CHAPTER

4 Stress and Integral Formulations of General Principles

In the previous chapter, we considered the purely kinematic description of the motion of a continuum without any consideration of the forces that cause the motion and deformation. In this chapter, we consider a means of describing the forces in the interior of a body idealized as a continuum. It is generally accepted that matter is formed of molecules, which in turn consist of atoms and subatomic particles. Therefore, the internal forces in real matter are those between these particles. In the classical continuum theory where matter is assumed to be continuously distributed, the forces acting at every point inside a body are introduced through the concept of body forces and surface forces. Body forces are those that act throughout a volume (e.g., gravity, electrostatic force) by a long-range interaction with matter or charges at a distance. Surface forces are those that act on a surface (real or imagined), separating parts of the body. We assume that it is adequate to describe the surface forces at a point on a surface through the definition of a *stress vector*, discussed in Section 4.1, which pays no attention to the curvature of the surface at the point. Such an assumption is known as *Cauchy's stress principle* and is one of the basic axioms of classical continuum mechanics.

4.1 STRESS VECTOR

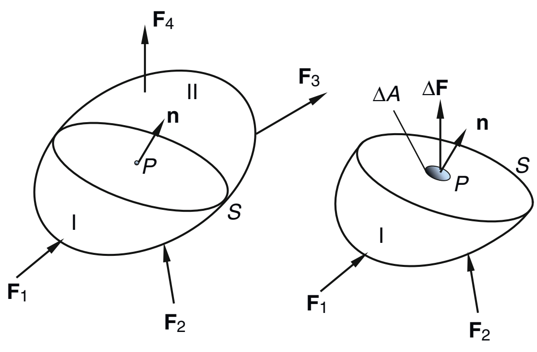

FIGURE 4.1-1

Let us consider a body depicted in Figure 4.1-1. Imagine a plane such as S, which passes through an arbitrary internal point P and which has a unit normal vector $\mathbf{n}$. The plane cuts the body into two portions. One portion lies on the side of the arrow of $\mathbf{n}$ (designated by II in the figure) and the other portion on the tail of $\mathbf{n}$ (designated by I). Considering portion I as a free body, there will be on plane S a resultant force $\Delta\mathbf{F}$ acting on a small area ΔA containing P. We define the stress vector (acting from II to I) at the point P on the plane S as the limit of the ratio $\Delta\mathbf{F}/\Delta A$ as $\Delta A \to 0$. That is, with $\mathbf{t_n}$ denoting the stress vector,

$$\mathbf{t_n} = \lim_{\Delta A \to 0} \frac{\Delta\mathbf{F}}{\Delta A}. \tag{4.1.1}$$

If portion II is considered as a free body, then by Newton's law of action and reaction, we shall have a stress vector (acting from I to II) $\mathbf{t_{-n}}$ at the same point on the same plane equal and opposite to that given by Eq. (4.1.1). That is,

$$\mathbf{t_n} = -\mathbf{t_{-n}}. \tag{4.1.2}$$

The subscript $-\mathbf{n}$ for $\mathbf{t}$ (i.e., $\mathbf{t_{-n}}$) indicates that outward normal for the portion II is in the negative direction of $\mathbf{n}$.

Next, let S be a surface (instead of a plane) passing the point P. Let $\Delta\mathbf{F}$ be the resultant force on a small area ΔS on the surface S. The *Cauchy stress vector* at P on S is defined as

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{F}}{\Delta S}. \tag{4.1.3}$$

We now state the following principle, known as the *Cauchy's stress principle:* The stress vector at any given place and time has a common value on all parts of material having a common tangent plane at P and lying on the same side of it. In other words, if $\mathbf{n}$ is the unit outward normal (i.e., a vector of unit length pointing outward, away from the material) to the tangent plane, then

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}), \tag{4.1.4}$$

where the scalar t denotes time.

In the following section, we show from Newton's second law that this dependence of the Cauchy's stress vector on the outward normal vector $\mathbf{n}$ can be expressed as

$$\mathbf{t} = \mathbf{T}(\mathbf{x}, t)\mathbf{n}, \tag{4.1.5}$$

where $\mathbf{T}$ is a linear transformation.

4.2 STRESS TENSOR

According to Eq. (4.1.4), the stress vector on a plane passing through a given spatial point $\mathbf{x}$ at a given time t depends only on the unit normal vector $\mathbf{n}$ to the plane. Thus, let $\mathbf{T}$ be the transformation such that

$$\mathbf{t_n} = \mathbf{Tn}. \tag{4.2.1}$$

We wish to show that this transformation is linear. Let a small tetrahedron be isolated from the body with the point P as one of its vertices (see Figure 4.2-1). The size of the tetrahedron will ultimately be made to approach zero volume so that, in the limit, the inclined plane will pass through the point P. The outward normal to the face PAB is $-\mathbf{e}_1$. Thus, the stress vector on this face is denoted by $\mathbf{t}_{-\mathbf{e}_1}$ and the force on the face is $\mathbf{t}_{-\mathbf{e}_1}\Delta A_1$, where ΔA_1 is the area of PAB. Similarly, the force acting on PBC, PAC and the inclined face ABC are $\mathbf{t}_{-\mathbf{e}_2}\Delta A_2$, $\mathbf{t}_{-\mathbf{e}_3}\Delta A_3$, and $\mathbf{t_n}\,\Delta A_n$, respectively. Thus, from Newton's second law written for the tetrahedron, we have

$$\sum \mathbf{F} = \mathbf{t}_{-\mathbf{e}_1}(\Delta A_1) + \mathbf{t}_{-\mathbf{e}_2}(\Delta A_2) + \mathbf{t}_{-\mathbf{e}_3}(\Delta A_3) + \mathbf{t_n}\Delta A_n = m\mathbf{a}. \tag{4.2.2}$$

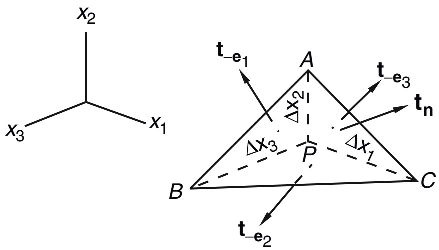

FIGURE 4.2-1

Since the mass m = (density)(volume) and the volume of the tetrahedron is proportional to the product of three infinitesimal lengths (in fact, the volume equals $(1/6)\Delta x_1 \Delta x_2 \Delta x_3$), when the size of the tetrahedron approaches zero, the right-hand side of Eq. (4.2.2) will approach zero faster than the terms on the left, where the stress vectors are multiplied by areas, the product of two infinitesimal lengths. Thus, in the limit, the acceleration term drops out exactly from Eq. (4.2.2). (We note that any body force, e.g., weight, that is acting will be of the same order of magnitude as that of the acceleration term and will also drop out.) Thus,

$$\sum \mathbf{F} = \mathbf{t}_{-\mathbf{e}_1}(\Delta A_1) + \mathbf{t}_{-\mathbf{e}_2}(\Delta A_2) + \mathbf{t}_{-\mathbf{e}_3}(\Delta A_3) + \mathbf{t}_\mathbf{n}\Delta A_n = 0. \tag{4.2.3}$$

Let the unit normal vector of the inclined plane ABC be

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3. \tag{4.2.4}$$

The areas ΔA_1, ΔA_2 and ΔA_3, being the projections of ΔA_n on the coordinate planes, are related to ΔA_n by

$$\Delta A_1 = n_1 \Delta A_n, \quad \Delta A_2 = n_2 \Delta A_n, \quad \Delta A_3 = n_3 \Delta A_n. \tag{4.2.5}$$

Using Eq. (4.2.5), Eq. (4.2.3) becomes

$$\mathbf{t}_{-\mathbf{e}_1} n_1 + \mathbf{t}_{-\mathbf{e}_2} n_2 + \mathbf{t}_{-\mathbf{e}_3} n_3 + \mathbf{t}_\mathbf{n} = 0. \tag{4.2.6}$$

But from the law of the action and reaction,

$$\mathbf{t}_{-\mathbf{e}_1} = -\mathbf{t}_{\mathbf{e}_1}, \quad \mathbf{t}_{-\mathbf{e}_2} = -\mathbf{t}_{\mathbf{e}_2}, \quad \mathbf{t}_{-\mathbf{e}_3} = -\mathbf{t}_{\mathbf{e}_3}, \tag{4.2.7}$$

therefore, Eq. (4.2.6) becomes

$$\mathbf{t}_\mathbf{n} = n_1 \mathbf{t}_{\mathbf{e}_1} + n_2 \mathbf{t}_{\mathbf{e}_2} + n_3 \mathbf{t}_{\mathbf{e}_3}. \tag{4.2.8}$$

Now, using Eq. (4.2.4) and Eq. (4.2.8), Eq. (4.2.1) becomes

$$\mathbf{T}(n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3) = n_1\mathbf{T}\mathbf{e}_1 + n_2\mathbf{T}\mathbf{e}_2 + n_3\mathbf{T}\mathbf{e}_3. \tag{4.2.9}$$

That is, the transformation $\mathbf{T}$, defined by

$$\mathbf{t}_\mathbf{n} = \mathbf{T}\mathbf{n}, \tag{4.2.10}$$

is a linear transformation. It is called the *stress tensor* or the *Cauchy stress tensor*.

4.3 COMPONENTS OF STRESS TENSOR

According to Eq. (4.2.10) of the previous section, the stress vectors $\mathbf{t}_{\mathbf{e}_i}$ on the three coordinate planes (the $\mathbf{e}_i$-planes) are related to the stress tensor $\mathbf{T}$ by

$$\mathbf{t}_{\mathbf{e}_1} = \mathbf{T}\mathbf{e}_1, \quad \mathbf{t}_{\mathbf{e}_2} = \mathbf{T}\mathbf{e}_2, \quad \mathbf{t}_{\mathbf{e}_3} = \mathbf{T}\mathbf{e}_3. \tag{4.3.1}$$

By the definition of the components of a tensor [see Eq. (2.7.2)], we have

$$\mathbf{T}\mathbf{e}_i = \mathbf{T}_{mi}\mathbf{e}_m. \tag{4.3.2}$$

Thus,

$$\begin{aligned} \mathbf{t}_{\mathbf{e}_1} &= T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3, \\ \mathbf{t}_{\mathbf{e}_2} &= T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3, \\ \mathbf{t}_{\mathbf{e}_3} &= T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3. \end{aligned} \tag{4.3.3}$$

Since $\mathbf{t}_{\mathbf{e}_1}$ is the stress vector acting on the plane whose outward normal is $\mathbf{e}_1$, it is clear from the first equation of Eq. (4.3.3) that T_{11} is its normal component and T_{21} and T_{31} are its tangential components. Similarly, T_{22} is the normal component on the $\mathbf{e}_2$-plane and T_{12} and T_{32} are the tangential components on the same plane, and so on.

We note that for each stress component T_{ij}, the second index j indicates the plane on which the stress component acts and the first index indicates the direction of the component; e.g., T_{12} is the stress component in the direction of $\mathbf{e}_1$ acting on the plane whose outward normal is in the direction of $\mathbf{e}_2$. We also note that the positive normal stresses are also known as *tensile stresses,* and negative normal stresses are known as *compressive stresses*. *Tangential stresses* are also known as *shearing stresses*. Both T_{21} and T_{31} are shearing stress components acting on the same plane (the $\mathbf{e}_1$-plane). Thus, the resultant shearing stress on this plane is given by

$$\boldsymbol{\tau}_1 = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3. \tag{4.3.4}$$

The magnitude of this shearing stress is given by

$$|\boldsymbol{\tau}_1| = \sqrt{T_{21}^2 + T_{31}^2}. \tag{4.3.5}$$

Similarly, on $\mathbf{e}_2$-plane,

$$\boldsymbol{\tau}_2 = T_{12}\mathbf{e}_1 + T_{32}\mathbf{e}_3, \tag{4.3.6}$$

and on $\mathbf{e}_3$-plane,

$$\boldsymbol{\tau}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2. \tag{4.3.7}$$

From $\mathbf{t} = \mathbf{T}\mathbf{n}$, the components of $\mathbf{t}$ are related to those of $\mathbf{T}$ and $\mathbf{n}$ by the equation

$$t_i = T_{ij}n_j, \tag{4.3.8}$$

or, in a form more convenient for computation,

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}]. \tag{4.3.9}$$

Thus, it is clear that if the matrix of $\mathbf{T}$ is known, the stress vector $\mathbf{t}$ on any inclined plane is uniquely determined from Eq. (4.3.9). In other words, the state of stress at a point is completely characterized by the stress tensor $\mathbf{T}$. Also, since $\mathbf{T}$ is a second-order tensor, any one matrix of $\mathbf{T}$ determines the other matrices of $\mathbf{T}$ (see Section 2.18).

We should also note that some authors use the convention $\mathbf{t} = \mathbf{T}^{\mathrm{T}}\mathbf{n}$ so that $\mathbf{t}_{\mathbf{e}_i} = T_{ij}\mathbf{e}_j$. Under that convention, for example, T_{21} and T_{23} are tangential components of the stress vector on the plane whose normal is $\mathbf{e}_2$, and so on. These differences in meaning regarding the nondiagonal elements of $\mathbf{T}$ disappear if the stress tensor is symmetric.

4.4 SYMMETRY OF STRESS TENSOR: PRINCIPLE OF MOMENT OF MOMENTUM

By the use of the moment of momentum equation for a differential element, we shall now show that the stress tensor is generally a symmetric tensor.* Consider the free body diagram of a differential parallelepiped isolated from a body, as shown in Figure 4.4-1. Let us find the moment of all the forces about an axis passing through the center point A and parallel to the x_3-axis:

$$\begin{aligned}\sum(M_A)_3 = {} & T_{21}(\Delta x_2)(\Delta x_3)(\Delta x_1/2) + (T_{21} + \Delta T_{21})(\Delta x_2)(\Delta x_3)(\Delta x_1/2) \\ & - T_{12}(\Delta x_1)(\Delta x_3)(\Delta x_2/2) + (T_{12} + \Delta T_{12})(\Delta x_1)(\Delta x_3)(\Delta x_2/2).\end{aligned} \tag{4.4.1}$$

In writing Eq. (4.4.1), we have assumed the absence of body moments. Dropping the terms containing small quantities of higher order, we obtain

$$\sum(M_A)_3 = (T_{21} - T_{12})(\Delta x_1)(\Delta x_2)(\Delta x_3). \tag{4.4.2}$$

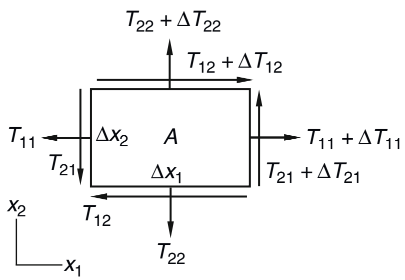

FIGURE 4.4-1

Now, whether the element is in static equilibrium or not,

$$\sum(M_A)_3 = I_{33}\alpha = 0. \tag{4.4.3}$$

This is because the angular acceleration term, $I_{33}\alpha$, is proportional to the moment of inertia I_{33}, which is given by $(1/12)(\text{density})\Delta x_1 \Delta x_2 \Delta x_3[(\Delta x_1)^2 + (\Delta x_2)^2]$ and is therefore a small quantity of higher order compared with the term $(T_{21} - T_{12})(\Delta x_1)(\Delta x_2)(\Delta x_3)$. Thus,

$$\sum(M_A)_3 = (T_{21} - T_{12})(\Delta x_1)(\Delta x_2)(\Delta x_3) = 0. \tag{4.4.4}$$

*See Prob. 4.29 for a case in which the stress tensor is not symmetric.

With similar derivations for the moments about the other two axes, we have

$$T_{12} = T_{21}, \quad T_{13} = T_{31}, \quad T_{23} = T_{32}. \tag{4.4.5}$$

These equations state that the stress tensor is symmetric, i.e., $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. Therefore, there are only six independent stress components.

Example 4.4.1

The state of stress at a certain point is $\mathbf{T} = -p\mathbf{I}$, where p is a scalar. Show that there is no shearing stress on any plane containing this point.

Solution

The stress vector on any plane passing through the point with normal $\mathbf{n}$ is

$$\mathbf{t_n} = \mathbf{Tn} = -p\mathbf{In} = -p\mathbf{n}.$$

Therefore, it is normal to the plane. This simple stress state is called a *hydrostatic state of stress*.

Example 4.4.2

With reference to a rectangular Cartesian coordinate system, the matrix of a state of stress at a certain point in a body is given by

$$[\mathbf{T}] = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} MPa.$$

(a) Find the stress vector and the magnitude of the normal stress on a plane that passes through the point and is parallel to the plane $x_1 + 2x_2 + 2x_3 - 6 = 0$.

(b) If $\mathbf{e}'_1 = \frac{1}{3}(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$ and $\mathbf{e}'_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$, find T'_{12}.

Solution

(a) The plane $x_1 + 2x_2 + 2x_3 - 6 = 0$ has a unit normal given by

$$\mathbf{n} = \frac{1}{3}(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3).$$

The stress vector is obtained from Eq. (4.3.9) as

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] = \frac{1}{3}\begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix},$$

or

$$\mathbf{t} = \frac{1}{3}(16\mathbf{e}_1 + 4\mathbf{e}_2 + \mathbf{e}_3)\ MPa.$$

The magnitude of the normal stress is, with $T_n \equiv T_{(n)(n)}$,

$$T_n = \mathbf{t}\cdot\mathbf{n} = \frac{1}{9}(16 + 8 + 2) = 2.89\ MPa.$$

(b) To find the primed components of the stress tensor, we have

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \frac{1}{3\sqrt{2}}[2 \quad 2 \quad 1]\begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{7}{3\sqrt{2}} = 1.65\ MPa.$$

Example 4.4.3

The distribution of stress inside a body is given by the matrix

$$[\mathbf{T}] = \begin{bmatrix} -p+\rho gy & 0 & 0 \\ 0 & -p+\rho gy & 0 \\ 0 & 0 & -p+\rho gy \end{bmatrix},$$

where p, ρ, and g are constants. Figure 4.4-2(a) shows a rectangular block inside the body.

(a) What is the distribution of the stress vector on the six faces of the block?
(b) Find the total resultant force acting on the face $y = 0$ and $x = 0$.

Solution

(a) From $\mathbf{t} = \mathbf{T}\mathbf{n}$, we have

$$\begin{aligned}
&\text{On } x=0, && [\mathbf{n}] = [-1 \quad 0 \quad 0], && [\mathbf{t}] = [\,p-\rho gy \quad 0 \quad 0\,],\\
&\text{On } x=a, && [\mathbf{n}] = [+1 \quad 0 \quad 0], && [\mathbf{t}] = [\,-p+\rho gy \quad 0 \quad 0\,],\\
&\text{On } y=0, && [\mathbf{n}] = [0 \quad -1 \quad 0], && [\mathbf{t}] = [\,0 \quad p \quad 0\,],\\
&\text{On } y=b, && [\mathbf{n}] = [0 \quad +1 \quad 0], && [\mathbf{t}] = [\,0 \quad -p+\rho gb \quad 0\,],\\
&\text{On } z=0, && [\mathbf{n}] = [0 \quad 0 \quad -1], && [\mathbf{t}] = [\,0 \quad 0 \quad p-\rho gy\,],\\
&\text{On } z=c, && [\mathbf{n}] = [0 \quad 0 \quad +1], && [\mathbf{t}] = [\,0 \quad 0 \quad -p+\rho gy\,].
\end{aligned}$$

The distribution of the stress vector on four faces of the cube is shown in Figure 4.4-2(b).

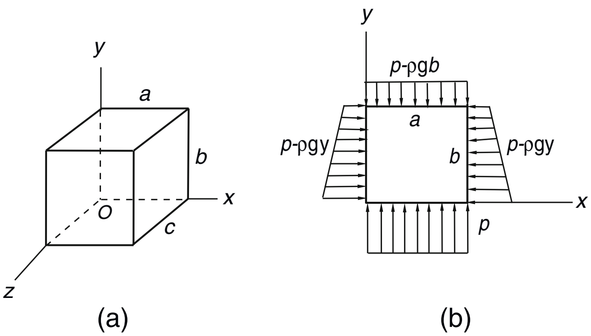

FIGURE 4.4-2

(b) On the face $y = 0$, the resultant force is

$$\mathbf{F}_1 = \int \mathbf{t} dA = \left(p \int dA\right)\mathbf{e}_2 = pac\mathbf{e}_2.$$

On the face $x = 0$, the resultant force is

$$\mathbf{F}_2 = \left[\int (p - \rho gy) dA\right]\mathbf{e}_1 = \left[\int p dA - \rho g \int y dA\right]\mathbf{e}_1.$$

The second integral can be evaluated directly by replacing dA by cdy and integrating from $y = 0$ to $y = b$. Or, since $\int y dA$ is the first moment of the face area about the z-axis, it is therefore equal to the product of the centroidal distance and the total area. Thus,

$$\mathbf{F}_2 = \left[pbc - \frac{\rho g b^2 c}{2}\right]\mathbf{e}_1.$$

4.5 PRINCIPAL STRESSES

From Section 2.23, we know that for any real symmetric stress tensor, there exist at least three mutually perpendicular principal directions (the eigenvectors of $\mathbf{T}$). The planes having these directions as their normals are known as the *principal planes*. On these planes, the stress vector is normal to the plane (i.e., no shearing stresses) and the normal stresses are known as the *principal stresses*. Thus, the principal stresses (eigenvalues of $\mathbf{T}$) include the maximum and the minimum values of normal stresses among all planes passing through a given point.

The principal stresses are to be obtained from the characteristic equation of $\mathbf{T}$, which may be written:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{4.5.1}$$

where

$$\begin{aligned} I_1 &= T_{11} + T_{22} + T_{33}, \\ I_2 &= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix}, \\ I_3 &= \det \mathbf{T}, \end{aligned} \tag{4.5.2}$$

are the three principal scalar invariants of the stress tensor. For the computations of these principal directions, refer to Section 2.22.

4.6 MAXIMUM SHEARING STRESSES

In this section, we show that the maximum shearing stress is equal to one-half the difference between the maximum and the minimum principal stresses and acts on the plane that bisects the right angle between the plane of maximum principal stress and the plane of minimum principal stress.

Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the principal directions of $\mathbf{T}$ and let T_1, T_2 and T_3 be the principal stresses. If $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ is the unit normal to a plane, the components of the stress vector on the plane is given by

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} T_1 n_1 \\ T_2 n_2 \\ T_3 n_3 \end{bmatrix}, \tag{4.6.1}$$

i.e.,

$$\mathbf{t} = n_1 T_1 \mathbf{e}_1 + n_2 T_2 \mathbf{e}_2 + n_3 T_3 \mathbf{e}_3, \tag{4.6.2}$$

and the normal stress on the same plane is given by

$$T_n = \mathbf{n} \cdot \mathbf{t} = n_1^2 T_1 + n_2^2 T_2 + n_3^2 T_3. \tag{4.6.3}$$

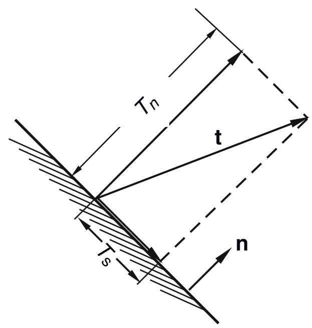

FIGURE 4.6-1

Thus, if T_s denotes the magnitude of the total shearing stress on the plane, we have (see Figure 4.6-1)

$$T_s^2 = |\mathbf{t}|^2 - T_n^2, \tag{4.6.4}$$

i.e.,

$$T_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - \left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)^2. \tag{4.6.5}$$

For a given set of values of $(T_1,\ T_2,\ T_3)$, we would like to find the maximum value of shearing stress T_s and the plane(s), described by $(n_1,\ n_2,\ n_3)$, on which it acts. Looking at Eq. (4.6.5), it is clear that working with T_s^2 is easier than working with T_s. For known values of $(T_1,\ T_2,\ T_3)$, Eq. (4.6.5) states that T_s^2 is a function of n_1, n_2 and n_3, i.e.,

$$T_s^2 = f(n_1,\ n_2,\ n_3). \tag{4.6.6}$$

We wish to find the triples $(n_1,\ n_2,\ n_3)$ for which the value of the function f is a maximum, subject to the constraint that

$$n_1^2 + n_2^2 + n_3^2 = 1. \tag{4.6.7}$$

Once the maximum value of T_s^2 is obtained, the maximum value of T_s is also obtained. We also note that when $(n_1,\ n_2,\ n_3) = (\pm 1, 0, 0)$, or $(0, \pm 1, 0)$, or $(0, 0, \pm 1)$, Eq. (4.6.5) gives $T_s = 0$. This is simply because these are principal planes on which the shearing stress is zero. Clearly, $T_s = 0$ is the minimum value for the function in Eq. (4.6.5).

Taking the total derivative of the function in Eq. (4.6.6), we obtain, for stationary values of T_s^2,

$$dT_s^2 = \frac{\partial T_s^2}{\partial n_1} dn_1 + \frac{\partial T_s^2}{\partial n_2} dn_2 + \frac{\partial T_s^2}{\partial n_3} dn_3 = 0. \tag{4.6.8}$$

If dn_1, dn_2 and dn_3 can vary independently of one another, then Eq. (4.6.8) gives the familiar condition for the determination of the triple $(n_1,\ n_2,\ n_3)$ for the stationary value of T_s^2,

$$\frac{\partial T_s^2}{\partial n_1} = 0, \quad \frac{\partial T_s^2}{\partial n_2} = 0, \quad \frac{\partial T_s^2}{\partial n_3} = 0. \tag{4.6.9}$$

But dn_1, dn_2 and dn_3 cannot vary independently. Indeed, taking the total derivative of Eq. (4.6.7), i.e., $n_1^2 + n_2^2 + n_3^2 = 1$, we obtain

$$n_1 dn_1 + n_2 dn_2 + n_3 dn_3 = 0. \tag{4.6.10}$$

Comparing Eq. (4.6.10) with Eq. (4.6.8), we arrive at the following equations:

$$\frac{\partial T_s^2}{\partial n_1} = \lambda n_1, \quad \frac{\partial T_s^2}{\partial n_2} = \lambda n_2, \quad \frac{\partial T_s^2}{\partial n_3} = \lambda n_3. \tag{4.6.11}$$

The three equations in Eq. (4.6.11), together with the equation $n_1^2 + n_2^2 + n_3^2 = 1$ [i.e., Eq. (4.6.7)], are four equations for the determination of the four unknowns n_1, n_2, n_3 and λ. The multiplier λ is known as the *Lagrange multiplier*, and this method of determining the stationary value of a function subject to a constraint is known as the *Lagrange multiplier method*.

Using Eq. (4.6.5), we have, from Eqs. (4.6.11),

$$2n_1\left[T_1^2 - 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)T_1\right] = n_1 \lambda, \tag{4.6.12}$$

$$2n_2\left[T_2^2 - 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)T_2\right] = n_2 \lambda, \tag{4.6.13}$$

$$2n_3\left[T_3^2 - 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)T_3\right] = n_3 \lambda. \tag{4.6.14}$$

The four nonlinear algebraic equations, Eqs. (4.6.12), (4.6.13), (4.6.14), and (4.6.7), for the four unknowns $(n_1,\ n_2,\ n_3,\lambda)^\dagger$ have many sets of solution for a given set of values of $(T_1,\ T_2,\ T_3)$. Corresponding to each set of solution, the stationary value T_s^2, on the plane whose normal is given by $(n_1,\ n_2,\ n_3)$, can be obtained from Eq. (4.6.5), i.e.,

$$T_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - \left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)^2.$$

Among the stationary values will be the maximum and the minimum values of T_s^2. Table 4.1 summarizes the solutions. (See Appendix 4.1 for details.)

We note that $(n_1,\ n_2, 0)$ and $(-n_1, -n_2, 0)$ represent the same plane. On the other hand, $(n_1,\ n_2, 0)$ and $(n_1, -n_2, 0)$ are two distinct planes that are perpendicular to each other. Thus, although there are mathematically 18 sets of roots, there are only nine distinct planes.

---

†The value of the Lagrangean multiplier λ does not have any significance and can be simply ignored once the solutions to the system of equations are obtained.

Table 4.1 Stationary Values of T_s^2 and the Corresponding Planes

(n_1, n_2, n_3), $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ **Are Principal Directions**	**The Plane**	**Stationary Value of T_s^2**
$(1,0,0)$ and $(-1,0,0)$, i.e., $\mathbf{n} = \pm\mathbf{e}_1$	$\mathbf{e}_1$-plane	0
$(0,1,0)$ and $(0,-1,0)$ i.e., $\mathbf{n} = \pm\mathbf{e}_2$	$\mathbf{e}_2$-plane	0
$(0,0,1)$ and $(0,0,-1)$ i.e., $\mathbf{n} = \pm\mathbf{e}_3$	$\mathbf{e}_3$-plane	0
$(1/\sqrt{2})(1,1,0)$ and $(1/\sqrt{2})(-1,-1,0)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$	The plane bisects $\mathbf{e}_1$-plane and $\mathbf{e}_2$-plane in the first and third quadrant	$\left(\frac{T_1 - T_2}{2}\right)^2$
$(1/\sqrt{2})(1,-1,0)$ and $(1/\sqrt{2})(-1,1,0)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_1 - \mathbf{e}_2)$	The plane bisects $\mathbf{e}_1$-plane and $\mathbf{e}_2$-plane in the second and fourth quadrant	$\left(\frac{T_1 - T_2}{2}\right)^2$
$(1/\sqrt{2})(1,0,1)$ and $(1/\sqrt{2})(-1,0,-1)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_3)$	The plane bisects $\mathbf{e}_1$-plane and $\mathbf{e}_3$-plane in the first and third quadrant	$\left(\frac{T_1 - T_3}{2}\right)^2$
$(1/\sqrt{2})(1,0,-1)$ and $(1/\sqrt{2})(-1,0,1)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_1 - \mathbf{e}_3)$	The plane bisects $\mathbf{e}_1$-plane and $\mathbf{e}_3$-plane in the second and fourth quadrant	$\left(\frac{T_1 - T_3}{2}\right)^2$
$(1/2)(0,1,1)$ and $(1/2)(0,-1,-1)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_2 + \mathbf{e}_3)$	The plane bisects $\mathbf{e}_2$-plane and $\mathbf{e}_3$-plane in the first and third quadrant	$\left(\frac{T_2 - T_3}{2}\right)^2$
$(1/\sqrt{2})(0,1,-1)$ and $(1/\sqrt{2})(0,-1,1)$ i.e., $\mathbf{n} = \pm(1/\sqrt{2})(\mathbf{e}_2 - \mathbf{e}_3)$	The plane bisects $\mathbf{e}_2$-plane and $\mathbf{e}_3$-plane in the second and fourth quadrant	$\left(\frac{T_2 - T_3}{2}\right)^2$

Three of the planes are the principal planes, on each of which the shearing stress is zero (as it should be), which is the minimum value of the magnitude of shearing stress. The other six planes in general have nonzero shearing stresses. We also note from the third column of the table that those two planes that are perpendicular to each other have the same magnitude of shearing stresses. This is because the stress tensor is symmetric. The values of T_s^2 given in the third column are the stationary values T_s^2, of which zero is the minimum. The maximum value of T_s^2 is the maximum of the values in the third column. Thus, the maximum magnitude of shearing stress is given by the maximum of the following three values:

$$\frac{|T_1 - T_2|}{2}, \quad \frac{|T_1 - T_3|}{2}, \quad \frac{|T_2 - T_3|}{2}. \tag{4.6.15}$$

In other words,

$$(T_s)_{\max} = \frac{(T_n)_{\max} - (T_n)_{\min}}{2}, \tag{4.6.16}$$

where $(T_n)_{\max}$ and $(T_n)_{\min}$ are the largest and the smallest normal stresses, respectively. The two mutually perpendicular planes, on which this maximum shearing stress acts, bisect the planes of the largest and the smallest normal stress.

It can also be shown that on the plane of maximum shearing stress, the normal stress is

$$T_n = \frac{(T_n)_{\max} + (T_n)_{\min}}{2}. \tag{4.6.17}$$

If two of the principal stresses are equal, say, $T_1 = T_2 \neq T_3$, then, in addition to the solutions listed in the table, infinitely many other solutions can be obtained by rotating $\mathbf{e}_1$ and $\mathbf{e}_2$ axes about the $\mathbf{e}_3$ axis. Their stationary values of T_s, however, remain the same as those before the rotation. Finally, if $T_1 = T_2 = T_3$, then there is zero shearing stress on all the planes.

Example 4.6.1

If the state of stress is such that the components T_{13}, T_{23} and T_{33} are equal to zero, it is called a *state of plane stress*. (a) For this state of plane stress, find the principal values and the corresponding principal directions. (b) Determine the maximum shearing stress.

Solution

(a) For the stress matrix

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.6.18}$$

the characteristic equation is

$$\lambda[\lambda^2 - (T_{11} + T_{22})\lambda + (T_{11}T_{22} - T_{12}^2)] = 0. \tag{4.6.19}$$

Therefore, $\lambda = 0$ is an eigenvalue and its corresponding eigenvector is obviously $\mathbf{n} = \mathbf{e}_3$. The remaining eigenvalues are

$$\left\{ \begin{matrix} T_1 \\ T_2 \end{matrix} \right. = \frac{(T_{11} + T_{22}) \pm \sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}}{2}. \tag{4.6.20}$$

To find the corresponding eigenvectors, we set $\left(T_{ij} - \lambda\delta_{ij}\right)n_j = 0$ and obtain, for either $\lambda = T_1$ or T_2,

$$\begin{aligned} &(T_{11} - \lambda)n_1 + T_{12}n_2 = 0 \\ &T_{12}n_1 + (T_{22} - \lambda)n_2 = 0 \\ &(0 - \lambda)n_3 = 0 \end{aligned} \tag{4.6.21}$$

The last equation gives $n_3 = 0$. Let $\mathbf{n} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$ (see Figure 4.6-2); then, from the first of Eq. (4.6.21), we have

$$\tan\theta = \frac{n_2}{n_1} = -\frac{T_{11} - \lambda}{T_{12}} \tag{4.6.22}$$

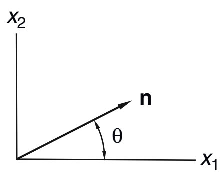

FIGURE 4.6-2

(b) Since the third eigenvalue T_3 is zero, the maximum shearing stress will be the greatest of the following three values:

$$\frac{|T_1|}{2}, \quad \frac{|T_2|}{2}, \quad \text{and} \quad \left|\frac{T_1 - T_2}{2}\right| = \frac{\sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}}{2} \tag{4.6.23}$$

Example 4.6.2

Do the previous example for the following state of stress: $T_{12} = T_{21} = 1000\ MPa$. All other T_{ij} are zero.

Solution

From Eq. (4.6.20), we have

$$\left\{\begin{matrix} T_1 \\ T_2 \end{matrix}\right. = \pm\frac{\sqrt{4(1000)^2}}{2} = \pm 1000\ MPa$$

Corresponding to the maximum normal stress $T_1 = 1000\ MPa$, Eq. (4.6.22) gives

$$\tan\theta_1 = -\frac{0 - 1000}{1000} = +1, \text{ i.e., } \theta_1 = 45°,$$

and corresponding to the minimum normal stress $T_2 = -1000\ MPa$ (i.e., maximum compressive stress),

$$\tan\theta_2 = -\frac{0 - (-1000)}{1000} = -1, \text{ i.e., } \theta_1 = -45°.$$

The maximum shearing stress is given by

$$(T_s)_{\max} = \frac{1000 - (-1000)}{2} = 1000\ MPa,$$

which acts on the plane bisecting the planes of maximum and minimum normal stress, i.e., it acts on the $\mathbf{e}_1$-plane and the $\mathbf{e}_2$-plane.

Example 4.6.3

Given $[\mathbf{T}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 500 \end{bmatrix}$ *MPa*, Determine the maximum shearing stress and the planes on which it acts.

Solution

Here we have $T_1 = T_2 = 100\ MPa$, $\mathrm{T}_3 = 500\ MPa$. Thus, the maximum shearing stress is

$$T_s = \frac{500 - 100}{2} = 200\ MPa.$$

The planes on which it acts include not only the four planes $(\mathbf{e}_1 \pm \mathbf{e}_3)/\sqrt{2}$ and $(\mathbf{e}_2 \pm \mathbf{e}_3)/\sqrt{2}$ but also any plane $\left(n_1\mathbf{e}_1 + n_2\mathbf{e}_2 \pm \frac{1}{\sqrt{2}}\mathbf{e}_3\right)$, where $n_1^2 + n_2^2 + \frac{1}{2} = 1$. In other words, these planes are tangent to the conical surface of the right circular cone, with $\mathbf{e}_3$ as its axis and with an angle of 45° between the generatrix and the axis.

4.7 EQUATIONS OF MOTION: PRINCIPLE OF LINEAR MOMENTUM

In this section, we derive the differential equations of motion for any continuum in motion. The basic postulate is that each particle of the continuum must satisfy Newton's law of motion.

Figure 4.7-1 shows the stress vectors that act on the six faces of a small rectangular element isolated from the continuum in the neighborhood of the position designated by x_i.

Let $\mathbf{B} = B_i\mathbf{e}_i$ be the body force (such as weight) per unit mass, ρ be the mass density at x_i, and $\mathbf{a}$ be the acceleration of a particle currently at the position x_i; then Newton's law of motion takes the form, valid in rectangular Cartesian coordinate systems,

$$\begin{aligned}&\{\mathbf{t}_{\mathbf{e}_1}(x_1+\Delta x_1,\ x_2,\ x_3)+\mathbf{t}_{-\mathbf{e}_1}(x_1,\ x_2,\ x_3)\}(\Delta x_2\Delta x_3)+\{\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2+\Delta x_2,\ x_3)+\mathbf{t}_{-\mathbf{e}_2}(x_1,\ x_2,\ x_3)\}(\Delta x_1\Delta x_3)\\&+\{\mathbf{t}_{\mathbf{e}_3}(x_1,\ x_2,\ x_3+\Delta x_3)+\mathbf{t}_{-\mathbf{e}_3}(x_1,\ x_2,\ x_3)\}(\Delta x_1\Delta x_2)+\rho\mathbf{B}\Delta x_1\Delta x_2\Delta x_3=(\rho\Delta x_1\Delta x_2\Delta x_3)\mathbf{a}.\end{aligned}\tag{i}$$

Since $\mathbf{t}_{-\mathbf{e}_1} = -\mathbf{t}_{\mathbf{e}_1}$,

$$\mathbf{t}_{\mathbf{e}_1}(x_1+\Delta x_1,\ x_2,\ x_3)+\mathbf{t}_{-\mathbf{e}_1}(x_1,\ x_2,\ x_3)=\left\{\frac{\mathbf{t}_{\mathbf{e}_1}(x_1+\Delta x_1,\ x_2,\ x_3)-\mathbf{t}_{\mathbf{e}_1}(x_1,\ x_2,\ x_3)}{\Delta x_1}\right\}\Delta x_1.\tag{ii}$$

Similarly,

$$\{\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2+\Delta x_2,\ x_3)+\mathbf{t}_{-\mathbf{e}_2}(x_1,\ x_2,\ x_3)\}=\left\{\frac{\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2+\Delta x_2,\ x_3)-\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2,\ x_3)}{\Delta x_2}\right\}\Delta x_2,\ \text{etc.}\tag{iii}$$

Thus, Eq. (i) becomes

$$\begin{aligned}&\left\{\frac{\mathbf{t}_{\mathbf{e}_1}(x_1+\Delta x_1,\ x_2,\ x_3)-\mathbf{t}_{\mathbf{e}_1}(x_1,\ x_2,\ x_3)}{\Delta x_1}\right\}+\left\{\frac{\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2+\Delta x_2,\ x_3)-\mathbf{t}_{\mathbf{e}_2}(x_1,\ x_2,\ x_3)}{\Delta x_2}\right\}\\&+\left\{\frac{\mathbf{t}_{\mathbf{e}_3}(x_1,\ x_2,\ x_3+\Delta x_3)-\mathbf{t}_{\mathbf{e}_3}(x_1,\ x_2,\ x_3)}{\Delta x_3}\right\}+\rho\mathbf{B}=\rho\mathbf{a}.\end{aligned}\tag{4.7.1}$$

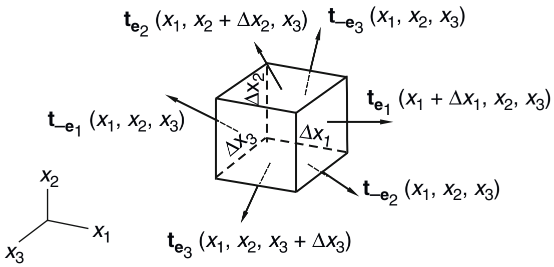

FIGURE 4.7-1

Letting $\Delta x_i \to 0$, we obtain from the preceding equation,

$$\frac{\partial\mathbf{t}_{\mathbf{e}_1}}{\partial x_1}+\frac{\partial\mathbf{t}_{\mathbf{e}_2}}{\partial x_2}+\frac{\partial\mathbf{t}_{\mathbf{e}_3}}{\partial x_3}+\rho\mathbf{B}=\rho\mathbf{a}\quad\text{or}\quad\frac{\partial\mathbf{t}_{\mathbf{e}_j}}{\partial x_j}+\rho B_j\mathbf{e}_j=\rho a_j\mathbf{e}_j.\tag{4.7.2}$$

Since $\mathbf{t}_{\mathbf{e}_j} = \mathbf{T}\mathbf{e}_j = T_{ij}\mathbf{e}_i$, we have (noting that all $\mathbf{e}_i$ are of fixed directions in Cartesian coordinates)

$$\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i + \rho B_i\mathbf{e}_i = \rho a_i\mathbf{e}_i. \tag{4.7.3}$$

In invariant form, the preceding equation is

$$\operatorname{div}\mathbf{T} + \rho\mathbf{B} = \rho\mathbf{a}, \tag{4.7.4}$$

and in Cartesian component form

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i. \tag{4.7.5}$$

These are the equations that must be satisfied for any continuum in motion, whether it is a solid or a fluid. They are called *Cauchy's equations of motion*. If the acceleration vanishes, then Eq. (4.7.5) reduces to the static equilibrium equation:

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0. \tag{4.7.6}$$

Example 4.7.1

In the absence of body forces, does the following stress distribution satisfy the equations of equilibrium? In these equations ν is a constant.

$$T_{11} = x_2^2 + \nu(x_1^2 - x_2^2),\quad T_{12} = -2\nu x_1 x_2,\quad T_{22} = x_1^2 + \nu(x_2^2 - x_1^2),$$
$$T_{23} = T_{13} = 0,\quad T_{33} = \nu(x_1^2 + x_2^2).$$

Solution

We have

$$\frac{\partial T_{1j}}{\partial x_j} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 2\nu x_1 - 2\nu x_1 + 0 = 0,$$

$$\frac{\partial T_{2j}}{\partial x_j} = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = -2\nu x_2 + 2\nu x_2 + 0 = 0,$$

and

$$\frac{\partial T_{3j}}{\partial x_j} = \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0 + 0 + 0 = 0.$$

Therefore, the given stress distribution does satisfy the equilibrium equations.

Example 4.7.2

Write the equations of motion for the case where the stress components have the form $T_{ij} = -p\delta_{ij}$, where $p = p(x_1, x_2, x_3, t)$.

Solution

For the given T_{ij},

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} = -\frac{\partial p}{\partial x_i}.$$

Therefore, from Eq. (4.7.6), we have

$$-\frac{\partial p}{\partial x_i} + \rho B_i = \rho a_i, \tag{4.7.7}$$

or

$$-\nabla p + \rho \mathbf{B} = \rho \mathbf{a}. \tag{4.7.8}$$

4.8 EQUATIONS OF MOTION IN CYLINDRICAL AND SPHERICAL COORDINATES

In Chapter 2, we presented the components of div **T** in cylindrical and in spherical coordinates. Using those formulas [Eqs. (2.34.8) to (2.34.10) and Eqs. (2.35.33) to (2.35.35)], we have the following equations of motion (see also Prob. 4.36).

Cylindrical coordinates:

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} + \rho B_r = \rho a_r, \tag{4.8.1}$$

$$\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} + \rho B_\theta = \rho a_\theta, \tag{4.8.2}$$

$$\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} + \rho B_z = \rho a_z. \tag{4.8.3}$$

For symmetric stress tensors, $T_{r\theta} + T_{\theta r} = 2T_{r\theta}$ in Eq. (4.8.2).

Spherical coordinates:

$$\frac{1}{r^2}\frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r} + \rho B_r = \rho a_r, \tag{4.8.4}$$

$$\frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta} - T_{\theta r} - T_{\phi\phi}\cot\theta}{r} + \rho B_\theta = \rho a_\theta, \tag{4.8.5}$$

$$\frac{1}{r^3}\frac{\partial\left(r^3 T_{\phi r}\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(T_{\phi\theta}\sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{r\phi} - T_{\phi r} + T_{\theta\phi}\cot\theta}{r} + \rho B_\phi = \rho a_\phi. \tag{4.8.6}$$

For symmetric stress tensors, $T_{r\theta} - T_{\theta r} = 0$ and $T_{r\phi} - T_{\phi r} = 0$ in the preceding equations.

Example 4.8.1

The stress field for the problem of an infinite elastic space loaded by a concentrated force at the origin (the Kelvin problem) is given by the following stress distribution in cylindrical coordinates:

$$T_{rr} = A\left(\frac{z}{R^3} - \frac{3r^2 z}{R^5}\right), \quad T_{\theta\theta} = \frac{Az}{R^3}, \quad T_{zz} = -A\left(\frac{z}{R^3} + \frac{3z^3}{R^5}\right),$$

$$T_{rz} = -A\left(\frac{r}{R^3} + \frac{3rz^2}{R^5}\right), \quad T_{r\theta} = T_{z\theta} = 0,$$

where $R^2 = r^2 + z^2$ and A is a constant related to the load. Verify that the given distribution of stress is in equilibrium in the absence of body forces.

Solution

From $R^2 = r^2 + z^2$, we obtain $\dfrac{\partial R}{\partial r} = \dfrac{r}{R}, \ \dfrac{\partial R}{\partial z} = \dfrac{z}{R}.$
Thus,

$$\frac{\partial T_{rr}}{\partial r} = A\left(-\frac{3z}{R^4}\frac{\partial R}{\partial r} - \frac{6rz}{R^5} + \frac{15r^2 z}{R^6}\frac{\partial R}{\partial r}\right) = A\left(-\frac{3zr}{R^5} - \frac{6rz}{R^5} + \frac{15r^3 z}{R^7}\right),$$

$$\frac{T_{rr} - T_{\theta\theta}}{r} = -A\left(\frac{3rz}{R^5}\right)$$

$$\frac{\partial T_{rz}}{\partial z} = -A\left(-\frac{3r}{R^4}\frac{\partial R}{\partial z} + \frac{6rz}{R^5} - \frac{15rz^2}{R^6}\frac{\partial R}{\partial z}\right) = A\left(\frac{3zr}{R^5} - \frac{6rz}{R^5} + \frac{15rz^3}{R^7}\right).$$

The left-hand side of Eq. (4.8.1) becomes

$$A\left(-\frac{3zr}{R^5} - \frac{6rz}{R^5} + \frac{15r^3 z}{R^7} - \frac{3rz}{R^5} + \frac{3zr}{R^5} - \frac{6rz}{R^5} + \frac{15rz^3}{R^7}\right) = A\left(-\frac{15rz}{R^5} + \frac{15rz}{R^7}\{r^2 + z^2\}\right)$$

$$= A\left(-\frac{15rz}{R^5} + \frac{15rz}{R^5}\right) = 0.$$

In other words, the r-equation of equilibrium is satisfied. Since $T_{r\theta} = T_{\theta z} = 0$ and $T_{\theta\theta}$ is independent of θ, the second equation of equilibrium is also satisfied. The third equation of equilibrium can be similarly verified (see Prob. 4.37).

4.9 BOUNDARY CONDITION FOR THE STRESS TENSOR

If on the boundary of some body there are applied distributive forces, we call them *surface tractions*. We wish to find the relation between the surface tractions and the stress field that is defined within the body.

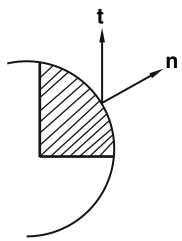

FIGURE 4.9-1

If we consider an infinitesimal tetrahedron cut from the boundary of a body with its inclined face coinciding with the plane tangent to the boundary face (Figure 4.9-1), then, as in Section 4.1, we obtain

$$\mathbf{t} = \mathbf{T}\mathbf{n}, \tag{4.9.1}$$

where $\mathbf{n}$ is the unit outward normal vector to the boundary, $\mathbf{T}$ is the stress tensor evaluated at the boundary, and $\mathbf{t}$ is the force vector per unit area on the boundary. Equation (4.9.1) is called the *stress boundary condition*. The special case of $\mathbf{t} = 0$ is known as the traction-free condition.

Example 4.9.1

Given the following stress field in a thick-wall elastic cylinder:

$$T_{rr} = A + \frac{B}{r^2}, \quad T_{\theta\theta} = A - \frac{B}{r^2}, \quad T_{r\theta} = T_{rz} = T_{\theta z} = T_{zz} = 0,$$

where A and B are constants. (a) Verify that the given state of stress satisfies the equations of equilibrium in the absence of body forces. (b) Find the stress vector on a cylindrical surface $r = a$, and (c) if the surface traction on the inner surface $r = r_i$ is a uniform pressure p_i and the outer surface $r = r_o$ is free of surface traction, find the constant A and B.

Solution

(a) With $T_{r\theta} = T_{rz} = T_{\theta z} = T_{zz} = 0$ and $T_{\theta\theta}$ depending only on r, we only need to check the r-equation of equilibrium. We have

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = -\frac{2B}{r^3} + 0 + \frac{2B}{r^3} + 0 = 0.$$

Thus, all equations of equilibrium are satisfied.

(b) The unit outward normal vector to a cylindrical surface at $r = a$ is $\mathbf{n} = \mathbf{e}_r$. Thus, the stress vector on this surface is given by

$$\begin{bmatrix} t_r \\ t_\theta \\ t_z \end{bmatrix} = \begin{bmatrix} T_{rr} & 0 & 0 \\ 0 & T_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{rr} \\ 0 \\ 0 \end{bmatrix},$$

i.e.,

$$\mathbf{t} = T_{rr}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_z = \left(A + \frac{B}{a^2}\right)\mathbf{e}_r.$$

(c) The boundary conditions are:

$$\text{At } r = r_o,\ T_{rr} = 0 \quad \text{and} \quad \text{at} \quad r = r_i,\ T_{rr} = -p_i.$$

Thus,

$$A + \frac{B}{r_i^2} = -p_i \quad \text{and} \quad A + \frac{B}{r_o^2} = 0.$$

The preceding two equations give

$$A = \frac{p_i r_i^2}{r_o^2 - r_i^2}, \quad B = -\frac{p_i r_i^2 r_o^2}{r_o^2 - r_i^2},$$

and the state of stress is given by

$$T_{rr} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right), \quad T_{\theta\theta} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right).$$

Example 4.9.2

It is known that the equilibrium stress field in an elastic spherical shell under the action of external and internal pressure in the absence of body forces is of the form

$$T_{rr} = A - \frac{2B}{r^3}, \quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3}, \quad T_{r\theta} = T_{r\phi} = T_{\theta\phi} = 0.$$

(a) Verify that the stress field satisfies the equations of equilibrium in the absence of body forces.
(b) Find the stress vector on a spherical surface $r = a$.
(c) Determine the constants A and B if the inner surface of the shell is subject to a uniform pressure p_i and the outer surface is free of surface traction.

Solution

(a) With

$$r^2 T_{rr} = Ar^2 - \frac{2B}{r}, \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) = \frac{2A}{r} + \frac{2B}{r^4}, \quad T_{r\theta} = T_{r\phi} = 0 \quad \text{and} \quad \frac{T_{\theta\theta} + T_{\phi\phi}}{r} = \frac{2A}{r} + \frac{2B}{r^4},$$

the left-hand side of the r-equation of equilibrium [see Eq. (4.8.4)] is

$$\frac{1}{r^2}\frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r}$$
$$= \left(\frac{2A}{r} + \frac{2B}{r^4}\right) + 0 + 0 - \left(\frac{2A}{r} + \frac{2B}{r^4}\right) = 0,$$

i.e., the r-equation of equilibrium is satisfied. The other two equations can be similarly verified (see Prob. 4.40).

(b) The unit outward normal vector to the spherical surface $r = a$ is $\mathbf{n} = \mathbf{e}_r$. Thus, the stress vector on this surface is given by

$$\begin{bmatrix} t_r \\ t_\theta \\ t_\phi \end{bmatrix} = \begin{bmatrix} T_{rr} & 0 & 0 \\ 0 & T_{\theta\theta} & 0 \\ 0 & 0 & T_{\phi\phi} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{rr} \\ 0 \\ 0 \end{bmatrix},$$

i.e.,

$$\mathbf{t} = T_{rr}\mathbf{e}_r + 0\mathbf{e}_\theta + 0\mathbf{e}_\phi = \left(A - \frac{2B}{a^3}\right)\mathbf{e}_r.$$

(c) The boundary conditions are

$$\text{At } r = r_o,\ T_{rr} = 0 \quad \text{and} \quad \text{at} \quad r = r_i,\ T_{rr} = -p_i.$$

Thus,

$$A - \frac{2B}{r_o^3} = 0 \quad \text{and} \quad A - \frac{2B}{r_i^3} = -p_i.$$

The preceding two equations give

$$A = \frac{p_i r_i^3}{(r_o^3 - r_i^3)} \quad \text{and} \quad B = \frac{p_i r_i^3 r_o^3}{2(r_o^3 - r_i^3)}.$$

The state of stress is

$$T_{rr} = \frac{p_i r_i^3}{(r_o^3 - r_i^3)}\left(1 - \frac{r_o^3}{r^3}\right), \quad T_{\theta\theta} = T_{\phi\phi} = \frac{p_i r_i^3}{(r_o^3 - r_i^3)}\left(1 + \frac{r_o^3}{2r^3}\right).$$

4.10 PIOLA KIRCHHOFF STRESS TENSORS

Cauchy stress tensor is defined in Section 4.2 based on the differential area at the current position. Stress tensors based on the undeformed area can also be defined. They are known as the *first* and *second Piola-Kirchhoff stress tensors*. It is useful to be familiar with them not only because they appear in many works on continuum mechanics but also because one particular tensor may be more suitable in a particular problem.

For example, there may be situations in which it is more convenient to formulate equations of motion (or equilibrium) with respect to the reference configuration instead of the current configuration. In this case, the use of the first Piola-Kirchhoff stress tensor results in the equations that are of the same form as the familiar Cauchy equations of motion (see Section 4.11). As another example, in finite deformations, depending on whether $\mathbf{D}$ (the rate of deformation) or $D\mathbf{F}/Dt$ ($\mathbf{F}$ being the deformation gradient) or $D\mathbf{E}^*/Dt$ ($\mathbf{E}^*$ being Lagrangian deformation tensor) are used, the calculation of stress power (the rate at which work is done to change the volume and shape of a particle of unit volume) is most conveniently obtained using the Cauchy stress tensor, the first Piola-Kirchhoff stress tensor, or the second Piola-Kirchhoff stress tensor, respectively (see Section 4.13).

Also, in Example 5.57.3 of Chapter 5, we will see that $\mathbf{T} = \mathbf{f}(\mathbf{C})$, where $\mathbf{T}$ is Cauchy's stress tensor and $\mathbf{C}$ is the right Cauchy-Green deformation tensor, is not an acceptable form of constitutive equation. On the other hand, $\tilde{\mathbf{T}} = \mathbf{f}(\mathbf{C})$ is acceptable, where $\tilde{\mathbf{T}}$ is the second Piola-Kirchhoff stress tensor.

Let dA_o and dA be the same differential material area at the reference time t_o and the current time t, respectively. We may refer to dA_o as the undeformed area and dA as the deformed area. These two areas in general have different orientations. We let the unit normal to the undeformed area be $\mathbf{n}_o$ and to the deformed area be $\mathbf{n}$. We may consider each area as a vector having a magnitude and a direction. For example, $d\mathbf{A}_o = dA_o\mathbf{n}_o$ and $d\mathbf{A} = dA\mathbf{n}$. Let $d\mathbf{f}$ be the force acting on the deformed area $d\mathbf{A} = dA\mathbf{n}$. In Section 4.1, we defined the Cauchy stress vector $\mathbf{t}$ and the associated Cauchy stress tensor $\mathbf{T}$ based on the deformed area $d\mathbf{A} = dA\mathbf{n}$, that is,

$$d\mathbf{f} = \mathbf{t}dA, \tag{4.10.1}$$

and

$$\mathbf{t} = \mathbf{T}\mathbf{n}. \tag{4.10.2}$$

In this section, we define two other pairs of (pseudo) stress vectors and tensors, based on the undeformed area $d\mathbf{A}_o = dA_o\mathbf{n}_o$.

(A) *The first Piola-Kirchhoff stress tensor*. Let

$$d\mathbf{f} \equiv \mathbf{t}_o dA_o. \tag{4.10.3}$$

The stress vector $\mathbf{t}_o$, defined by the preceding equation, is a pseudo-stress vector in that, being based on the undeformed area, it does not describe the actual intensity of the force $d\mathbf{f}$, which acts on the deformed area $d\mathbf{A} = dA\mathbf{n}$. We note that $\mathbf{t}_o$ has the same direction as the Cauchy stress vector $\mathbf{t}$.

The *first Piola-Kirchhoff stress tensor* (also known as the *Lagrangian stress tensor*) is a linear transformation $\mathbf{T}_o$ such that

$$\mathbf{t}_o = \mathbf{T}_o\mathbf{n}_o. \tag{4.10.4}$$

The relation between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor can be obtained as follows: From

$$d\mathbf{f} = \mathbf{t}dA = \mathbf{t}_o dA_o, \tag{4.10.5}$$

we have

$$\mathbf{t}_o = \left(\frac{dA}{dA_o}\right)\mathbf{t}. \tag{4.10.6}$$

Using Eq. (4.10.2) and Eq. (4.10.4), Eq. (4.10.6) becomes

$$\mathbf{T}_o\mathbf{n}_o = \left(\frac{dA}{dA_o}\right)\mathbf{T}\mathbf{n} = \frac{\mathbf{T}(dA\mathbf{n})}{dA_o}. \tag{4.10.7}$$

In Section 3.27, we obtained the relation between $d\mathbf{A}_o = dA_o\mathbf{n}_o$ and $d\mathbf{A} = dA\mathbf{n}$ as

$$dA\mathbf{n} = dA_o J(\mathbf{F}^{-1})^{\mathrm{T}}\mathbf{n}_o. \tag{4.10.8}$$

where $J = |\det\mathbf{F}|$. Thus,

$$\mathbf{T}_o\mathbf{n}_o = J\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}\mathbf{n}_o. \tag{4.10.9}$$

The preceding equation is to be true for all $\mathbf{n}_o$; therefore,

$$\mathbf{T}_o = J\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}, \tag{4.10.10}$$

and

$$\mathbf{T} = \frac{1}{J}\mathbf{T}_o\mathbf{F}^{\mathrm{T}}. \tag{4.10.11}$$

These are the desired relationships. In Cartesian component form, we have

$$(T_o)_{ij} = JT_{im}F^{-1}_{jm}, \tag{4.10.12}$$

and

$$T_{ij} = \frac{1}{J}(T_o)_{im}F_{jm}. \tag{4.10.13}$$

When Cartesian coordinates are used for both the reference and the current configuration,

$$F_{im} = \frac{\partial x_i}{\partial X_m} \quad \text{and} \quad F_{im}^{-1} = \frac{\partial X_i}{\partial x_m}.$$

We note that the first Piola-Kirchhoff stress tensor is in general not symmetric.

(B) The *second Piola-Kirchhoff stress tensor*. Let

$$d\tilde{\mathbf{f}} = \tilde{\mathbf{t}}dA_o, \tag{4.10.14}$$

where

$$d\mathbf{f} = \mathbf{F}d\tilde{\mathbf{f}}. \tag{4.10.15}$$

In Eq. (4.10.15), $d\tilde{\mathbf{f}}$ is the (pseudo) differential force that transforms, under the deformation gradient $\mathbf{F}$, into the (actual) differential force $d\mathbf{f}$ at the deformed position; thus, the pseudo-vector $\tilde{\mathbf{t}}$ is in general in a different direction than that of the Cauchy stress vector $\mathbf{t}$.

The second Piola-Kirchhoff stress tensor is a linear transformation $\tilde{\mathbf{T}}$ such that

$$\tilde{\mathbf{t}} = \tilde{\mathbf{T}}\mathbf{n}_o, \tag{4.10.16}$$

where we recall that $\mathbf{n}_o$ is the unit normal to the undeformed area. From Eqs. (4.10.14), (4.10.15), and (4.10.16), we have

$$d\mathbf{f} = \mathbf{F}\tilde{\mathbf{T}}\mathbf{n}_o dA_o. \tag{4.10.17}$$

We also have [see Eqs. (4.10.3) and (4.10.4)]

$$d\mathbf{f} \equiv \mathbf{t}_o dA_o = \mathbf{T}_o\mathbf{n}_o dA_o. \tag{4.10.18}$$

Comparing Eq. (4.10.17) with Eq. (4.10.18), we have

$$\tilde{\mathbf{T}}\mathbf{n}_o = \mathbf{F}^{-1}\mathbf{T}_o\mathbf{n}_o. \tag{4.10.19}$$

Again, this is to be valid for all $\mathbf{n}_o$; therefore,

$$\tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{T}_o. \tag{4.10.20}$$

Equation (4.10.20) gives the relationship between the first Piola-Kirchhoff stress tensor $\mathbf{T}_o$ and the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{T}}$. The relationship between the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor can be obtained from Eqs. (4.10.10) and (4.10.20). We have

$$\tilde{\mathbf{T}} = J\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}} \quad \text{where} \quad J = |\det\ \mathbf{F}|. \tag{4.10.21}$$

We note that the second Piola-Kirchhoff stress tensor is a symmetric tensor if the Cauchy stress tensor is a symmetric one.

Example 4.10.1

The deformed configuration of a body is described by

$$x_1 = 4X_1, \quad x_2 = -\frac{1}{2}X_2, \quad x_3 = -\frac{1}{2}X_3. \tag{i}$$

If the Cauchy stress tensor for this body is

$$[\mathbf{T}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa. \tag{ii}$$

(a) What is the corresponding first Piola-Kirchhoff stress tensor?
(b) What is the corresponding second Piola-Kirchhoff stress tensor?

Solution

(a) From Eq. (i), we have

$$[\mathbf{F}] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \quad [\mathbf{F}^{-1}] = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ det } \mathbf{F} = 1. \tag{iii}$$

Thus, the first Piola-Kirchhoff stress tensor is, from Eqs. (4.10.10), (ii), and (iii)

$$[\mathbf{T}_o] = (1)[\mathbf{T}]\left[(\mathbf{F}^{-1})^{\mathrm{T}}\right] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa. \tag{iv}$$

(b) From Eqs. (4.10.20) and (iv),

$$[\tilde{\mathbf{T}}] = [\mathbf{F}^{-1}]\,[\mathbf{T}_o] = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 25/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa. \tag{v}$$

Example 4.10.2

The equilibrium configuration of a body is described by

$$x_1 = \frac{1}{2}X_1, \quad x_2 = -\frac{1}{2}X_3, \quad x_3 = 4X_2. \tag{i}$$

If the Cauchy stress tensor for this body is

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix} MPa. \tag{ii}$$

(a) What is the corresponding first Piola-Kirchhoff stress tensor?
(b) What is the corresponding second Piola-Kirchhoff stress tensor?

(c) Calculate the pseudo-stress vector associated with the first Piola-Kirchhoff stress tensor on the $\mathbf{e}_3$-plane in the deformed state.

(d) Calculate the pseudo-stress vector associated with the second Piola-Kirchhoff stress tensor on the $\mathbf{e}_3$-plane in the deformed state.

Solution

From Eq. (i), we have

$$[\mathbf{F}] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/4 \\ 0 & -2 & 0 \end{bmatrix}, \ \det \mathbf{F} = 1. \tag{iii}$$

(a) The first Piola-Kirchhoff stress tensor is, from Eqs. (4.10.10), (ii), and (iii)

$$[\mathbf{T}_o] = (1)[\mathbf{T}]\,[(\mathbf{F}^{-1})^{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1/4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 25 & 0 \end{bmatrix} MPa. \tag{iv}$$

(b) The second Piola-Kirchhoff stress tensor is, from Eqs. (4.10.20) and (iv),

$$[\tilde{\mathbf{T}}] = [\mathbf{F}^{-1}]\,[\mathbf{T}_o] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1/4 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 25 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 25/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa. \tag{v}$$

(c) For a unit area in the deformed state in the $\mathbf{e}_3$ direction, its undeformed area $dA_o\mathbf{n}_o$ is given by [see Eq. (3.27.12)]:

$$dA_o\mathbf{n}_o = \frac{1}{|\det \mathbf{F}|}\mathbf{F}^{T}\mathbf{n}. \tag{vi}$$

Using Eq. (iii) in Eq. (vi), we have, with $\mathbf{n} = \mathbf{e}_3$,

$$[dA_o\mathbf{n}_o] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}. \tag{vii}$$

That is,

$$\mathbf{n}_o = \mathbf{e}_2 \quad \text{and} \quad dA_o = 4. \tag{viii}$$

Thus, the stress vector associated with the first Piola-Kirchhoff stress tensor is

$$[\mathbf{t}_o] = [\mathbf{T}_o]\,[\mathbf{n}_o] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 25 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix} MPa. \tag{ix}$$

That is, $\mathbf{t}_o = 25\mathbf{e}_3$ *MPa*. We note that this vector is in the same direction as the Cauchy stress vector; its magnitude is one fourth of that of the Cauchy stress vector because the undeformed area is four times that of the deformed area.

(d) The stress vector associated with the second Piola-Kirchhoff stress tensor is

$$[\tilde{\mathbf{t}}] = \left[\tilde{\mathbf{T}}\right][\mathbf{n}_o] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 25/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 25/4 \\ 0 \end{bmatrix} MPa. \tag{x}$$

That is, $\tilde{\mathbf{t}} = (25/4)\mathbf{e}_2$ *MPa*. We see that this pseudo-stress vector is in a different direction from that of the Cauchy stress vector.

Example 4.10.3

Given the following identity for any tensor function $\mathbf{A}(X_1, X_2, X_3)$ (see Prob. 3.73):

$$\frac{\partial}{\partial X_m}\det\mathbf{A} = (\det\mathbf{A})(\mathbf{A}^{-1})_{nj}\frac{\partial A_{jn}}{\partial X_m}. \tag{4.10.22}$$

Show that for the deformation gradient tensor $\mathbf{F}$

$$\frac{\partial}{\partial x_j}\left(\frac{F_{jm}}{J}\right) = 0, \tag{4.10.23}$$

where $F_{jm} = \dfrac{\partial x_j}{\partial X_m}$, $x_j = \hat{x}_j(X_1, X_2, X_3, t), J = \det\mathbf{F} > 0$.

Solution

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(\frac{F_{jm}}{J}\right) &= \frac{1}{J}\frac{\partial F_{jm}}{\partial x_j} - \frac{F_{jm}}{J^2}\frac{\partial J}{\partial x_j} = \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{1}{J^2}\left(\frac{\partial x_j}{\partial X_m}\right)\frac{\partial J}{\partial X_n}\frac{\partial X_n}{\partial x_j} \\ &= \frac{1}{J}\frac{\partial F_{jm}}{\partial X_n}\frac{\partial X_n}{\partial x_j} - \frac{1}{J^2}\delta_{nm}\frac{\partial J}{\partial X_n} = \frac{1}{J}\left(\frac{\partial^2 x_j}{\partial X_n \partial X_m}\right)\frac{\partial X_n}{\partial x_j} - \frac{1}{J^2}\frac{\partial J}{\partial X_m}.\end{aligned} \tag{i}$$

Now, from the given identity Eq. (4.10.22), with $\mathbf{A} \equiv \mathbf{F}$, $(\mathbf{A}^{-1})_{nj} = (\mathbf{F}^{-1})_{nj} = \dfrac{\partial X_n}{\partial x_j}$, we have

$$\frac{\partial J}{\partial X_m} = J\frac{\partial X_n}{\partial x_j}\frac{\partial F_{jn}}{\partial X_m} = J\frac{\partial X_n}{\partial x_j}\frac{\partial^2 x_j}{\partial X_m \partial X_n}. \tag{ii}$$

Thus,

$$\frac{\partial}{\partial x_j}\left(\frac{F_{jm}}{J}\right) = \frac{1}{J}\left(\frac{\partial^2 x_j}{\partial X_n \partial X_m}\right)\frac{\partial X_n}{\partial x_j} - \frac{1}{J}\frac{\partial X_n}{\partial x_j}\left(\frac{\partial^2 x_j}{\partial X_m \partial X_n}\right) = 0. \tag{iii}$$

4.11 EQUATIONS OF MOTION WRITTEN WITH RESPECT TO THE REFERENCE CONFIGURATION

In Section 4.7, we derive the equations of motion in terms of the Cauchy stress tensor as follows:

$$\operatorname{div}\mathbf{T} + \rho\mathbf{B} = \rho\mathbf{a} \quad \text{or} \quad \frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i, \tag{4.11.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{B}$ is the body force per unit mass, $\mathbf{a}$ is the acceleration, and ρ is the density in the deformed state. Here the partial derivative $\partial T_{ij}/\partial x_j$ is with respect to the spatial coordinates x_i.

In this section we show that the equations of motion written in terms of the first Piola-Kirchhoff stress tensor have the same form as those written in terms of Cauchy stress tensor. That is,

$$\operatorname{Div}\mathbf{T}_o + \rho_o\mathbf{B} = \rho_o\mathbf{a} \quad \text{or} \quad \frac{\partial (T_o)_{im}}{\partial X_m} + \rho_o B_i = \rho_o a_i. \tag{4.11.2}$$

We note, however, here X_i are the material coordinates and ρ_o is the density at the reference state.

To derive Eq. (4.11.2), we use Eq. (4.10.13), i.e.,

$$T_{ij} = \frac{1}{J}(T_o)_{im}F_{jm} \quad \text{where } J = \det \mathbf{F}, \tag{i}$$

to obtain

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{(T_o)_{im}F_{jm}}{J} = \frac{F_{jm}}{J}\frac{\partial (T_o)_{im}}{\partial x_j} + (T_o)_{im}\frac{\partial}{\partial x_j}\frac{F_{jm}}{J} = \frac{F_{jm}}{J}\frac{\partial (T_o)_{im}}{\partial x_j}, \tag{ii}$$

where we have used the result of the previous example (Example 4.10.3) that $\frac{\partial}{\partial x_j}\frac{F_{jm}}{J} = 0$. Now,

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{F_{jm}}{J}\frac{\partial (T_o)_{im}}{\partial x_j} = \frac{1}{J}\frac{\partial x_j}{\partial X_m}\frac{\partial (T_o)_{im}}{\partial X_n}\frac{\partial X_n}{\partial x_j} = \frac{1}{J}\delta_{mn}\frac{\partial (T_o)_{im}}{\partial X_n}. \tag{iii}$$

Thus,

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{1}{J}\frac{\partial (T_o)_{ij}}{\partial X_j}. \tag{iv}$$

Using the preceding equation in the Cauchy equations of motion, i.e., $\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i$, we obtain

$$\frac{\partial (T_o)_{ij}}{\partial X_j} + (J\rho)B_i = (J\rho)a_i. \tag{v}$$

Now, $dV = (\det \mathbf{F})dV_o$ [see Eq. (3.28.3)]; therefore,

$$\rho_o = (\det \mathbf{F})\rho = J\rho, \tag{vi}$$

and Eq. (v) becomes

$$\frac{\partial (T_o)_{ij}}{\partial X_j} + \rho_o B_i = \rho_o a_i. \tag{vii}$$

4.12 STRESS POWER

Referring to the infinitesimal rectangular parallelepiped of Figure 4.12-1 (which is the same as Figure 4.7-1, repeated here for convenience), the rate at which work is done by the stress vectors $\mathbf{t}_{-\mathbf{e}_1}$ and $\mathbf{t}_{\mathbf{e}_1}$ on the pair of faces having $-\mathbf{e}_1$ and $\mathbf{e}_1$ as their respective normal is

$$\begin{aligned}\left[(\mathbf{t}_{\mathbf{e}_1}\cdot\mathbf{v})_{x_1+dx_1,\,x_2,\,x_3} + (\mathbf{t}_{-\mathbf{e}_1}\cdot\mathbf{v})_{x_1,\,x_2,\,x_3}\right]dx_2dx_3 &= \left[(\mathbf{t}_{\mathbf{e}_1}\cdot\mathbf{v})_{x_1+dx_1,\,x_2,\,x_3} - (\mathbf{t}_{\mathbf{e}_1}\cdot\mathbf{v})_{x_1,\,x_2,\,x_3}\right]dx_2dx_3 \\ &= \left[\frac{\partial}{\partial x_1}(\mathbf{t}_{\mathbf{e}_1}\cdot\mathbf{v})dx_1\right]dx_2dx_3 = \frac{\partial (T_{j1}v_j)}{\partial x_1}dV,\end{aligned} \tag{i}$$

where we have used the result that $\mathbf{t}_{\mathbf{e}_1}\cdot\mathbf{v} = \mathbf{T}\mathbf{e}_1\cdot\mathbf{v} = \mathbf{e}_1\cdot\mathbf{T}^{\mathrm{T}}\mathbf{v} = \mathbf{e}_1\cdot T_{ji}v_j\mathbf{e}_i = T_{ji}v_j(\mathbf{e}_1\cdot\mathbf{e}_i) = T_{j1}v_j$ and $dx_1dx_2dx_3 = dV$. Similarly, the rate at which work is done by the stress vectors on the other two pairs of faces are $\frac{\partial (T_{j2}v_j)}{\partial x_2}dV$ and $\frac{\partial (T_{j3}v_j)}{\partial x_3}dV$. Including the rate of work done by the body forces, which is $(\rho\mathbf{B}dV)\cdot\mathbf{v} = \rho B_i v_i dV$, the total rate of work done on the particle is

$$P = \left[\frac{\partial}{\partial x_j}(v_iT_{ij}) + \rho B_i v_i\right]dV = \left[v_i\left(\frac{\partial T_{ij}}{\partial x_j} + \rho B_i\right) + T_{ij}\frac{\partial v_i}{\partial x_j}\right]dV = \left[\rho v_i\frac{Dv_i}{Dt} + T_{ij}\frac{\partial v_i}{\partial x_j}\right]dV. \tag{ii}$$

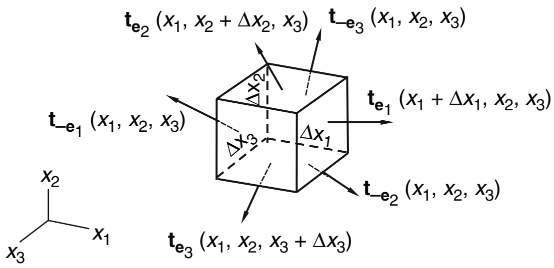

FIGURE 4.12-1

Now, $\dfrac{D}{Dt}(\rho dV) = 0$ (conservation of mass principle); therefore,

$$\rho v_i \frac{Dv_i}{Dt} dV = \rho dV \frac{D}{Dt}\left(\frac{v_i v_i}{2}\right) = \frac{D}{Dt}\left(\frac{v_i v_i}{2}\rho dV\right) = \frac{D}{Dt}\left(dm\frac{v^2}{2}\right) = \frac{D}{Dt}(KE). \qquad \text{(iii)}$$

where (KE) is the kinetic energy. We can now write

$$P = \frac{D}{Dt}(KE) + P_s dV, \qquad (4.12.1)$$

where

$$P_s = T_{ij}\frac{\partial v_i}{\partial x_j} = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\nabla \mathbf{v}). \qquad (4.12.2)$$

Since

$$T_{ij}\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(T_{ij}\frac{\partial v_i}{\partial x_j} + T_{ij}\frac{\partial v_i}{\partial x_j}\right) = \frac{1}{2}\left(T_{ij}\frac{\partial v_i}{\partial xj} + T_{ji}\frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}T_{ij}\left(\frac{\partial v_i}{\partial xj} + \frac{\partial v_j}{\partial x_i}\right) = T_{ij}D_{ij}, \qquad (4.12.3)$$

in terms of the symmetric stress tensor **T** and the rate of deformation tensor **D**, the stress power is

$$P_s = T_{ij}D_{ij} = \mathrm{tr}(\mathbf{TD}). \qquad (4.12.4)$$

The *stress power* P_s represents the rate at which work is done to change the volume and shape of a particle of unit volume.

4.13 STRESS POWER IN TERMS OF THE PIOLA-KIRCHHOFF STRESS TENSORS

In the previous section, we obtained the stress power in terms of the Cauchy stress tensor **T** and the rate of deformation tensor **D** [Eq. (4.12.4)]. In this section we obtain the stress power (a) in terms of the first Piola-Kirchhoff stress tensor $\mathbf{T}_o$ and the deformation gradient **F** and (b) in terms of the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{T}}$ and the Lagrangian deformation tensor $\mathbf{E}^*$. The pairs $(\mathbf{T}, \mathbf{D})$, $(\mathbf{T}_o, \mathbf{F})$ and $\left(\tilde{\mathbf{T}}, \mathbf{E}^*\right)$ are sometimes known as the *conjugate pairs*.

(a) In Section 3.12 we obtained [see Eq. (3.12.6)]

$$\frac{D}{Dt}d\mathbf{x} = (\nabla_{\mathbf{x}}\mathbf{v})d\mathbf{x}. \tag{4.13.1}$$

Since $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ [see Eq. (3.18.3)], Eq. (4.13.1) becomes

$$\frac{D}{Dt}\mathbf{F}d\mathbf{X} = \frac{D\mathbf{F}}{Dt}d\mathbf{X} = \nabla_{\mathbf{x}}\mathbf{v}\mathbf{F}d\mathbf{X}. \tag{4.13.2}$$

This equation is to be true for all $d\mathbf{X}$, thus

$$\frac{D\mathbf{F}}{Dt} = (\nabla_{\mathbf{x}}\mathbf{v})\mathbf{F}, \tag{4.13.3}$$

or

$$(\nabla_{\mathbf{x}}\mathbf{v}) = \frac{D\mathbf{F}}{Dt}\mathbf{F}^{-1}. \tag{4.13.4}$$

Now, from Eqs. (4.12.2) and (4.13.4), we have

$$P_s = \mathrm{tr}\left(\mathbf{T}^{\mathrm{T}}\frac{D\mathbf{F}}{Dt}\mathbf{F}^{-1}\right). \tag{4.13.5}$$

Since the Cauchy stress tensor $\mathbf{T}$ is related to the first Piola-Kirchhoff stress tensor $\mathbf{T}_{\mathrm{o}}$ by the equation $\mathbf{T} = \dfrac{1}{\det \mathbf{F}}\mathbf{T}_{\mathrm{o}}\mathbf{F}^{\mathrm{T}}$, [Eq. (4.10.11)], therefore,

$$P_s = \frac{1}{\det \mathbf{F}}\mathrm{tr}\left(\mathbf{F}\mathbf{T}_{\mathrm{o}}^{\mathrm{T}}\frac{D\mathbf{F}}{Dt}\mathbf{F}^{-1}\right). \tag{4.13.6}$$

Using the identity $\mathrm{tr}(\mathbf{ABCD}) = \mathrm{tr}(\mathbf{BCDA}) = \mathrm{tr}(\mathbf{CDAB})$ and the relation $\det \mathbf{F} = \rho_{\mathrm{o}}/\rho$, we have

$$P_s = \frac{\rho}{\rho_{\mathrm{o}}}\mathrm{tr}\left(\mathbf{T}_{\mathrm{o}}^{\mathrm{T}}\frac{D\mathbf{F}}{Dt}\right) = \frac{\rho}{\rho_{\mathrm{o}}}\mathrm{tr}\left((T_{\mathrm{o}})_{ij}\frac{DF_{ij}}{Dt}\right). \tag{4.13.7}$$

(b) The Cauchy stress tensor $\mathbf{T}$ is related to the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{T}}$ by the equation $\mathbf{T} = \dfrac{1}{\det \mathbf{F}}\mathbf{F}\tilde{\mathbf{T}}\mathbf{F}^{\mathrm{T}}$ [see Eq. (4.10.21)], therefore,

$$P_s = \mathrm{tr}(\mathbf{TD}) = \frac{1}{\det \mathbf{F}}\mathrm{tr}\left(\mathbf{F}\tilde{\mathbf{T}}\mathbf{F}^{\mathrm{T}}\mathbf{D}\right) = \frac{1}{\det \mathbf{F}}\mathrm{tr}\left(\tilde{\mathbf{T}}\mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}\right). \tag{4.13.8}$$

We now show that

$$\left(\frac{D\mathbf{E}^*}{Dt}\right) = \mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}. \tag{4.13.9}$$

We had [see Eq. (3.24.3)]

$$ds^2 = dS^2 + 2d\mathbf{X}\cdot\mathbf{E}^*d\mathbf{X}, \tag{4.13.10}$$

therefore,

$$\frac{D}{Dt}ds^2 = 2d\mathbf{X}\cdot\left(\frac{D\mathbf{E}^*}{Dt}\right)d\mathbf{X}. \tag{4.13.11}$$

But we also had [see Eq. (3.13.11)]

$$\frac{D}{Dt}ds^2 = 2d\mathbf{x}\cdot\mathbf{D}d\mathbf{x} = 2\mathbf{F}d\mathbf{X}\cdot\mathbf{D}\mathbf{F}d\mathbf{X} = 2d\mathbf{X}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}d\mathbf{X}. \tag{4.13.12}$$

Comparing Eq. (4.13.11) with Eq. (4.13.12), we obtain

$$\left(\frac{D\mathbf{E}^*}{Dt}\right) = \mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}. \tag{4.13.13}$$

Using Eq. (4.13.13), Eq. (4.13.8) becomes

$$P_s = \frac{1}{\det\mathbf{F}}\operatorname{tr}\left(\tilde{\mathbf{T}}\frac{D\mathbf{E}^*}{Dt}\right) = \frac{\rho}{\rho_{\mathrm{o}}}\operatorname{tr}\left(\tilde{\mathbf{T}}\frac{D\mathbf{E}^*}{Dt}\right). \tag{4.13.14}$$

4.14 RATE OF HEAT FLOW INTO A DIFFERENTIAL ELEMENT BY CONDUCTION

Let $\mathbf{q}$ be a vector whose magnitude gives the rate of heat flow across a unit area by conduction and whose direction gives the direction of the heat flow; then the net heat flow by conduction Q_c into a differential element can be computed as follows:

Referring to the infinitesimal rectangular parallelepiped of Figure 4.12-1, the net rate at which heat flows *into* the element across the pair of faces with $\mathbf{e}_1$ and $-\mathbf{e}_1$ as their outward normal vectors is

$$\left[-(\mathbf{q}\cdot\mathbf{e}_1)_{x_1+dx_1,\,x_2,\,x_3} + (\mathbf{q}\cdot\mathbf{e}_1)_{x_1,\,x_2,\,x_3}\right]dx_2dx_3 = -\left[\frac{\partial}{\partial x_1}(\mathbf{q}\cdot\mathbf{e}_1)dx_1\right]dx_2dx_3 = -\left(\frac{\partial q_1}{\partial x_1}dx_1\right)dx_2dx_3. \tag{i}$$

Including the contributions from the other two pairs of faces, the total net rate of heat *inflow* by conduction into the element is

$$-\left(\frac{\partial q_1}{\partial x_1}dx_1\right)dx_2dx_3 - \left(\frac{\partial q_2}{\partial x_2}dx_2\right)dx_1dx_3 - \left(\frac{\partial q_3}{\partial x_3}dx_3\right)dx_1dx_2 = -\left(\frac{\partial q_1}{\partial x_1}+\frac{\partial q_2}{\partial x_2}+\frac{\partial q_3}{\partial x_3}\right)dx_1dx_2dx_3. \tag{ii}$$

That is,

$$Q_c = -\left(\frac{\partial q_1}{\partial x_1}+\frac{\partial q_2}{\partial x_2}+\frac{\partial q_3}{\partial x_3}\right)dV = -(\operatorname{div}\mathbf{q})dV, \tag{4.14.1}$$

where dV is the differential volume of the element.

Example 4.14.1

Using the Fourier heat conduction law

$$\mathbf{q} = -\kappa\nabla\Theta, \tag{4.14.2}$$

where Θ is the temperature and κ is the coefficient of thermal conductivity, find the equation governing the steady-state temperature distribution in a heat-conducting body.

Solution

Using Eq. (4.14.1), we obtain, the net rate of heat inflow per unit volume at a point in the body as

$$-\left[\frac{\partial}{\partial x_1}\left(\kappa\frac{\partial\Theta}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa\frac{\partial\Theta}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\kappa\frac{\partial\Theta}{\partial x_3}\right)\right]$$

For a steady-state temperature distribution in the body, there should be no net rate of heat flow (either in or out) at every point in the body. Therefore, the governing equation is

$$\frac{\partial}{\partial x_1}\left(\kappa\frac{\partial\Theta}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\kappa\frac{\partial\Theta}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\kappa\frac{\partial\Theta}{\partial x_3}\right)=0. \tag{4.14.3}$$

For constant κ, the preceding equation reduces to the Laplace equation:

$$\nabla^2\Theta=\frac{\partial^2\Theta}{\partial x_1^2}+\frac{\partial^2\Theta}{\partial x_2^2}+\frac{\partial^2\Theta}{\partial x_3^2}=0. \tag{4.14.4}$$

4.15 ENERGY EQUATION

Consider a particle with a differential volume dV at position $\mathbf{x}$ at time t. Let U denote its internal energy, KE its kinetic energy, Q_c the net rate of heat inflow by conduction from its surroundings, Q_s the heat supply (rate of heat input due, e.g., to radiation), and P the rate of work done on the particle by body forces and surface forces. Then, in the absence of other forms of energy input, the fundamental postulate of conservation of energy states that *the rate of increase of internal and kinetic energy for a particle equals the work done on the material plus heat input through conduction across its boundary surface and heat supply throughout its volume*. That is,

$$\frac{D}{Dt}(U+KE)=P+Q_c+Q_s, \tag{4.15.1}$$

where (D/Dt) is material derivative, $P=\dfrac{D}{Dt}(KE)+T_{ij}\dfrac{\partial v_i}{\partial x_j}dV$ and $Q_c=-\dfrac{\partial q_i}{\partial x_i}dV$. [See Eqs. (4.12.1), (4.12.2), and (4.14.1)]. Thus,

$$\frac{DU}{Dt}=T_{ij}\frac{\partial v_i}{\partial x_j}dV-\frac{\partial q_i}{\partial x_i}dV+Q_s. \tag{4.15.2}$$

If we let u be the internal energy per unit mass, then

$$\frac{DU}{Dt}=\frac{D}{Dt}(u\rho dV)=\rho dV\frac{Du}{Dt}, \tag{4.15.3}$$

where we have used the conservation of mass equation $\dfrac{D}{Dt}(\rho dV)=0$. The energy equation then becomes

$$\rho\frac{Du}{Dt}=T_{ij}\frac{\partial v_i}{\partial x_j}-\frac{\partial q_i}{\partial x_i}+\rho q_s, \tag{4.15.4}$$

where q_s is heat supply per unit mass. In direct notation, the preceding equation reads

$$\rho\frac{Du}{Dt}=\mathrm{tr}(\mathbf{TD})-\mathrm{div}\,\mathbf{q}+\rho q_s. \tag{4.15.5}$$

4.16 ENTROPY INEQUALITY

Let $\eta(\mathbf{x},t)$ denote the entropy per unit mass for the continuum. Then the entropy in a particle of volume dV is $\rho\eta dV$, where ρ is density. The rate of increase of entropy following the particle as it is moving is

$$\frac{D}{Dt}(\rho\eta dV) = \rho dV\frac{D\eta}{Dt} + \eta\frac{D}{Dt}(\rho dV) = \rho dV\frac{D\eta}{Dt}, \tag{4.16.1}$$

where we have used the equation $(D/Dt)(\rho dV) = 0$ in accordance with the conservation of mass principle. Thus, per unit volume, the rate of increase of entropy is given by $\rho(D\eta/Dt)$. The entropy inequality law states that *the rate of increase of entropy in a particle is always greater than or equal to the entropy inflow across its boundary surface plus entropy supply throughout the volume*. That is,

$$\rho\frac{D\eta}{Dt} \geq -\mathrm{div}\left(\frac{\mathbf{q}}{\Theta}\right) + \frac{\rho q_s}{\Theta}, \tag{4.16.2}$$

where Θ is absolute temperature, $\mathbf{q}$ is heat flux vector, and q_s is heat supply.

Example 4.16.1

The temperature at $x_1 = 0$ of a body is kept at a constant Θ_1 and that at $x_1 = L$ is kept at a constant Θ_2. (a) Using the Fourier heat conduction law $\mathbf{q} = -\kappa\nabla\Theta$, where κ is a constant, find the temperature distribution. (b) Show that κ must be positive in order to satisfy the entropy inequality law.

Solution

(a) This is a one-dimensional steady-state temperature problem. The equation governing the temperature distribution is given by [see Eq. (4.14.4)]:

$$\frac{d^2\Theta}{dx_1^2} = 0. \tag{4.16.3}$$

Thus,

$$\Theta = \frac{\Theta_2 - \Theta_1}{L}x_1 + \Theta_1. \tag{4.16.4}$$

(b) With $\frac{D\eta}{Dt} = 0$ and $q_s = 0$, the inequality [Eq. (4.16.2)] becomes

$$0 \geq -\frac{d}{dx_1}\left[\frac{1}{\Theta}\left(-\kappa\frac{d\Theta}{dx_1}\right)\right] = \kappa\frac{d}{dx_1}\left[\frac{1}{\Theta}\left(\frac{d\Theta}{dx_1}\right)\right]. \tag{4.16.5}$$

Now,

$$\kappa\frac{d}{dx_1}\left[\frac{1}{\Theta}\left(\frac{d\Theta}{dx_1}\right)\right] = \kappa\left[\frac{1}{\Theta}\left(\frac{d^2\Theta}{dx_1^2}\right) - \frac{1}{\Theta^2}\left(\frac{d\Theta}{dx_1}\right)^2\right] = -\kappa\frac{1}{\Theta^2}\left(\frac{\partial\Theta}{\partial x_1}\right)^2.$$

Therefore, we have

$$\kappa\frac{1}{\Theta^2}\left(\frac{\partial\Theta}{\partial x_1}\right)^2 \geq 0. \tag{4.16.6}$$

Thus,

$$\kappa \geq 0, \tag{4.16.7}$$

and heat flows from high temperature to low temperature.

4.17 ENTROPY INEQUALITY IN TERMS OF THE HELMHOLTZ ENERGY FUNCTION

The Helmholtz energy per unit mass A is defined by the equation

$$A = u - \Theta\eta, \tag{4.17.1}$$

where u and η are internal energy per unit mass and entropy per unit mass, respectively, and Θ is absolute temperature. From Eq. (4.17.1), $u = A + \Theta\eta$, so that the energy equation, [Eq. (4.15.4)], i.e., $\rho\dfrac{Du}{Dt} = T_{ij}\dfrac{\partial v_i}{\partial x_j} - \dfrac{\partial q_i}{\partial x_i} + \rho q_s$, can be written as

$$\rho\Theta\frac{D\eta}{Dt} = -\left(\rho\frac{DA}{Dt} + \rho\eta\frac{D\Theta}{Dt}\right) + T_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s, \tag{4.17.2}$$

and the entropy inequality, [Eq. (4.16.2)], i.e., $\rho\dfrac{D\eta}{Dt} \geq -\mathrm{div}\left(\dfrac{\mathbf{q}}{\Theta}\right) + \dfrac{\rho q_s}{\Theta}$, can be written as

$$\rho\Theta\frac{D\eta}{Dt} \geq -\Theta\frac{\partial}{\partial x_i}\left(\frac{q_i}{\Theta}\right) + \rho q_s. \tag{4.17.3}$$

Using Eq. (4.17.2), the inequality Eq. (4.17.3) becomes

$$-\left(\rho\frac{DA}{Dt} + \rho\eta\frac{D\Theta}{Dt}\right) + T_{ij}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s \geq -\frac{\partial q_i}{\partial x_i} + \frac{q_i}{\Theta}\frac{\partial\Theta}{\partial x_i} + \rho q_s.$$

That is,

$$-\left(\rho\frac{DA}{Dt} + \rho\eta\frac{D\Theta}{Dt}\right) + T_{ij}D_{ij} - \frac{q_i}{\Theta}\frac{\partial\Theta}{\partial x_i} \geq 0, \tag{4.17.4}$$

where D_{ij} are components of the rate of deformation tensor and we have used the equation $T_{ij}\dfrac{\partial v_i}{\partial x_j} = T_{ij}D_{ij}$ for symmetric tensor T_{ij}. Equation (4.17.4) is the entropy law in terms of the Helmholtz energy function.

Example 4.17.1

In linear thermo-elasticity, one assumes that the Helmholtz function depends on the infinitesimal strain E_{ij} and absolute temperature Θ. That is,

$$A = A(E_{ij}, \Theta). \tag{4.17.5}$$

Derive the relationship between the stress tensor and the Helmholtz energy function.

Solution

From Eq. (4.17.5), we have

$$\frac{DA}{Dt} = \frac{\partial A}{\partial E_{ij}}\frac{DE_{ij}}{Dt} + \frac{\partial A}{\partial\Theta}\frac{D\Theta}{Dt}. \tag{4.17.6}$$

For small strain, $\dfrac{DE_{ij}}{Dt} = \dfrac{1}{2}\dfrac{D}{Dt}\left(\dfrac{\partial u_i}{\partial X_j} + \dfrac{\partial u_j}{\partial X_i}\right) = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial X_j} + \dfrac{\partial v_j}{\partial X_i}\right) \approx \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right) = D_{ij}.$

Thus, $\dfrac{DA}{Dt} = D_{ij}\dfrac{\partial A}{\partial E_{ij}} + \dfrac{\partial A}{\partial \Theta}\dfrac{D\Theta}{Dt}$, and the inequality (4.17.4) becomes

$$\left(-\rho\frac{\partial A}{\partial E_{ij}} + T_{ij}\right)D_{ij} - \left(\rho\frac{\partial A}{\partial \Theta} + \rho\eta\right)\frac{D\Theta}{Dt} - \frac{q_i}{\Theta}\frac{\partial \Theta}{\partial x_i} \geq 0. \tag{4.17.7}$$

This inequality must be satisfied for whatever values of D_{ij} and $\dfrac{D\Theta}{Dt}$. It follows that

$$\left(-\rho\frac{\partial A}{\partial E_{ij}} + T_{ij}\right) = 0, \quad \left(\rho\frac{\partial A}{\partial \Theta} + \rho\eta\right) = 0 \quad \text{and} \quad -\frac{q_i}{\Theta}\frac{\partial \Theta}{\partial x_i} \geq 0. \tag{4.17.8}$$

That is,

$$T_{ij} = \rho\frac{\partial A}{\partial E_{ij}}, \tag{4.17.9}$$

$$\eta = -\frac{\partial A}{\partial \Theta}, \tag{4.17.10}$$

and

$$-\frac{q_i}{\Theta}\frac{\partial \Theta}{\partial x_i} \geq 0. \tag{4.17.11}$$

The first equation states that the stress is derivable from a potential function; the last inequality states that heat must flow from high temperature to low temperature.

4.18 INTEGRAL FORMULATIONS OF THE GENERAL PRINCIPLES OF MECHANICS

In Section 3.15 of Chapter 3 and in Sections 4.4, 4.7, 4.15, and 4.16 of the current chapter, the field equations expressing the principles of conservation of mass, moment of momentum, linear momentum, energy, and the entropy inequality were derived using a differential element approach, and each of them was derived whenever the relevant tensors (e.g., the rate of deformation tensor, the Cauchy stress tensors, and so on) had been defined. In this section, all these principles are presented together and derived using the integral formulation by considering an arbitrary fixed part of the material. In the form of differential equations, the principles are sometimes referred to as *local principles*. In the form of integrals, they are known as *global principles*. Under the assumption of smoothness of functions involved, the two forms are completely equivalent, and in fact the requirement that the global theorem is to be valid for each and every part of the continuum results in the same differential form of the principles, as shown in this section. The purpose of this section is simply to provide an alternate approach to the formulation of the field equations and to group all the field equations for a continuum in one section for easy reference. We begin by deriving the conservation of mass equation by following a fixed part of the material.

(I) The conservation of mass principle states that the rate of increase of mass in a fixed part of a material is always zero. That is, the material derivative of the mass in any fixed part of the material is zero:

$$\frac{D}{Dt}\int_{V_m} \rho dV = 0. \tag{4.18.1}$$

In the preceding equation, ρ denotes density and V_m denotes the material volume that moves with the material. Now,

$$\frac{D}{Dt}\int_{V_m}\rho dV=\int_{V_m=V_c}\left[\frac{D}{Dt}(\rho dV)\right]=\int_{V_c}\left[\frac{D\rho}{Dt}dV+\rho\frac{DdV}{Dt}\right]=0. \tag{4.18.2}$$

In the preceding equation, V_c denotes the so-called control volume, which instantaneously coincides with the material volume V_m. In Section 3.13, we had [see Eq. (3.13.14)]

$$\frac{1}{dV}\frac{D}{Dt}dV=\frac{\partial v_i}{\partial x_i}=\text{div }\mathbf{v}. \tag{4.18.3}$$

Thus, Eq. (4.18.2) becomes

$$\int_{V_c}\left(\frac{D\rho}{Dt}+\rho\,\text{div }\mathbf{v}\right)dV=0 \quad \text{or} \quad \int_{V_c}\left(\frac{\partial\rho}{\partial t}+\text{div}\,(\rho\mathbf{v})\right)dV=0. \tag{4.18.4}$$

Equation (4.18.4) must be valid for all V_c, therefore, the integrand must be zero. That is,

$$\frac{D\rho}{Dt}+\rho\,\text{div }\mathbf{v}=0 \quad \text{or} \quad \frac{\partial\rho}{\partial t}+\text{div}(\rho\mathbf{v})=0. \tag{4.18.5}$$

This is the same as Eq. (3.15.4).

To derive the other four principles by considering a fixed part of a material, we will need the divergence theorem, which we state as follows without proof:

$$\int_{V_c}\text{div }\mathbf{v}dV=\int_{S_c}\mathbf{v}\cdot\mathbf{n}dS \quad \text{or} \quad \int_{V_c}\frac{\partial v_j}{\partial x_j}dV=\int_{S_c}v_jn_jdS, \tag{4.18.6}$$

$$\int_{V_c}\text{div }\mathbf{T}dV=\int_{S_c}\mathbf{Tn}dS \quad \text{or} \quad \int_{V_c}\frac{\partial T_{ij}}{\partial x_j}dV=\int_{S_c}T_{ij}n_jdV. \tag{4.18.7}$$

For a discussion of this theorem, refer to the first two sections of Chapter 7.

In the preceding equations, $\mathbf{v}$ and $\mathbf{T}$ are vector and tensor, respectively; $\mathbf{n}$ is a unit *outward* normal vector, and V_c and S_c denote control volume and the corresponding control surface. We note that using the divergence theorem, the second equation in Eq. (4.18.4) becomes

$$\frac{\partial}{\partial t}\int_{V_c}\rho dV=-\int_{S_c}(\rho\mathbf{v}\cdot\mathbf{n})dS, \tag{4.18.8}$$

which states that the rate of increase of mass inside a control volume must be equal the rate at which the mass enters the control volume. Eq. (4.18.8) is often used as the starting point to derive Eq. (4.18.5) by using the divergence theorem.

(II) The principle of linear momentum states that the forces acting on a fixed part of a material must equal the rate of change of linear momentum of the part:

$$\frac{D}{Dt}\int_{V_m}\rho\mathbf{v}dV=\int_{S_c}\mathbf{t}dS+\int_{V_c}\rho\mathbf{B}dV=\int_{S_c}\mathbf{Tn}dS+\int_{V_c}\rho\mathbf{B}dV, \tag{4.18.9}$$

where $\mathbf{t}$, $\mathbf{T}$, $\mathbf{B}$ and $\mathbf{v}$ are stress vector, stress tensor, body force per unit mass and velocity, respectively. Now

$$\frac{D}{Dt}\int_{V_m}\rho\mathbf{v}dV=\int_{V_m}\left[\frac{D}{Dt}(\rho\mathbf{v}dV)\right]=\int_{V_m=V_c}\left[\mathbf{v}\frac{D}{Dt}(\rho dV)+\frac{D\mathbf{v}}{Dt}\rho dV\right]=\int_{V_c}\frac{D\mathbf{v}}{Dt}\rho dV, \tag{4.18.10}$$

where $(D/Dt)(\rho dV)=0$ in accordance with the principle of conservation of mass.

Using the divergence theorem, the right side of Eq. (4.18.9) becomes

$$\int_{V_c} \operatorname{div} \mathbf{T} dV + \int_{V_c} \rho \mathbf{B} dV,$$

so that Eq. (4.18.9) becomes

$$\int_{V_c} \left[\rho \frac{D\mathbf{v}}{Dt} - \operatorname{div} \mathbf{T} - \rho \mathbf{B} \right] dV = 0. \tag{4.18.11}$$

This equation is to be valid for all V_c, therefore,

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{B}. \tag{4.18.12}$$

This is the same as Eq. (4.7.4).

(III) The principle of moment of momentum states that the moments about a fixed point of all the forces acting on a fixed part of a material must equal the rate of change of moment of momentum of the part about the same point:

$$\frac{D}{Dt}\int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV = \int_{S_c} \mathbf{x} \times \mathbf{t} dS + \int_{V_c} \mathbf{x} \times \rho \mathbf{B} dV = \int_{S_c} (\mathbf{x} \times \mathbf{T}\mathbf{n}) dS + \int_{V_c} \mathbf{x} \times \rho \mathbf{B} dV, \tag{4.18.13}$$

where $\mathbf{x}$ is the position vector. Again, since $(D/Dt)(\rho dV) = 0$, the left side of Eq. (4.18.13) becomes

$$\begin{aligned} \frac{D}{Dt}\int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV &= \int_{V_m = V_c} \left[\frac{D}{Dt}(\mathbf{x} \times \rho \mathbf{v} dV) \right] = \int_{V_c} \left[\mathbf{v} \times \rho \mathbf{v} dV + \mathbf{x} \times \frac{D}{Dt}(\rho \mathbf{v} dV) \right] \\ &= \int_{V_c} \left[\mathbf{x} \times \mathbf{v} \frac{D}{Dt}(\rho dV) + \mathbf{x} \times \frac{D\mathbf{v}}{Dt} \rho dV \right] = \int_{V_c} \mathbf{x} \times \frac{D\mathbf{v}}{Dt} \rho dV. \end{aligned} \tag{4.18.14}$$

Since $\mathbf{x} \times \mathbf{T}\mathbf{n} = \mathbf{e}_i \varepsilon_{ijk} x_j (\mathbf{T}\mathbf{n})_k = \mathbf{e}_i \varepsilon_{ijk} x_j T_{km} n_m$, by using the divergence theorem we obtain

$$\int_{S_c} \mathbf{x} \times \mathbf{T}\mathbf{n} dS = \mathbf{e}_i \int_{S_c} (\varepsilon_{ijk} x_j T_{km}) n_m dS = \mathbf{e}_i \int_{V_c} \frac{\partial \varepsilon_{ijk} x_j T_{km}}{\partial x_m} dV. \tag{4.18.15}$$

Now, $\partial x_i / \partial x_m = \delta_{im}$; therefore,

$$\begin{aligned} \int_{S_c} \mathbf{x} \times \mathbf{T}\mathbf{n} dS &= \mathbf{e}_i \int_{V_c} \frac{\partial \varepsilon_{ijk} x_j T_{km}}{\partial x_m} dV = \int_{V_c} \mathbf{e}_i \varepsilon_{ijk} x_j \frac{\partial T_{km}}{\partial x_m} dV + \int_{V_c} \mathbf{e}_i \varepsilon_{ijk} T_{kj} dV \\ &= \textstyle\int_{V_c} \mathbf{x} \times \operatorname{div} \mathbf{T} dV + \int_{V_c} \mathbf{e}_i \varepsilon_{ijk} T_{kj} dV. \end{aligned} \tag{4.18.16}$$

Thus, Eq. (4.18.13) becomes

$$\int_{V_c} \mathbf{x} \times \frac{D\mathbf{v}}{Dt} \rho dV = \int_{V_c} \mathbf{x} \times (\operatorname{div} \mathbf{T} + \rho \mathbf{B}) dV + \int_{V_c} \mathbf{e}_i \varepsilon_{ijk} T_{kj} dV, \tag{4.18.17}$$

or

$$\int_{V_c} \mathbf{x} \times \left(\rho \frac{D\mathbf{v}}{Dt} - \operatorname{div} \mathbf{T} - \rho \mathbf{B} \right) dV + \int_{V_c} \mathbf{e}_i \varepsilon_{ijk} T_{kj} dV = 0. \tag{4.18.18}$$

But the linear momentum equation gives $\rho \dfrac{D\mathbf{v}}{Dt} - \text{div}\mathbf{T} - \rho\mathbf{B} = \mathbf{0}$. Thus, Eq. (4.18.18) becomes $\int_{V_c} \mathbf{e}_i \varepsilon_{ijk} T_{kj} dV = 0$, so that

$$\varepsilon_{ijk} T_{kj} = 0. \tag{4.18.19}$$

From which we arrive at the symmetry of stress tensor. That is,

$$T_{12} - T_{21} = 0, \quad T_{23} - T_{32} = 0, \quad T_{31} - T_{13} = 0. \tag{4.18.20}$$

This same result was obtained in Section 4.4.

(IV) The conservation of energy principle states that the rate of increase of kinetic energy and internal energy in a fixed part of a material must equal the sum of the rate of work by surface and body forces, rate of heat inflow across the boundary, and heat supply within:

$$\frac{D}{Dt}\int_{V_m}\left(\frac{\rho v^2}{2} + \rho u\right)dV = \int_{S_c}(\mathbf{t}\cdot\mathbf{v})dS + \int_{V_c}\rho\mathbf{B}\cdot\mathbf{v}dV - \int_{S_c}(\mathbf{q}\cdot\mathbf{n})dS + \int_{V_c}\rho q_s dV, \tag{4.18.21}$$

where u is the internal energy per unit mass, $\mathbf{q}$ the heat flux vector, and q_s the heat supply per unit mass. We note that with $\mathbf{n}$ being an outward unit normal vector, $(-\mathbf{q}\cdot\mathbf{n})$ represents rate of heat inflow. Again, $(D/Dt)(\rho dV) = 0$; therefore, the left side becomes

$$\frac{D}{Dt}\int_{V_m}\rho\left(\frac{v^2}{2} + u\right)dV = \int_{V_m=V_c}\left[\frac{D}{Dt}\left(\frac{v^2}{2} + u\right)\right]\rho dV. \tag{4.18.22}$$

Now,

$$\int_{S_c}\mathbf{t}\cdot\mathbf{v}dS = \int_{S_c}\mathbf{Tn}\cdot\mathbf{v}dS = \int_{S_c}\mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{v}dS = \int_{V_c}\text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)dV, \tag{4.18.23}$$

$$\text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) = \frac{\partial T_{ji}v_j}{\partial x_i} = \frac{\partial T_{ji}}{\partial x_i}v_j + T_{ji}\frac{\partial v_j}{\partial x_i} = (\text{div }\mathbf{T})\cdot\mathbf{v} + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{v}\right), \tag{4.18.24}$$

and $\int_{S_c}\mathbf{q}\cdot\mathbf{n}dS = \int_{V_c}(\text{div }\mathbf{q})dV$, therefore, Eq. (4.18.21) becomes

$$\int_{V_c}\rho\frac{D}{Dt}\left(\frac{v^2}{2} + u\right)dV = \int_{V_c}\left[(\text{div }\mathbf{T} + \rho\mathbf{B})\cdot\mathbf{v} + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{v}\right) - \text{div }\mathbf{q} + \rho q_s\right]dV. \tag{4.18.25}$$

But $(\text{div }\mathbf{T} + \rho\mathbf{B})\cdot\mathbf{v} = \rho(D\mathbf{v}/Dt)\cdot\mathbf{v} = (1/2)\rho(Dv^2/Dt)$, therefore, Eq. (4.18.25) becomes

$$\int_{V_c}\rho\frac{Du}{Dt}dV = \int_{V_c}\left[\text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{v}\right) - \text{div }\mathbf{q} + \rho q_s\right]dV. \tag{4.18.26}$$

For this equation to be valid for all V_c, we must have

$$\rho\frac{Du}{Dt} = \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{v}\right) - \text{div }\mathbf{q} + \rho q_s. \tag{4.18.27}$$

This is the same as Eq. (4.15.4).

(V) The entropy inequality states that the rate of increase of entropy in a fixed part of a material is not less than the influx of entropy, $\mathbf{q}/\Theta$, across the surface of the part plus the entropy supply within the volume:

$$\frac{D}{Dt}\int_{V_m}\rho\eta dV \geq -\int_{S_c}\frac{\mathbf{q}}{\Theta}\cdot\mathbf{n}dS + \int_{V_c}\frac{\rho q_s}{\Theta}dV, \tag{4.18.28}$$

where η is the entropy per unit mass, and other symbols have the same meanings as before. Now, again, $(D/Dt)(\rho dV) = 0$, therefore,

$$\frac{D}{Dt}\int_{V_m} \rho\eta dV = \int_{V_c} \frac{D\eta}{Dt}\rho dV. \tag{4.18.29}$$

Using the divergence theorem, we have $\int_{S_c} (\mathbf{q}/\Theta)\cdot \mathbf{n} dS = \int_{V_c} \operatorname{div}(\mathbf{q}/\Theta)dV$; thus, the inequality (4.18.29) becomes

$$\int_{V_c} \rho\frac{D\eta}{Dt}dV \geq -\int_{V_c} \operatorname{div}\left(\frac{\mathbf{q}}{\Theta}\right)dV + \int_{V_c}\frac{\rho q_s}{\Theta}dV, \tag{4.18.30}$$

so that

$$\rho\frac{D\eta}{Dt} \geq -\operatorname{div}\left(\frac{\mathbf{q}}{\Theta}\right) + \frac{\rho q_s}{\Theta}. \tag{4.18.31}$$

This is the same as Eq. (4.16.2).

We remark that later, in Chapter 7, we revisit the derivations of the integral form of the principles with emphasis on Reynold's transport theorem and its applications to obtain the approximate solutions of engineering problems using the concept of moving as well as fixed control volumes.

APPENDIX 4.1: DETERMINATION OF MAXIMUM SHEARING STRESS AND THE PLANES ON WHICH IT ACTS

This appendix gives the details of solving the following system of four nonlinear algebraic equations in $n_1,\ n_2,\ n_3$ and λ:

$$2n_1\left[T_1^2 - 2\left(T_1n_1^2 + T_2n_2^2 + T_3n_3^2\right)T_1\right] = n_1\lambda, \tag{i}$$

$$2n_2\left[T_2^2 - 2\left(T_1n_1^2 + T_2n_2^2 + T_3n_3^2\right)T_2\right] = n_2\lambda, \tag{ii}$$

$$2n_3\left[T_3^2 - 2\left(T_1n_1^2 + T_2n_2^2 + T_3n_3^2\right)T_3\right] = n_3\lambda, \tag{iii}$$

$$n_1^2 + n_2^2 + n_3^2 = 1. \tag{iv}$$

These are Eqs. (4.6.12), (4.6.13), (4.6.14), and (4.6.7) in Section 4.6 for the determination of the maximum shearing stress and the plane(s) on which it acts. This system of equations determines all stationary values of T_s^2 from Eq. (4.6.5), which is repeated here:

$$T_s^2 = T_1^2n_1^2 + T_2^2n_2^2 + T_3^2n_3^2 - \left(T_1n_1^2 + T_2n_2^2 + T_3n_3^2\right)^2. \tag{v}$$

From the stationary values of T_s^2, the maximum and the minimum values of T_s are obtained. The following are the details:

1. Case I: $T_1 = T_2 = T_3 = T$. In this case, Eqs. (i), (ii), and (iii) reduce to the following three equations:

$$-2n_1T^2 = n_1\lambda, \quad -2n_2T^2 = n_2\lambda, \quad -2n_3T^2 = n_3\lambda.$$

These equations show that (i), (ii), and (iii) are satisfied for arbitrary values of (n_1, n_2, n_3) with $\lambda = -2T^2$ and $n_1^2 + n_2^2 + n_3^2 = 1$. Eq. (v) gives $T_s^2 = 0$ for this case. This is to be expected because with

$T_1 = T_2 = T_3$, every plane is a principal plane having zero shearing stress on it. In this case, $T_s^2 = 0$ is both the maximum and the minimum value of T_s^2 and of T_s. We note that although we get a value for the Lagrangian multiplier $\lambda = -2T^2$, it does not have any significance and can be simply ignored.

2. Case II: Only two of the T_is are the same.

(a) If $T_1 = T_2 \neq T_3$,

$$\text{Equation (i) becomes } 2n_1\left[-T_1^2 + 2(T_1 - T_3)T_1 n_3^2\right] = n_1\lambda. \tag{vi}$$

$$\text{Equation (ii) becomes } 2n_2\left[-T_1^2 + 2(T_1 - T_3)T_1 n_3^2\right] = n_2\lambda. \tag{vii}$$

$$\text{Equation (iii) becomes } 2n_3\left[T_3^2 - 2T_1T_3 + \left(2T_1T_3 - 2T_3^2\right)n_3^2\right] = n_3\lambda. \tag{viii}$$

From the preceding three equations, we see that if $n_3 = 0$, any $(n_1, n_2, 0)$ with $n_1^2 + n_2^2 = 1$ is a solution with $\lambda = -2T_1^2$ and $T_s^2 = 0$ [from Eq. (v)]. We note that all these planes are principal planes, including (1, 0, 0) and (0, 1, 0).

If $n_3 \neq 0$, in addition to the obvious solution $(0, 0, \pm 1)$, there are also solutions from the following [see Eqs. (vi) and (viii)]:

$$2\left[-T_1^2 + 2(T_1 - T_3)T_1 n_3^2\right] = 2\left[T_3^2 - 2T_1T_3 + \left(2T_1T_3 - 2T_3^2\right)n_3^2\right] = \lambda.$$

Rearranging the preceding equation, we have

$$\left[2(T_1 - T_3)T_1 n_3^2\right] = \left[(T_1 - T_3)^2 + 2(T_1 - T_3)T_3 n_3^2\right],$$

which leads to

$$2n_3^2 = 1,$$

and

$$T_s^2 = T_1^2\left(1 - n_3^2\right) + T_3^2 n_3^2 - \left(T_1\left(1 - n_3^2\right) + T_3 n_3^2\right)^2 = \frac{(T_1 - T_3)^2}{4} = \frac{(T_2 - T_3)^2}{4}.$$

Thus, if $T_1 = T_2 \neq T_3$, the solutions are

$$(n_1,\ n_2, 0),\ \text{any } n_1, n_2 \text{ satisfying } n_1^2 + n_2^2 = 1, T_s^2 = 0, \tag{ix}$$

and

$$\left(n_1, n_2, \pm\sqrt{1/2}\right),\ \text{any } n_1,\ n_2 \text{ satisfying } n_1^2 + n_2^2 + 1/2 = 1,\ T_s^2 = \frac{(T_1 - T_3)^2}{4} = \frac{(T_2 - T_3)^2}{4}. \tag{x}$$

(b) If $T_2 = T_3 \neq T_1$, the solutions are

$$(0, n_2, n_3),\ \text{for any } n_2, n_3 \text{ satisfying } n_2^2 + n_3^2 = 1 \text{ and } T_s^2 = 0 \text{ on those planes.} \tag{xi}$$

$$\left(\pm\sqrt{1/2}, n_2, n_3\right),\ \text{for any } n_2, n_3 \text{ satisfying } 1/2 + n_2^2 + n_3^2 = 1 \text{ and}$$
$$T_s^2 = \frac{(T_2 - T_1)^2}{4} = \frac{(T_3 - T_1)^2}{4} \text{ on those planes.} \tag{xii}$$

(c) If $T_3 = T_1 \neq T_2$, the solutions are

$$(n_1, 0, n_3),\ \text{for any } n_1, n_3 \text{ satisfying } n_1^2 + n_3^2 = 1 \text{ and } T_s^2 = 0 \ \text{ on those planes}, \tag{xiii}$$

$$\left(n_1, \pm\sqrt{1/2}, n_3\right),\ \text{for any } n_1, n_3 \ \text{ satisfying } \ n_1^2 + 1/2 + n_3^2 = 1 \text{ and}$$
$$T_s^2 = \frac{(T_3 - T_2)^2}{4} = \frac{(T_1 - T_2)^2}{4} \text{ on those planes.} \tag{xiv}$$

3. Case III: All three T_i are distinct. In this case, at least one of the three n_1, n_2, n_3 must be zero. To show this, we first assume that neither n_1 nor n_2 are zero; then Eqs. (i) and (ii) give

$$2\left[T_1^2 - 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)T_1\right] = 2\left[T_2^2 - 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)T_2\right] = \lambda,$$

thus,

$$T_1^2 - T_2^2 = 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right)(T_1 - T_2).$$

Since $T_1 \neq T_2$,

$$T_1 + T_2 = 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right).$$

If n_3 is also not zero, then we also have

$$T_1 + T_3 = 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right) \quad \text{and} \quad T_2 + T_3 = 2\left(T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2\right).$$

In other words,

$$T_1 + T_2 = T_1 + T_3 = T_2 + T_3.$$

from which we see that $T_1 = T_2 = T_3$, which contradicts the assumption that all three T_i are distinct. Therefore, if all three T_is are distinct, at least one of the three n_is must be zero. If two of the n_is are zero, we obviously have the following three cases:

$$\textbf{(a)}\ (n_1, n_2, n_3) = (\pm 1, 0, 0), \lambda = -2T_1^2, T_s = 0. \tag{xv}$$

$$\textbf{(b)}\ (n_1, n_2, n_3) = (0, \pm 1, 0), \lambda = -2T_2^2, T_s = 0. \tag{xvi}$$

$$\textbf{(c)}\ (n_1, n_2, n_3) = (0, 0, \pm 1), \lambda = -2T_3^2, T_s = 0. \tag{xvii}$$

If only n_3 is zero, then Eqs. (i) and (ii) give

$$2\left[T_1^2 - 2\left(T_1 n_1^2 + T_2 n_2^2\right)T_1\right] = 2\left[T_2^2 - 2\left(T_1 n_1^2 + T n_2^2\right)T_2\right] = \lambda,$$

or

$$T_1^2 - T_2^2 = 2\left(T_1 n_1^2 + T_2 n_2^2\right)(T_1 - T_2).$$

Since $T_1 \neq T_2$ and $n_1^2 + n_2^2 = 1$, the preceding equation becomes

$$T_1 + T_2 = 2\left(T_1 n_1^2 + T_2 n_2^2\right) = 2\left[T_1 n_1^2 + T_2\left(1 - n_1^2\right)\right].$$

Thus,

$$T_1 - T_2 = 2n_1^2(T_1 - T_2) \quad \text{or} \quad 1 = 2n_1^2.$$

Therefore, $n_1 = \pm\sqrt{1/2}$ and $n_2 = \pm\sqrt{1/2}$, i.e.,

$$\textbf{(d)}\ (n_1, n_2, n_3) = \pm(1/\sqrt{2}, \pm 1/\sqrt{2}, 0),\ T_s^2 = \frac{(T_1 - T_2)^2}{4}. \tag{xviii}$$

Similarly, we also have

$$\textbf{(e)}\ (n_1, n_2, n_3) = \pm(1/\sqrt{2}, 0, \pm 1/\sqrt{2}),\ T_s^2 = \frac{(T_1 - T_3)^2}{4}. \tag{xix}$$

$$\textbf{(f)}\ (n_1, n_2, n_3) = \pm(0, 1/\sqrt{2}, \pm 1/\sqrt{2}),\ T_s^2 = \frac{(T_2 - T_3)^2}{4}. \tag{xx}$$

PROBLEMS FOR CHAPTER 4

4.1 The state of stress at a certain point in a body is given by

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}_{\mathbf{e}_i} MPa.$$

On each of the coordinate planes (with normal in $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ directions), (a) what is the normal stress? (b) What is the total shearing stress?

4.2 The state of stress at a certain point in a body is given by

$$[\mathbf{T}] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}_{\mathbf{e}_i} MPa.$$

(a) Find the stress vector at a point on the plane whose normal is in the direction of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$.
(b) Determine the magnitude of the normal and shearing stresses on this plane.

4.3 Do the previous problem for a plane passing through the point and parallel to the plane $x_1 - 2x_2 + 3x_3 = 4$.

4.4 The stress distribution in a certain body is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100x_1 & -100x_2 \\ 100x_1 & 0 & 0 \\ -100x_2 & 0 & 0 \end{bmatrix} MPa.$$

Find the stress vector acting on a plane that passes through the point $\left(1/2, \sqrt{3}/2, 3\right)$ and is tangent to the circular cylindrical surface $x_1^2 + x_2^2 = 1$ at that point.

4.5 Given $T_{11} = 1\ MPa$, $T_{22} = -1\ MPa$, and all other $T_{ij} = 0$ at a point in a continuum.
(a) Show that the only plane on which the stress vector is zero is the plane with normal stress in the $\mathbf{e}_3$ direction.
(b) Give three planes on which no normal stress is acting.

4.6 For the following state of stress:

$$[\mathbf{T}] = \begin{bmatrix} 10 & 50 & -50 \\ 50 & 0 & 0 \\ -50 & 0 & 0 \end{bmatrix} MPa.$$

Find T'_{11} and T'_{13}, where $\mathbf{e}'_1$ is in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{e}'_2$ is in the direction of $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$.

4.7 Consider the following stress distribution:

$$[\mathbf{T}] = \begin{bmatrix} \alpha x_2 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where α and β are constants.

(a) Determine and sketch the distribution of the stress vector acting on the square in the $x_1 = 0$ plane with vertices located at $(0, 1, 1)$, $(0, -1, 1)$, $(0, 1, -1)$, and $(0, -1, -1)$.

(b) Find the total resultant force and moment about the origin of the stress vectors acting on the square of part (a).

4.8 Do the previous problem if the stress distribution is given by $T_{11} = \alpha x_2^2$ and all other $T_{ij} = 0$.

4.9 Do Prob. 4.7 for the stress distribution $T_{11} = \alpha$, $T_{12} = T_{21} = \alpha X_3$ and all other $T_{ij} = 0$.

4.10 Consider the following stress distribution for a circular cylindrical bar:

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}.$$

(a) What is the distribution of the stress vector on the surfaces defined by (i) the lateral surface $x_2^2 + x_3^2 = 4$, (ii) the end face $x_1 = 0$, and (iii) the end face $x_1 = l$?

(b) Find the total resultant force and moment on the end face $x_1 = l$.

4.11 An elliptical bar with lateral surface defined by $x_2^2 + 2x_3^2 = 1$ has the following stress distribution:

$$[\mathbf{T}] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}.$$

(a) Show that the stress vector at any point (x_1, x_2, x_3) on the lateral surface is zero.

(b) Find the resultant force, and resultant moment, about the origin O of the stress vector on the left end face $x_1 = 0$.

Note:

$$\int x_2^2 dA = \frac{\pi}{4\sqrt{2}} \quad \text{and} \quad \int x_3^2 dA = \frac{\pi}{8\sqrt{2}}.$$

4.12 For any stress state $\mathbf{T}$ we define the deviatoric stress $\mathbf{S}$ to be $\mathbf{S} = \mathbf{T} - (T_{kk}/3)\mathbf{I}$, where T_{kk} is the first invariant of the stress tensor $\mathbf{T}$.

(a) Show that the first invariant of the deviatoric stress vanishes.

(b) Evaluate **S** for the stress tensor:

$$[\mathbf{T}] = 100\begin{bmatrix} 6 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 9 \end{bmatrix} kPa.$$

(c) Show that the principal directions of the stress tensor coincide with those of the deviatoric stress tensor.

4.13 An octahedral stress plane is one whose normal makes equal angles with each of the principal axes of stress.
(a) How many independent octahedral planes are there at each point?
(b) Show that the normal stress on an octahedral plane is given by one-third the first stress invariant.
(c) Show that the shearing stress on the octahedral plane is given by

$$T_s = \frac{1}{3}\left[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2\right]^{1/2},$$

where T_1, T_2, T_3 are principal values of the stress tensor.

4.14 **(a)** Let **m** and **n** be two unit vectors that define two planes M and N that pass through a point P. For an arbitrary state of stress defined at the point P, show that the component of the stress vector $\mathbf{t_m}$ in the **n** direction is equal to the component of the stress vector $\mathbf{t_n}$ in the **m** direction.
(b) If $\mathbf{m} = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_2$, what do the results of (a) reduce to?

4.15 Let **m** be a unit vector that defines a plane M passing through a point P. Show that the stress vector on any plane that contains the stress traction $\mathbf{t_m}$ lies in the M plane.

4.16 Let $\mathbf{t_m}$ and $\mathbf{t_n}$ be stress vectors on planes defined by the unit vector **m** and **n**, respectively, and pass through the point P. Show that if **k** is a unit vector that determines a plane that contains $\mathbf{t_m}$ and $\mathbf{t_n}$, then $\mathbf{t_k}$ is perpendicular to **m** and **n**.

4.17 Given the function $f(x, y) = 4 - x^2 - y^2$, find the maximum value of f subjected to the constraint that $x + y = 2$.

4.18 True or false:
(i) Symmetry of stress tensor is not valid if the body has an angular acceleration.
(ii) On the plane of maximum normal stress, the shearing stress is always zero.

4.19 True or false:
(i) On the plane of maximum shearing stress, the normal stress is always zero.
(ii) A plane with its normal in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$ has a stress vector $\mathbf{t} = 50\mathbf{e}_1 + 100\mathbf{e}_2 - 100\mathbf{e}_3$ MPa. It is a principal plane.

4.20 Why can the following two matrices not represent the same stress tensor?

$$\begin{bmatrix} 100 & 200 & 40 \\ 200 & 0 & 0 \\ 40 & 0 & -50 \end{bmatrix} MPa \quad \begin{bmatrix} 40 & 100 & 60 \\ 100 & 100 & 0 \\ 60 & 0 & 20 \end{bmatrix} MPa.$$

4.21 Given:

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa.$$

(a) Find the magnitude of shearing stress on the plane whose normal is in the direction of $\mathbf{e}_1 + \mathbf{e}_2$.
(b) Find the maximum and minimum normal stresses and the planes on which they act.
(c) Find the maximum shearing stress and the plane on which it acts.

4.22 Show that the equation for the normal stress on the plane of maximum shearing stress is

$$T_n = \frac{(T_n)_{\max} + (T_n)_{\min}}{2}$$

4.23 The stress components at a point are given by $T_{11} = 100\,MPa$, $T_{22} = 300\,MPa$, $T_{33} = 400\,MPa$, $T_{12} = T_{13} = T_{23} = 0$.
(a) Find the maximum shearing stress and the planes on which they act.
(b) Find the normal stress on these planes.
(c) Are there any plane(s) on which the normal stress is 500 MPa?

4.24 The principal values of a stress tensor $\mathbf{T}$ are $T_1 = 10\,MPa$, $T_2 = -10\,MPa$, and $T_3 = 30\,MPa$. If the matrix of the stress is given by

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{bmatrix} \times 10\,MPa,$$

find the values of T_{11} and T_{33}.

4.25 If the state of stress at a point is

$$[\mathbf{T}] = \begin{bmatrix} 300 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & 400 \end{bmatrix} kPa,$$

find (a) the magnitude of the shearing stress on the plane whose normal is in the direction of $(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$ and (b) the maximum shearing stress.

4.26 Given:

$$[\mathbf{T}] = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} MPa.$$

(a) Find the stress vector on the plane whose normal is in the direction of $\mathbf{e}_1 + \mathbf{e}_2$.
(b) Find the normal stress on the same plane.
(c) Find the magnitude of the shearing stress on the same plane.
(d) Find the maximum shearing stress and the planes on which this maximum shearing stress acts.

4.27 The stress state in which the only nonvanishing stress components are a single pair of shearing stresses is called simple shear. Take $T_{12} = T_{21} = \tau$ and all other $T_{ij} = 0$.
(a) Find the principal values and principal directions of this stress state.
(b) Find the maximum shearing stress and the planes on which it acts.

4.28 The stress state in which only the three normal stress components do not vanish is called a *triaxial state of stress*. Take $T_{11} = \sigma_1$, $T_{22} = \sigma_2$, $T_{33} = \sigma_3$ with $\sigma_1 > \sigma_2 > \sigma_3$ and all other $T_{ij} = 0$. Find the maximum shearing stress and the plane on which it acts.

4.29 Show that the symmetry of the stress tensor is not valid if there are body moments per unit volume, as in the case of a polarized anisotropic dielectric solid.

4.30 Given the following stress distribution:

$$[\mathbf{T}] = \begin{bmatrix} x_1 + x_2 & T_{12}(x_1, x_2) & 0 \\ T_{12}(x_1, x_2) & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix},$$

find T_{12} so that the stress distribution is in equilibrium with zero body force and so that the stress vector on the plane $x_1 = 1$ is given by $\mathbf{t} = (1 + x_2)\mathbf{e}_1 + (5 - x_2)\mathbf{e}_2$.

4.31 Consider the following stress tensor:

$$[\mathbf{T}] = \alpha \begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T_{33} \end{bmatrix}.$$

Find an expression for T_{33} such that the stress tensor satisfies the equations of equilibrium in the presence of the body force $\mathbf{B} = -g\mathbf{e}_3$, where g is a constant.

4.32 In the absence of body forces, the equilibrium stress distribution for a certain body is

$$T_{11} = Ax_2, \quad T_{12} = T_{21} = x_1, \quad T_{22} = Bx_1 + Cx_2, \quad T_{33} = (T_{11} + T_{22})/2, \quad \text{all other} \quad T_{ij} = 0.$$

Also, the boundary plane $x_1 - x_2 = 0$ for the body is free of stress. (a) Find the value of C and (b) determine the value of A and B.

4.33 In the absence of body forces, do the following stress components satisfy the equations of equilibrium?

$$T_{11} = \alpha\left[x_2^2 + \nu\left(x_1^2 - x_2^2\right)\right], \quad T_{22} = \alpha\left[x_1^2 + \nu\left(x_2^2 - x_1^2\right)\right], \quad T_{33} = \alpha\nu\left(x_1^2 + x_2^2\right),$$
$$T_{12} = T_{21} = -2\alpha\nu x_1 x_2, \quad T_{13} = T_{31} = 0, \quad T_{23} = T_{32} = 0.$$

4.34 Repeat the previous problem for the stress distribution:

$$[\mathbf{T}] = \alpha \begin{bmatrix} x_1 + x_2 & 2x_1 - x_2 & 0 \\ 2x_1 - x_2 & x_1 - 3x_2 & 0 \\ 0 & 0 & x_1 \end{bmatrix}.$$

4.35 Suppose that the stress distribution has the form (called a *plane stress state*)

$$[\mathbf{T}] = \begin{bmatrix} T_{11}(x_1, x_2) & T_{12}(x_1, x_2) & 0 \\ T_{12}(x_1, x_2) & T_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) If the state of stress is in equilibrium, can the body forces be dependent on x_3?
(b) Demonstrate that if we introduce a function $\varphi(x_1, x_2)$ such that $T_{11} = \partial^2\varphi/\partial x_2^2$, $T_{22} = \partial^2\varphi/\partial x_1^2$ and $T_{12} = -\partial^2\varphi/\partial x_1 \partial x_2$, then the equations of equilibrium are satisfied in the absence of body forces for any $\varphi(x_1, x_2)$ that is continuous up to the third derivatives.

4.36 In cylindrical coordinates (r, θ, z), consider a differential volume of material bounded by the three pairs of faces: $r = r$ and $r = r + dr$; $\theta = \theta$ and $\theta = \theta + d\theta$; and $z = z$ and $z = z + dz$. Derive the r and θ equations of motion in cylindrical coordinates and compare the equations with those given in Section 4.8.

4.37 Verify that the following stress field satisfies the z-equation of equilibrium in the absence of body forces:

$$T_{rr} = A\left(\frac{z}{R^3} - \frac{3r^2z}{R^5}\right), \quad T_{\theta\theta} = \frac{Az}{R^3}, \quad T_{zz} = -A\left(\frac{z}{R^3} + \frac{3z^3}{R^5}\right), \quad T_{rz} = -A\left(\frac{r}{R^3} + \frac{3rz^2}{R^5}\right), \quad T_{r\theta} = T_{z\theta} = 0.$$

where $R^2 = r^2 + z^2$.

4.38 Given the following stress field in cylindrical coordinates:

$$T_{rr} = -\frac{3Pzr^2}{2\pi R^5}, \quad T_{zz} = -\frac{3Pz^3}{2\pi R^5}, \quad T_{rz} = -\frac{3Pz^2r}{2\pi R^5}, \quad T_{\theta\theta} = T_{r\theta} = T_{z\theta} = 0, \quad R^2 = r^2 + z^2.$$

Verify that the state of stress satisfies the equations of equilibrium in the absence of body forces.

4.39 For the stress field given in Example 4.9.1, determine the constants A and B if the inner cylindrical wall is subjected to a uniform pressure p_i and the outer cylindrical wall is subjected to a uniform pressure p_o.

4.40 Verify that Eqs. (4.8.4) to (4.8.6) are satisfied by the equilibrium stress field given in Example 4.9.2 in the absence of body forces.

4.41 In Example 4.9.2, if the spherical shell is subjected to an inner pressure p_i and an outer pressure p_o, determine the constant A and B.

4.42 The equilibrium configuration of a body is described by

$$x_1 = 16X_1, \quad x_2 = -\frac{1}{4}X_2, \quad x_3 = -\frac{1}{4}X_3$$

and the Cauchy stress tensor is given by $T_{11} = 1000\,MPa.$, and all other $T_{ij} = 0$.

(a) Calculate the first Piola-Kirchhoff stress tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane.

(b) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo-stress vector for the same plane.

4.43 Can the following equations represent a physically acceptable deformation of a body? Give reason(s).

$$x_1 = -\frac{1}{2}X_1, \quad x_2 = \frac{1}{2}X_3, \quad x_3 = -4X_2.$$

4.44 The deformation of a body is described by

$$x_1 = 4X_1, \quad x_2 = -(1/4)X_2, \quad x_3 = -(1/4)X_3.$$

(a) For a unit cube with sides along the coordinate axes, what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$-face of the cube?

(b) If the Cauchy stress tensor is given by $T_{11} = 100\,MPa$, and all other $T_{ij} = 0$, calculate the first Piola-Kirchhoff stress tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane.

(c) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane. Also calculate the pseudo-differential force for the same plane.

4.45 The deformation of a body is described by

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) For a unit cube with sides along the coordinate axes, what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$ face of the cube?

(b) If the Cauchy stress tensor is given by $T_{12} = T_{21} = 100\,MPa$, and all other $T_{ij} = 0$, calculate the first Piola-Kirchhoff stress tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane and compare it with the Cauchy stress vector in the deformed state.

(c) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane. Also calculate the pseudo-differential force for the same plane.

4.46 The deformation of a body is described by

$$x_1 = 2X_1, \quad x_2 = 2X_2, \quad x_3 = 2X_3.$$

(a) For a unit cube with sides along the coordinate axes, what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$ face of the cube?

(b) If the Cauchy stress tensor is given by

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} Mpa,$$

calculate the first Piola-Kirchhoff stress tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane and compare it with the Cauchy stress vector on its deformed plane.

(c) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo-stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$-plane. Also calculate the pseudo-differential force for the same plane.