^*$, is very close to $c_{\text{map}}(d_1,d_r)$, the MAP threshold of the underlying uncoupled $(d_1,d_r)$ ensemble. For doing so, [6] first established the coupled DE equations and the FP conditions, and discussed several properties of the non-trivial FPs of forward DE. We describe this below.

B. Existence of FP

Definition 3 (DE of $(d_1,d_r,L,w)$ Ensemble): Let $x_i, i \in \mathbb{Z}$, be the average erasure probability, which is emitted by variable nodes at position $i$. For $i \notin [-L,L]$, let $x_i = 0$. For $i \in [-L,L]$, the FP condition implied by DE is

$$x_i = \varepsilon \left(1 - \frac{1}{w} \sum_{j=0}^{w-1} \left(1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k}\right) d_{r-1}\right) d_{r-1}. \quad (3)$$

Discussion: Define $f_j = \left(1 - \frac{1}{w} \sum_{k=0}^{w-1} x_{i+j-k}\right) d_{r-1}$ so that the FP condition becomes $x_i = \varepsilon \left(1 - \frac{1}{w} \sum_{j=0}^{w-1} f_{i+j}\right) d_{r-1}$. Note that the average erasure probability emitted from a check node at position $i + j$ to a variable node at position $i$ is equal to $1 - f_{i+j}$. In what follows, it will be also convenient to write the FP condition as

$$x_i = \varepsilon g(x_{i-w+1}, \cdots, x_{i+w-1}).$$

Note that the function $f_j(x_{i-w+1}, \cdots, x_{i})$ is decreasing in all its argument, or equivalently, the function $g(x_{i-w+1}, \cdots, x_{i+w-1})$ is increasing in all its arguments.

Definition 4 (FPs of DE): Consider DE for the ensemble $(d_1,d_r,L,w)$. Let $\bar{x} = (x_{-L}, \cdots, x_L)$ be the constellation and let $\varepsilon = (\varepsilon_{-L}, \cdots, \varepsilon_0, \cdots, \varepsilon_L)$ where $\varepsilon_i \in \mathbb{R}^+$ for $i \in [-L,L]$. We say that $(\varepsilon, \bar{x})$ forms an FP if, for all $i \in [-L,L]$,

$$x_i = \varepsilon_i g(x_{i-w+1}, \cdots, x_{i+w-1}). \quad (4)$$

We say that $(\varepsilon, \bar{x})$ is a non-trivial FP if $\bar{x}$ is not zero.
Definition 5 (Forward DE): Consider DE for the ensemble $(d_l, d_r, L, w)$. Let $\epsilon \in [0, 1]$. Initialize $x(0) = (1, \ldots, 1)$. Let $x^{(t)}$ be the result of $t$ rounds of DE, i.e.,
\[
x^{(t+1)}_i = e \left( \lambda^{(t)}_i \right)_{i=-L}^L ,
\]
for all $i \in [-L, L]$. We call this the forward DE.

Lemma 6 (FPs of Forward DE): Consider forward DE for the ensemble $(d_l, d_r, L, w)$. The sequence of constellations $x^{(t)}$ converges to a limit $x^{(\infty)}$ of DE with parameter $\epsilon$.

Definition 7 (Entropy): Let $x$ be a constellation. We define the normalized entropy to be
\[
\chi(x) = \frac{1}{2L+1} \sum_{i=-L}^L x_i ,
\]
Lemma 8 (Nontrivial FPs of the Forward DE): Consider the ensemble $(d_l, d_r, L, w)$. Let $x$ be an FP of forward DE for the parameter $\epsilon$. For $\epsilon \in (\frac{d_l}{d_r}, 1]$ and $\chi \in (0, \epsilon^{(1/d_l-1)}(\epsilon - \frac{d_l}{d_r}))$, if
\[
L \geq \frac{2w}{2 \left( \frac{d_l}{d_r} \epsilon - \chi \epsilon^{\frac{1}{d_l-1}} - 1 \right)} ,
\]
then $\chi(x) \geq \chi$.

Discussion: As has been pointed out in [6], proving the existence of a unimodal FP by direct methods is very difficult. Instead, [6] considered proving the existence of a one-sided “increasing” FP, and then constructed the desired unimodal FP from the one-sided FP.

Definition 9 (One-Sided DE): Consider $(x_{-L}, \ldots, x_0)$ which has length $L+1$. The FP condition implied by one-sided DE is (3) with $x_i = 0$ for $i < -L$ and $x_i = x_0$ for $i > 0$.

Discussion: $x$ is called non-decreasing if $x_i \leq x_{i+1}$ for $-L \leq i \leq 0$, and the entropy of $x$ is defined as $\chi(x) = \frac{1}{L+1} \sum_{i=-L}^{0} x_i$. We say that $(\epsilon, x)$ is a proper one-sided FP if it is non-trivial and non-decreasing.

Lemma 10 (FPs of One-Sided DE): Consider the ensemble $(d_l, d_r, L, w)$ and let $\epsilon \in [0, 1]$. Let $x(0) = (1, \ldots, 1)$ be length $L+1$ and let $x(0)$ be the result of applying $\ell$ rounds of one-sided forward DE. Then, $x^{(t)}$ converges to a limit $x^{(\infty)}$ which is either proper or trivial. Furthermore, if the conditions of Lemma 8 is fulfilled, then $\chi(x^{(\infty)}) \geq \chi$.

Lemma 11 (Transition Length): Let $w \geq 2d_r^2d_l^2$. Let $(\epsilon, x)$,\n$\epsilon \in (\epsilon^{BP}, 1]$, be a proper one-sided FP of length $L+1$. Then, for all $0 < \delta < \frac{2d_r^2d_l^2}{(L+1)(w+2d_l^2)}$,\n\[
\begin{align*}
\{i : \delta < x_i < x_\epsilon(\epsilon) - \delta\} &\leq w \frac{e(d_l, d_r)}{\delta} ,
\end{align*}
\]
where $c(d_l, d_r)$ is a strictly positive constant independent of $L$ and $\epsilon$. Let $h(x) = \epsilon \left(1 - (1-x)e^{-\alpha} \right)^{d_r-1} - x$. Then $h(x) = 0$ has two roots in $[0, 1]$, the larger one is denoted by $x_\epsilon(\epsilon)$ and the smaller one is denoted by $x_r(\epsilon)$.

Discussion: The proper one-sided DE always satisfies $x_\epsilon(\epsilon) \leq x_0 \leq x_r(\epsilon)$ and the associated channel parameter $\epsilon > \epsilon^{BP}(d_l, d_r)$. Moreover, Lemma 11 implies that the transition part of the one-sided DE is a constant fraction of $L$.

Theorem 12 (Existence of One-Sided FPs): For the fixed parameters $(d_l, d_r, w)$, let $x_u(1) < \chi$ and $L \geq \lambda(\epsilon)(d_l, d_r, w, \chi)$, where
\[
L(d_l, d_r, w, \chi) \geq \max \left\{ 4d_l w, \left( \frac{1}{d_r} \right) (\chi - x_u(1)) \right\} ,
\]
and channel parameter $\epsilon = 1$.

Discussion: By choosing $(d_l, d_r, L, w)$ properly, the second alternative is not possible, i.e., for the range of parameters of interest, the proper one-sided FP exists and has entropy equal to $\chi$ and the associated channel parameter is bounded by $\epsilon^{BP}(d_l, d_r) < \epsilon < 1$. This FP will be the key to construct the EXIT curve. We describe this below.

C. Construction of EXIT Curve

Definition 13 (Interpolated FP Family): Let $(\epsilon^{*}, x^{*})$, where $0 \leq \epsilon^{*} \leq 1$, denote a proper one-sided FP of length $L'$ and entropy $\chi$. Fix $L$ such that $1 \leq L < L'$. Then, the interpolated family of constellations based on $(\epsilon^{*}, x^{*})$ is denoted by $(g(\alpha)(\epsilon^{*}), x^{*})_{\alpha=0}$. It is indexed from $-L$ to $L$. This family is constructed from the one-sided FP $(\epsilon^{*}, x^{*})$. By definition, each element $g(\alpha)(\epsilon^{*})$ is symmetric and thus it suffices to define the constellation in the range $[-L, 0]$, and set $x_r(\alpha) = x_u(\alpha)$ for $\alpha \leq 0$. Set $x_u(\alpha) = 0$ for $\alpha \not\in [-L, 0]$. For $\alpha \in [0, 1]$, define
\[
\begin{align*}
x_i(\alpha) &\triangleq \begin{cases} 
4\alpha x_i^{*} - (4\alpha - 3)x_i^{*}, & \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right] , \\
(4\alpha - 2)x_i^{*} - (4\alpha - 3)x_i^{*}, & \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right] , \\
a(i, \alpha), & \alpha \in \left[\frac{1}{4}, \frac{3}{4}\right] ,
\end{cases} \\
a(i, \alpha), & \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right] ,
\end{align*}
\]
where, for $\alpha \in \left(\frac{1}{2}, 1\right)$,
\[
a(i, \alpha) = \frac{x_i^{*^{(L'-L)\left(\frac{1}{2}-\alpha\right)}}^{\left(L'-L\right)^{\left\lfloor \frac{1}{2}-\alpha\right\rfloor} mod(1)}}{x_i^{*^{(L'-L)\left(\frac{1}{2}-\alpha\right)}}^{\left(L'-L\right)^{\left\lfloor \frac{1}{2}-\alpha\right\rfloor} + 1}} ,
\]
The constellations $g(\alpha)$ are componentwise increasing as a function of $\alpha$, and $g(0) = (0, \ldots, 0)$ and $g(1) = (1, \ldots, 1)$.

Discussion: The above interpolation is split into four phases. In phase (i), where $\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]$, the constellations decreases from $x_u(1) = 1$ to $x_u(\frac{1}{4}) = x^{*}_{u}$. In phase (ii), where $\alpha \in \left[\frac{1}{2}, \frac{3}{4}\right]$, the constellations decreases further so that at the end of
this phase, $x_i(\frac{L}{2}) = x_i^*$. In phase (iii), where $\alpha \in [\frac{1}{4}, \frac{1}{2}]$, the constellation $x_i^*$ is “moved in” by interpolating between two consecutive sections in the middle. In phase (iv), where $\alpha \in [0, \frac{1}{4}]$, the interpolated constellations decrease linearly towards zero. Moreover, if we look at the EBP EXIT curve of the constructed family of unimodal FPs, the constellation “moves in” or “collapses” once the channel parameter has reached a value close to $\epsilon_{MAP}(d_i, d_r)$. This phenomenon corresponds to the sharp transition of the EBP EXIT curve close to $\epsilon_{MAP}(d_i, d_r)$. We describe this phenomenon precisely by the following theorem.

**Theorem 14 (Fundamental Properties of EXIT Curve):** Let $(\epsilon^*, x^*)$, where $\epsilon^* \in (\epsilon_{BP}, 1]$, denote a proper one-sided FP of length $L'$ and entropy $\gamma > 0$. Then, for $1 \leq L < L'$, we have

(i) **Continuity:** The curve $\{x(\alpha), \epsilon(\alpha)\}_{\alpha=0}^1$ is continuous and differentiable for $\alpha \in [0, 1]$ except for a finite set of points.

(ii) **Bounds in phase (i):** For $\alpha \in [\frac{1}{4}, 1]$, 

$$
\epsilon_i(\alpha) \begin{cases} = \epsilon_0(\alpha), & \text{where } \epsilon(x) = \frac{x}{(1-(1-x)^{\frac{1}{\alpha}})} \frac{1}{w}, \\
\geq \epsilon \left(1 - \frac{1}{1+w^{1/\gamma}}\right), & \text{where } i = \{-L,0\}.
\end{cases}
$$

For $x_i(\alpha) > \gamma$,

$$
\epsilon_i(\alpha) \leq \epsilon^* \left(1 + \frac{1}{w^{1/\gamma}}\right), \quad \text{for } i \in [-L+w-1, -w+1];
$$

$$
\epsilon_i(\alpha) \geq \epsilon^* \left(1 - \frac{1}{1+w^{1/\gamma}}\right), \quad \text{for } i \in [-L,0].
$$

For $x_i(\alpha) \leq \gamma$ and $w > \max\{2^*d_i^2, 2^*d_r^2, 2^*d_i^2\}$, 

$$
\epsilon^*(\alpha) \geq \epsilon^* \left(1 - \frac{4}{w^{1/\gamma}}\right)^{(d_i-1)(d_r-1)},
$$

where $i \in [-L,0]$.

(v) **Area under EXIT curve:** The EXIT value at position $i \in [-L, L]$ is defined by

$$
h_i(\alpha) = (g(x_{i-w+1}(\alpha), \ldots, x_{i+w-1}(\alpha)))^{\frac{d_i}{d_r}}.
$$

The area of the EXIT integral is defined by

$$
A(d_i, d_r, L, w) \triangleq \int_0^1 \frac{1}{2L+1} \sum_{i=-L}^L h_i(\alpha) d\epsilon_i(\alpha). \quad (10)
$$

Then,

$$
A(d_i, d_r, L, w) - \left(1 - \frac{d_i}{d_r}\right) \leq \frac{w}{L} d_id_r.
$$

(vi) **Bounds on $\epsilon^*$:** For $w > \max\{2^*d_i^2, 2^*d_r^2, 2^*d_i^2\}$,

$$
\left|\epsilon_{MAP}(d_i, d_r) - \epsilon^*\right| \leq \frac{2d_id_r|x_0^* - x_+(\epsilon^*)| + c(d_i, d_r, L, w)}{(1 - (d_i - 1)^{-\frac{1}{d_r}})^{\frac{1}{d_r}}},
$$

where

$$
c(d_i, d_r, L, w) = 4d_id_r w + \frac{wd_i(d_r + 2)}{L} + d_r(x_{-L} + x_0^* - x_{-L}) + w^{-\frac{1}{d_r}} \frac{2d_i^2}{1 - 4w^{-\frac{1}{d_r}}} d_r
$$

**Discussion:** Theorem 14 asserts that the area enclosed within the constructed EBP EXIT curve is very close to the design rate of the underlying $(d_i, d_r)$ ensemble, by choosing the parameters $(d_i, d_r, L, w)$ properly. Furthermore, we have that $\epsilon^*$ is close to $\epsilon_{MAP}(d_i, d_r)$. In a next step, [6] proved that $\epsilon^*(1 - 4w^{-1/8})d_id_r$ is a lower bound on the BP threshold of the coupled code ensemble $BP(d_i, d_r, L, w)$, by choosing $L'$ sufficiently large and by choosing $L$ appropriately. Thus, upon providing an upper bound on $\epsilon_{BP}(d_i, d_r, L, w)$ in terms of $\epsilon_{MAP}(d_i, d_r)$ and combining this with Theorem 14, [6] gives the following main result.

**D. Main Result**

**Theorem 15 (BP Threshold of the $(d_i, d_r, L, w)$ Ensemble):** Consider transmission over BEC(\epsilon) using random elements from the ensemble $(d_i, d_r, L, w)$. Let $\epsilon_{BP}(d_i, d_r, L, w)$ denote the BP threshold and let $R(d_i, d_r, L, w)$ denote the design rate of this ensemble. In the limit as $M$ tends to infinity, and for

$$
w > \max\{2^*d_i^2, 2^*d_r^2, 2^*d_i^2\}, \quad \left(\frac{2d_id_r \left(1 + \frac{2d_i}{1-2^{-1/(d_r-1)}}\right)^8}{(1 - 2^{-1/(d_r-2)})16 \left(1 - \frac{d_i}{d_r}\right)^8}\right)
$$

we have

$$
\epsilon_{BP}(d_i, d_r, L, w) \leq \epsilon_{MAP}(d_i, d_r, L, w) \leq \epsilon_{MAP}(d_i, d_r) + \frac{w-1}{2L \left(1 - (1 - x_{MAP}(d_i, d_r))^{d_r-1}\right)},
$$

and

$$
\epsilon_{BP}(d_i, d_r, L, w) \geq \epsilon_{MAP}(d_i, d_r) - w^{-\frac{1}{d_r}} \frac{8d_id_r + \frac{4d_i d_r}{(1-4w^{-\frac{1}{d_r}})^{d_r}}}{\left(1 - 2^{-\frac{1}{d_r}}\right)^2} d_r
$$

$$
\times \left(1 - 4w^{-\frac{1}{d_r}}\right)^{d_r}.
$$
In the limit as $M$, $L$, and $w$ (in that order) tend to infinity,
\[
\lim_{w \to \infty} \lim_{L \to \infty} R(d_l, d_r, L, w) = 1 - \frac{d_l}{d_r};
\]
\[
\lim_{w \to \infty} \lim_{L \to \infty} \epsilon_{BP}(d_l, d_r, L, w) = \lim_{w \to \infty} \lim_{L \to \infty} \epsilon_{MAP}(d_l, d_r, L, w) = \epsilon_{BP}(d_l, d_r).
\]

IV. Threshold Saturation for Coupled Scalar Recursions [8]

A. Notation

The following notations are used throughout this section. Closed intervals are defined as $X = [0, x_{\text{max}}]$ with $x_{\text{max}} \in (0, +\infty)$, $Y = [0, y_{\text{max}}]$ with $y_{\text{max}} \in (0, +\infty)$, and $\mathcal{E} = [0, \epsilon_{\text{max}}]$ with $\epsilon_{\text{max}} \in (0, +\infty)$. Intervals of natural numbers are denoted by $[m : n] = \{m, m+1, \ldots, n\}$. A column vector $x \in \mathcal{X}^n$ has as its elements $[x]_1$ or $x_i$ for $i \in [1 : n]$. Scalar-valued functions are, e.g., denoted by $f(x)$ and $F(x)$, while vector-valued functions are denoted by $f(x)$. Also, $f'(x)$ denotes the Jacobian matrix of the vector-valued function.

B. Main Result

Let $f : Y \to X$ be a non-decreasing $C^1$ function and $g : X \to Y$ be a strictly increasing $C^2$ function; assume $0 = g(0)$ and $y_{\text{max}} = g(x_{\text{max}})$. Defined the uncoupled scalar recursion
\[
y_i^{(\ell+1)} = g(x_i^{(\ell)}),
\]
\[
x_i^{(\ell+1)} = f(y_i^{(\ell+1)}),
\]
(11)

which is initialized by choosing $x_i^{(0)} = x_{\text{max}}$. Because $f(g(X)) \subseteq X$ and $g(f(Y)) \subseteq Y$, this initialization implies $x_i^{(1)} \leq x_i^{(0)}$ and $y_i^{(1)} \leq y_i^{(0)}$. Mathematical induction shows that the sequence $(x_i^{(\ell)}$, $y_i^{(\ell)})$ converges to a limit $(x_i^{(\infty)}$, $y_i^{(\infty)})$ and this limit is a fixed point because $f$ and $g$ are continuous. Let $F \triangleq \{x \in X | x = f(g(x))\}$ be the family of fixed points.

Definition 16 (Potential Function $U_s$): The potential function $U_s : X \to \mathbb{R}$ of the uncoupled recursion (11) is defined as
\[
U_s(x) \triangleq \int_0^x g'(z)(z-f(g(z)))\,dz.
\]
(12)

An alternative form of the potential function which can be derived from Equation (12) is $U_s(x) = xg(x) - G(x) - F(g(x))$, where $G(x) \triangleq \int_0^x g(z)\,dz$ and $F(x) \triangleq \int_0^x f(z)\,dz$. Taking the derivative of $U_s(x)$ gives
\[
U'_s(x) = (x-f(g(x)))g'(x).
\]

Discussion: Since $g(\cdot)$ is strictly increasing, i.e., $g'(x) > 0$, we must have $U'_s(x) = 0$ if and only if $x = f(g(x))$, i.e., $x$ is a fixed point of the uncoupled scalar recursion. The potential value is also non-increasing with respect to iteration, i.e., $U_s(f(g(x))) \leq U_s(x)$, and is strictly decreasing if $x$ is not a fixed point.

Now consider the coupled scalar recursion
\[
y_i^{(\ell+1)} = g(x_i^{(\ell)});
\]
\[
x_i^{(\ell+1)} = \sum_{j=1}^N A_{j,i} f \left( \sum_{k=1}^M A_{j,k} g_k^{(\ell+1)} \right),
\]
(13)

where $i \in [1 : M]$ and $M \triangleq N + w - 1$. The recursion starts from $x_i^{(0)} = x_{\text{max}}$, for each $i \in [1 : M]$. The $N \times M$ matrix $A$ is defined as
\[
A_{j,k} = |A|_{j,k} = \begin{cases} \frac{1}{w} & \text{if } 1 \leq k - j + 1 \leq w; \\ 0 & \text{otherwise}. \end{cases}
\]

It is thus not difficult to see that $\sum_{k=1}^M A_{j,k} = 1$ and $\sum_{j=1}^N A_{j,k} \leq 1$, so that $x_i^{(1)} \leq x_i^{(0)}$, for all $i \in [1 : M]$. Mathematical induction then implies that, for each $i \in [1 : M]$, the sequence $(x_i^{(\ell)}$, $y_i^{(\ell)})$ converges to a limit $(x_i^{(\infty)}$, $y_i^{(\infty)})$ which is also a fixed point of the coupled recursion. Using the vector notation $x^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_M^{(\ell)})^T$, the coupled recursion defined in Equation (13) can be written compactly as
\[
x_i^{(\ell+1)} = h \left( x^{(\ell)} \right) = A^T f \left( Ag \left( x^{(\ell)} \right) \right),
\]
(14)

where $f : Y^N \to X^N$ and $g : X^N \to Y^M$ are, respectively, defined by $[f(x)]_i = f(x_i)$ and $[g(x)]_i = g(x_i)$.

Theorem 17 (Upper Bound on Coupled FP): For any $\delta > 0$, there is a $w_0 < \infty$ such that, for all $w > w_0$, the fixed point of the coupled recursion satisfies the upper bound
\[
\max_{i \in [1 : M]} x_i^{(\infty)} - \delta \leq x^* \triangleq \max \left( \arg \min_{x \in X} U_s(x) \right).
\]
(15)

Discussion: Theorem 17 implies that as $w$ tends to infinity, the maximum element of the fixed-point vector for the coupled recursion is upper bounded by the largest minimizer of the uncoupled potential function. To prove Theorem 17, we extend the definition of potential function to coupled recursion (14).

Definition 18 (Potential Function for Coupled Recursion): The potential function for the coupled recursion $U_c : \mathcal{X}^M \to \mathbb{R}$ is defined as
\[
U_c(x) \triangleq \int_C g'(z)(z - A^T f(Ag(z)))\,dz
\]
\[
= x^* g(x) - G(x) - F(Ag(x))
\]
\[
= \sum_{i=1}^M (x_i g(x_i) - G(x_i)) - \sum_{j=1}^N F \left( \sum_{k=1}^M A_{j,k} g(x_k) \right),
\]

where $C$ is an arbitrary smooth curve in $\mathcal{X}^M$ from 0 to $x$. Also, $g'(x) = \text{diag}(g'(x_i))$, $G(x) = \sum_{i=1}^M G(x_i)$, and $F(y) = \sum_{j=1}^N F(y_j)$.

Discussion: The Hessian matrix $U''_c(x)$ of $U_c(x)$ satisfies $\|U''_c(x)\|_p \leq K_{f,g}$, where
\[
K_{f,g} \triangleq \|g''\|_{\infty} x_{\text{max}} + \|g'\|_{\infty} + \|f'\|_{\infty} \|g''\|_{\infty}^2.
\]

for all $x \in \mathcal{X}^M$ and $p \in [1, 2, \infty]$. Moreover, as we have seen, for each $\ell > 0$, the coupled recursion vector is symmetric and
unimodal. Define $i_0 \triangleq \lceil M/2 \rceil$. Then, the one-sided coupled recursion is
\begin{equation}
\hat{x}(t+1) \triangleq q \left( h \left( \hat{x}(t) \right) \right),
\end{equation}
where $[q(x)]_i = [x]_i$ if $i \in [i_0 + 1 : M]$, and $[q(x)]_i = [x]_i$ otherwise. Recall that the one-sided recursion is an upper bound on the coupled recursion, and the one-sided recursion converges to a limit which is also a fixed point. Let $\hat{x}^{(\infty)}$ be the fixed point of the one-sided coupled recursion. Then,
$$U_i^0(\hat{x})^T (S \hat{x} - \hat{x}) = 0,$$
where the shift operator $S$ is defined by $[Sx]_i = x_{i-1}$ for $i \in [1 : M]$ with $x_0 \equiv 0$. Furthermore, for a one-sided coupled recursion vector $x$, we have the change of coupled potential $U_c(Sx) - U_c(x) \leq -U_s(x_{i_0})$. Moreover, $||Sx - x||_\infty \leq \frac{1}{w} x_{\max}$ and $||Sx - x||_\infty \leq x_{\max}$.

Now, we introduce a change of variables for the uncoupled recursion which allows us to translate any fixed point to 0. This translation is useful for proving Theorem 17. Specifically, for any fixed point $\hat{x} \in F$ of the uncoupled recursion, define the functions $\tilde{f}(y) \triangleq f(y + g(\hat{x})) - \hat{x}$ and $\tilde{g}(y) \triangleq g(x + \hat{x}) - g(\hat{x})$ such that
$$\tilde{f}(\tilde{g}(y)) = f(g(x + \hat{x})) - \hat{x}.$$

This new recursion operates on the new space $\tilde{X} \triangleq [0, \tilde{x}_{\max}]$, where $\tilde{x}_{\max} \triangleq x_{\max} - \hat{x}$. In particular, we have $\tilde{f}(0) = \tilde{g}(0) = 0$, and $\tilde{U}_s(x) = U_s(x + \hat{x}) - U_s(\hat{x})$. Consequently, it suffices to prove the following lemma, and then Theorem 17 follows.

**Lemma 19 (Upper Bound on the Coupled FP):** Consider the uncoupled recursion satisfying $f(0) = g(0) = 0$. For any $\delta > 0$, if $\Delta_0 \triangleq \inf\{U_s(x)|x \in F \cap \delta, x_{\max}\} > 0$ and $w > a_{\max} + \frac{1}{\Delta_0}$, then the fixed point of the coupled recursion must satisfy $\hat{x}^{(\infty)} \leq \delta$ for $i \in [1 : M]$.

**Proof:** The outline of the complete proof is the following. First, we know that the one-sided coupled recursion converges to a fixed point $\hat{x} = \hat{x}^{(\infty)}$. If $\hat{x}_{i_0} \leq \delta$, then the proof is done. Thus, in what follows, we assume on the contrary that $\hat{x}_{i_0} > \delta$. Further, from the above discussion, we know that the inequality $U_c(S\hat{x}) - U_c(\hat{x}) \leq -U_s(\hat{x}_{i_0})$ holds. The fact that $\hat{x}_{i_0} = h(\hat{x}_{i_0})$ implies that the recursion $\hat{x}^{(t+1)} = f(g(z(t)))$ with $z(0) = \hat{x}_{i_0}$ satisfies $\hat{x}^{(t+1)} \geq \hat{x}^{(t)}$ and converges to a limit $\hat{x}^{(\infty)}$ such that $\hat{x}^{(\infty)} \geq \hat{x}_{i_0} > \delta$. This implies that $U_s(\hat{x}_{i_0}) \geq U_s(Z(S\hat{x} - \hat{x}))$ and
$$U_s(\hat{x}^{(\infty)}) \geq \inf\{U_s(x)|x \in F \cap \delta, x_{\max}\} = \Delta_0.$$

Thus, $-U_s(\hat{x}_{i_0}) \leq -\Delta_0$ and $|U_c(S\hat{x}) - U_c(\hat{x})| \geq \Delta_0$. On the other hand, the fact that $U_c(S\hat{x})^T (S\hat{x} - \hat{x}) = 0$ simplifies the second-order Taylor expansion of $U_c(S\hat{x})$ around $\hat{x}$, i.e.,
$$U_c(S\hat{x}) - U_c(\hat{x}) = \frac{1}{2} (S\hat{x} - \hat{x})^T U''_c(\hat{x}) (S\hat{x} - \hat{x}),$$
where $z(t) = \hat{x} + t(S\hat{x} - \hat{x})$, for some $t \in (0, 1)$. Therefore,
$$|U_c(S\hat{x}) - U_c(\hat{x})| \leq \frac{1}{2} K_{f,g} ||S\hat{x} - \hat{x}||_1 ||S\hat{x} - \hat{x}||_\infty \leq \frac{1}{2w} K_{f,g} x_{\max}^2 \leq \Delta_0,$$
if $w > \frac{1}{\Delta_0} K_{f,g} x_{\max}^2$, which leads to a contradiction. Thus, we must have $\hat{x}_{i_0} \leq \delta$.

**Discussion:** The translated coupled system satisfies $\hat{x}^{(\infty)} \leq \delta$ for all $i \in [1 : M]$ by Lemma 19, where the translation is done by setting $\hat{x} = x^*$. Since $\hat{x}_{i_0} \leq \hat{x}^{(\infty)} + x^*$ by induction, it follows that $x_{i_0}^{(\infty)} \leq \delta + x^*$, i.e., $x_{i_0}^{(\infty)} - \delta \leq x^*$ for $i \in [1 : M]$.

### C. Dependence on a Parameter

In many cases, the system also depends on a parameter $\epsilon \in \mathcal{E}$, where $\mathcal{E} \triangleq [0, \epsilon_{\max}]$ and $\epsilon_{\max} \in (0, +\infty)$. In this case, the recursion is defined by two bivariate functions $f : \mathcal{Y} \times \mathcal{E} \rightarrow \mathcal{X}$ and $g : \mathcal{X} \times \mathcal{E} \rightarrow \mathcal{Y}$. Let $f(m, n)(x; \epsilon) \triangleq \frac{d_m}{dx} \frac{d_n}{d\epsilon} f(x; \epsilon)$ and let $h(x; \epsilon) \triangleq f(g(x; \epsilon); \epsilon)$ and $X_\epsilon \triangleq \{0, x_{\max}\}$.

**Definition 20 (Admissible System):** An admissible system is a system where the function $f(x; \epsilon)$ and $g(x; \epsilon)$ satisfy
(i) $f(x; \epsilon)$ and $g(x; \epsilon)$ are C1 functions on $\mathcal{X} \times \mathcal{E}$,
(ii) $f(x; \epsilon)$ and $g(x; \epsilon)$ are non-decreasing in both $x$ and $\epsilon$,
(iii) $g(x; \epsilon)$ is strictly increasing in $x$, and
(iv) $g^2(x; \epsilon) \times \text{exists and is jointly continuous on } \mathcal{X} \times \mathcal{E}$.

In addition, an admissible system is proper if $h(0, 1)(x; \epsilon) > 0$ for all $(x, \epsilon) \in X_\epsilon \times \mathcal{E}$.

**Discussion:** It is natural to extend the definitions related to potential functions to admissible systems that depend on an additional parameter. Particularly, let $U_s : \mathcal{X} \times \mathcal{E} \rightarrow \mathbb{R}$ be the uncoupled potential function which is defined by
$$U_s(x; \epsilon) \triangleq x g(x; \epsilon) - G(x; \epsilon) - F(g(x; \epsilon); \epsilon),$$
where $G(x; \epsilon) \triangleq \int_0^1 g(x; \epsilon) dz$ and $F(x; \epsilon) \triangleq \int_0^x f(z; \epsilon) dz$. Moreover, define $\Psi(\epsilon) \triangleq \min_{x \in X} U_s(x; \epsilon)$, define $X^*(\epsilon) \triangleq \{x \in X|U_s(x; \epsilon) = \Psi(\epsilon)\}$, and define $\tilde{x}^*(\epsilon) \triangleq \max X^*(\epsilon)$. Meanwhile, we are always interested in the $\epsilon$-threshold below which the uncoupled (resp. coupled) recursion converges to 0 (resp. 0). This leads to the following definition.

**Definition 21 (Thresholds):** Let the single-system threshold $\epsilon_0^s$ be defined as
$$\epsilon_0^s \triangleq \sup \{ \epsilon \in \mathcal{E}|h(x; \epsilon) < x, x \in X_\epsilon \}.$$  
(17)

The coupled threshold (or potential threshold) $\epsilon_0^c$ is defined as
$$\epsilon_0^c \triangleq \sup \{ \epsilon \in \mathcal{E}|\tilde{x}^*(\epsilon) = 0 \}.$$  
(18)

**Discussion:** As we can see from Theorem 17 and Definition 21, if $\epsilon < \epsilon_0^c$, then $\tilde{x}^*(\epsilon) = 0$ such that the fixed-point vector of the coupled recursion converges to 0 as $w \rightarrow \infty$. Next, we will see an alternative definition for the coupled threshold, and this alternative definition can then be used to prove that spatially coupled LDPC codes achieve the conjectured MAP threshold of the underlying uncoupled codes under iterative BP decoding.

**Theorem 22 (Minimizers of Potential):** The function $\Psi(\epsilon)$ is non-increasing and satisfies $\Psi(\epsilon) = \int_0^\epsilon \psi(t) dt$, with
$$\psi(t) = -G^{(0,1)}(\tilde{x}^*(t); t) - F^{(0,1)}(g(\tilde{x}^*(t); t); t).$$

For a proper admissible system, $\Psi(\epsilon)$ is strictly decreasing on $\epsilon \in [\epsilon_0^c, \epsilon_{\max}]$, or similarly, if $\tilde{x}^*(\epsilon) > 0$. 

Discussion: In order to prove Theorem 22, we also need the envelope theorem which states that \( v(t) = U_q^{(0,1)}(x^*(t); t). \) Now, define the subset of \( \mathcal{X}_f \) that supports a fixed point
\[
\mathcal{X}_f \triangleq \{ x \in \mathcal{X} | h(x; 0) \leq x, h(x; \epsilon_{\text{max}} \geq x) \}.
\]
For each \( x \in \mathcal{X}_f \), let \( \bar{c}(x) \triangleq \min \{ \epsilon \in \mathcal{E} | h(x; \epsilon) = x \} \) be the smallest \( \epsilon \) that has \( x \) as its fixed point.

Definition 23 (Fixed-Point Potential Function): The fixed-point potential function \( Q : \mathcal{X}_f \rightarrow \mathbb{R} \) is defined by
\[
Q(x) \triangleq U_s(x; \bar{c}(x)) = x g(x; \bar{c}(x)) - G(x; \bar{c}(x)) - F(g(x; \bar{c}(x)); \bar{c}(x)).
\]

Theorem 24 (Alternative Description of \( \epsilon^*_c \)): For any proper admissible system, \( \Psi(\epsilon) = 0 \) for \( \epsilon \leq \epsilon^*_c \) and \( \Psi(\epsilon) < 0 \) implies \( \bar{x}^*(\epsilon) > 0 \). Moreover, if \( \mathcal{X}_f \) closed, then
\[
\epsilon^*_c = \sup \{ \epsilon \in \mathcal{E} | \min_{x \in \mathcal{X}} U_s(x; \epsilon) = 0 \} = \inf \{ \bar{c}(x) | x \in \mathcal{X}_f, Q(x) = 0 \}.
\]

Proof: The outline of the complete proof is the following. First, if \( \epsilon < \epsilon^*_c \), then by the original definition of \( \epsilon^*_c \), we get \( \bar{x}^*(\epsilon) = 0 \) and \( \Psi(\epsilon) = U_s(\bar{x}^*(\epsilon); \epsilon) = 0 \). Moreover, by the continuity of \( \Psi \), \( \Psi(\epsilon^*_c) = 0 \). Next, if \( \Psi(\epsilon) < 0 = \Psi(\epsilon^*_c) \), then \( \epsilon > \epsilon^*_c \) by the monotonicity of \( \Psi \) and thus \( \bar{x}^*(\epsilon) > 0 \). Since \( \Psi(\epsilon) \) is strictly decreasing for \( \epsilon \in [\epsilon^*_c, \epsilon_{\text{max}}] \) and \( \Psi(\epsilon) = 0 \) for \( \epsilon \in [0, \epsilon^*_c] \), we have
\[
\epsilon^*_c = \sup \{ \epsilon \in \mathcal{E} | \Psi(\epsilon) = 0 \} = \sup \{ \epsilon \in \mathcal{E} | \min_{x \in \mathcal{X}} U_s(x; \epsilon) = 0 \}.
\]

Now if \( Q(x) < 0 \) for some \( x \in \mathcal{X}_f \), then we must have \( \Psi(\bar{c}(x)) = \min_{x \in \mathcal{X}} U_s(x^*(\epsilon); \bar{c}(x)) \leq Q(x) < 0 \), i.e., \( \bar{c}(x) > \epsilon^*_c \). Let \( A = \{ \epsilon \in \mathcal{E} | \Psi(\epsilon) < 0 \} \). If \( \epsilon^*_c \leq \epsilon_{\text{max}} \), then \( \epsilon^*_c = \inf A \). Define \( B = \{ \epsilon \in \mathcal{E} | \exists x \in \mathcal{X}_f, \bar{c}(x) = \epsilon, Q(x) < 0 \} \). It is not difficult to see that \( A = B \) so that \( \epsilon^*_c = \inf A = \inf B \). Now by using an approximation sequence approaching \( \epsilon^*_c \), we can also show that
\[
\epsilon^*_c = \inf \{ \bar{c}(x) | x \in \mathcal{X}_f, Q(x) = 0 \},
\]
and this completes the proof.

V. RESEARCH PLAN

As we have seen in Sections III and IV, spatially coupled LDPC code ensembles are able to achieve capacity under BP decoding. Remarkably, this achieveability is also universal [9], i.e., the same ensemble can achieve capacity for all binary memoryless symmetric (BMS) channels have that capacity or higher. Note that this achievable and universal result has been rigorously proved in [9] by spatial-coupling regular LDPC code ensembles where \( d_i \geq 3 \). This gives us a new procedure to achieve capacity. In a first step, for a given BMS channel, we need to find LDPC code ensembles that can achieve capacity under the optimal MAP decoding. Then, by spatial-coupling the individual ensembles, we are able to achieve capacity under low-complexity BP decoding. As we have seen in [3], devising capacity-achieving LDPC code ensembles for the BEC is relatively tractable. Thus, we want to answer the following question. Given a sequence of LDPC code ensembles which is capacity-achieving for the BEC under MAP decoding, can we prove or disprove that the same sequence of ensembles is still capacity-achieving for BMS channels other than the BEC under optimal MAP decoding. As we will see in the following, partial results imply that the answer is negative, e.g., not all capacity-achieving ensembles are universal even under optimal MAP decoding. We take as an example the well-known sequence of right-regular LDPC code ensembles [10] which is known to achieve capacity under MAP decoding for the BEC.

A. Right-Regular LDPC Code Ensemble

Proposed in [10], the sequence of right-regular LDPC code ensembles can be specified by the sequence of degree distributions \( \{ \lambda(N), \rho(N) \} \), where \( N \in \mathbb{N} \). For each particular \( N \) in the sequence, we have
\[
\lambda(N)_i = \sum_{i=1}^{N-1} \frac{\lambda(N)_i}{\lambda(N)_i}(-1)^{i-1} \frac{N}{N-1} x^i;
\]
\[
\rho(N)_x = x^{N-1} C,
\]
where \( \lambda(N)_i := \frac{1}{\sum_{i=1}^{N-1} \lambda(N)_i} \), and \( C \in (0, 1) \) is the capacity of BEC. For simplicity, in the following we might also use the notation \( r_N = \alpha_N^{-1} \). Denote by \( n \) the block-length and let \( n \) go to \( +\infty \) for fixed \( N \). This sequence is claimed to asymptotically achieve capacity in the sense that the sequence of design rates \( r(N) \) tends to \( C \) as \( N \) goes to \( +\infty \), and the sequence of BP thresholds tends to the Shannon threshold \( 1 - C \) as \( N \) goes to \( +\infty \). Since this sequence is capacity-achieving under BP decoding, it is also capacity-achieving under MAP decoding. One particularly important quantity is
\[
\mu_N := \lambda(N)_0 \rho(N)_1 = \frac{1}{1 - \sum_{i=1}^{N-1} \lambda(N)_i (-1)^{N-1}}.
\]
Moreover, by [10, Proposition 1], we have
\[
\mu_N := \lim_{N \to \infty} \mu_N = \frac{1}{1 - C}.
\]

B. Prior Work

We next explain why the sequence of right-regular LDPC code ensembles cannot achieve capacity under MAP decoding for BMS channels other than the BEC. The rationale depends on the following simple lemma.

Lemma 25 (Extremal Densities for Fixed \( \mathbb{H}(\cdot, [3]) \): Let \( c \) be a symmetric \( L \)-density. For a fixed \( \mathbb{H}(\epsilon) \), the symmetric density which maximizes/minimizes \( \mathbb{B}(c) \) is the BSC/BEC.

Discussion: Lemma 25 implies there is a BMS channel with \( L \)-density \( c \) such that \( \mathbb{B}(c) > \mathbb{B}(c_{\text{BEC}}) \) if \( \mathbb{H}(\epsilon) = \mathbb{H}(c_{\text{BEC}}) = 1 - C \), where \( C \in (0, 1) \). Since \( \lim_{N \to \infty} \mathbb{B}(c_{\text{BEC}})\mu_N = 1 \), we must have
\[
\lim_{N \to \infty} \mathbb{B}(c)\mu_N > 1.
\]

The next two lemmas simplifies the analysis of DE recursions. More precisely, instead of considering the original output
density of DE, we use an ungraded density as the input to DE for the next round.

**Lemma 26 (Linearization, Part I):** Let \( c \) be the \( L \)-density associated to a BMS channel and consider the right-regular degree distribution \( (\lambda(x), \rho(x) := x^L) \). Let \( T(\cdot) \) be the density evolution operator. Then, with \( b := \phi c + (1 - \phi) \Delta + \infty \) where \( 0 < \phi < 1 \), we have

\[
T(b) \leftarrow g(\phi)c^{\oplus 2} + (1 - g(\phi))\Delta + \infty,
\]

where \( g(\phi) := \lambda_2(1 - (1 - \phi)^T) \) and \( \lambda_2 \) is the fraction of edges connected to the variable nodes of degree 2.

**Lemma 27 (Linearization, Part II):** Let \( c \) be the \( L \)-density associated to a BMS channel and consider the sequence of right-regular degree distributions \( \{ (\lambda(N)(x), \rho(N)(x) = x^T) \} \). Let \( T(N)(\cdot) \) be the sequence of associated density evolution operators and let \( b(N) := \beta(N)c + (1 - \beta(N)\epsilon)\Delta + \infty \), where \( \beta(N) > 0 \), \( 0 < \beta(N) < 1 \), and \( \lim_{N \to \infty} \beta(N) > 0 \). Then, there exists a strictly positive integer \( N \) and a strictly positive constant \( \delta \) such that for all \( N \geq N \), we have

\[
T(N)(b(N)) \leftarrow (\mu N \beta(N) - \delta)c^{\oplus 2} + (1 - (\mu N \beta(N) - \delta)\epsilon)\Delta + \infty,
\]

where \( \mu N := \lambda_1(N)(0)\rho(N)(1) = \lambda_1(N)2T_{N} \).

**Discussion:** The next Theorem 28 asserts that the stability condition for a BMS channel and the sequence of right-regular LDPC code ensembles is essentially tight, i.e., if then channel entropy is above the stability threshold, BP decoding cannot converge to the zero probability of error.

**Theorem 28 (Stability Condition for BP):** Assume that we are given the sequence of right-regular degree distributions \( \{ (\lambda(N), \rho(N)) \} \) which achieves capacity \( C \) of the BEC. Consider using the same sequence \( \{ (\lambda(N), \rho(N)) \} \) for transmission over the BMS channel having the same capacity \( C \) with the associated \( L \)-density \( c \). For \( \ell \geq 1 \), define

\[
x(N, \ell) = c \oplus \lambda(N)\rho(N)(x(N, \ell - 1)),
\]

where \( x_0 \) is an arbitrary symmetric \( L \)-density. Furthermore, if \( \mathbb{E}(c)\mu_{N} > 1 \), then there exists a strictly positive constant \( \xi = \xi(c) \) and a strictly positive integer \( n_{\max} \) such that, for all \( N \geq n_{\max} \),

\[
\liminf_{\ell \to \infty} \mathbb{E}(x(N, \ell)) > \xi,
\]

for all \( x_0 \neq \Delta + \infty \).

**Discussion:** As we can see from Figure 1, as \( N \) tends to infinity, the slope at which the GEXIT curve starts to be non-zero does not vanishes, which also demonstrates the assertion in Theorem 28.

**C. Future Plan**

Since we have shown that the stability condition is tight, the next step towards proving the main universality result is to show that, for a given BMS channel and the sequence of right-regular LDPC code ensembles, the stability condition threshold is asymptotically at least as large as the MAP threshold. This would give us the desired result. We might also ask the question, except for the regular LDPC code ensemble, does there exist a second ensemble which universally achieve capacity under MAP decoding? Another interesting aspect is to investigate general one-dimensional or finite-dimensional spatially-coupled systems that share the same properties [8], [11], [12] and [13], which will also help us to gain better understanding of the underlying uncoupled systems.

**REFERENCES**


