SCHEDULE MODULE 3:

WK 1:	WK 2:	WK 3:	WK 4:
LECTURE	LAB WORK;	LAB WORK;	FINALIZE ANALYSIS & PREPARE REPORTS.
NEW GROUPS	LECTURE	ANALYSIS;	
EXERCISES (WK 1)		HYPOTHESES	
GRADE PEER REPORTS			

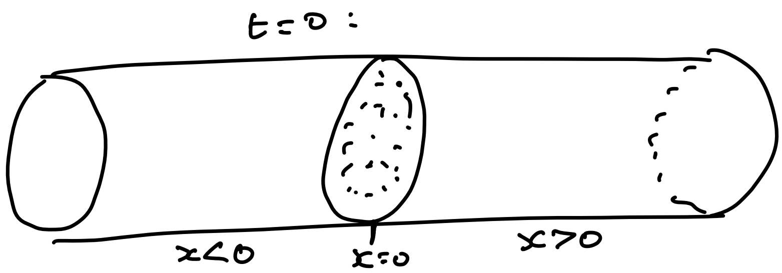
BROWNIAN MOTION:

THERMAL "KICKS" DUE TO MOLECULAR MOTION.

DEVELOP STOKES-EINSTEIN IN 2 STEPS:

1. EINSTEIN'S THEORY OF BROWNIAN MOTION
2. STOKES DRAG ON A SPHERE (LOW-REYNOLDS)

EINSTEIN'S THEORY:



N· BROWNIAN PARTICLES AT $x=0$, $t=0$

AT SOME LATER TIME THESE PARTICLES WILL MOVE ALONG x.

BECAUSE THIS MOTION IS RANDOM WE CAN END UP WITH MORE OR FEWER PARTICLES ON

EITHER SIDE OF THE PLANE AT $x=0$.

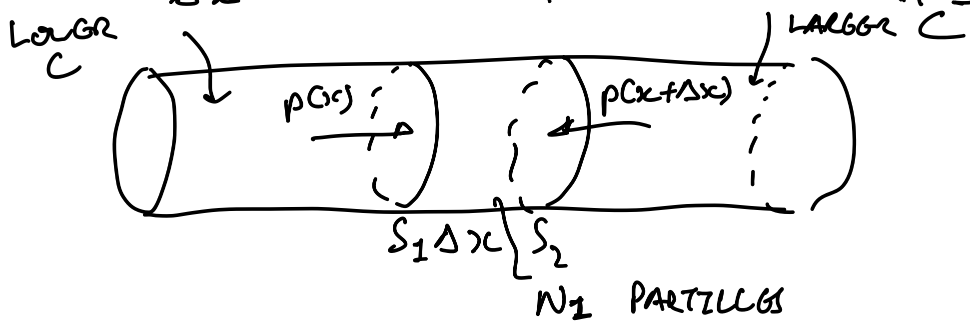
ASSUME MORE PARTICLES ON $x>0$ HALF.

THIS GENERATES AN OSMOTIC PRESSURE, $p(x)$,

DUE TO THE LARGER CONCENTRATION OF PARTICLES ON $x>0$ HALF OF CYLINDER

THIS PRESSURE IS APPLIED ON AN IMAGINARY SURFACE AT LOCATION x, SUCH THAT FOR S_1 AT x, S_2 at $x+\Delta x$, WE HAVE:

$$\frac{p(x) - p(x+\Delta x)}{\Delta x} = \frac{F(x) - F(x+\Delta x)}{A \cdot \Delta x} = \frac{\overset{\#\text{ OF PARTICLES}}{N_1} \; \overset{\text{FORCE/PARTICLE}}{F_p}}{A \cdot \Delta x}$$



TAKE THE LIMIT AS $\Delta x \to 0$:

$$\frac{\partial p}{\partial x} = -C_1 \cdot F_p$$

$$C_1 = \frac{N_1}{A \Delta x} \quad \text{(DEFINITION OF } C_1\text{)}$$

FOR EACH PARTICLE MOVING AT VELOCITY v, IT EXPERIENCES A RESISTANCE OR DRAG FORCE OPPOSITE TO ITS MOTION:

$$F_f = m\beta v$$

IF WE ASSUME THE DYNAMICS ARE OVER-DAMPED – NO INERTIA – THEN WE HAVE A BALANCE OF THESE FORCES:

$$F_p = F_f = m\beta v$$

THE N_1 BROWNIAN PARTICLES IN THE DIFFERENTIAL VOLUME ΔV, SATISFY THE EQUATION OF STATE (IDEAL GAS LAW):

\# MOLES OF BROWNIAN PARTICLES

$$N_1 = n\, N_0$$

AVAGADRO'S CONST – $\frac{\text{MOLECULES}}{\text{MOLE}}$

IDEAL GAS: $pV = nN_0 kT$

IN THE DIFFERENTIAL VOLUME:

$$p\,\Delta V = N_1 kT \rightsquigarrow p = \underbrace{C}_{\frac{N_1}{\Delta V}} kT$$

DIFFERENTIATING W.R.T. x:

$$\frac{\partial p}{\partial x} = kT\frac{\partial C}{\partial x} = -C F_p = -C m\beta v$$

FICK'S LAW OF DIFFUSION: \# OF PARTICLES PASSING AREA ΔA IN TIME Δt IS:

$$\Delta N = -D\cdot\frac{\partial C}{\partial x}\cdot\Delta A\cdot\Delta t \Leftrightarrow \frac{N_1}{A\,\Delta t} = -D\frac{\partial C}{\partial x}$$

$$\Rightarrow C v = \frac{N_1}{A\,\Delta x}\cdot\frac{\Delta x}{\Delta t} = \frac{N_1}{A\,\Delta t} = -D\frac{\partial C}{\partial x}$$

FINALLY, WE OBTAIN:

$$kT\frac{\partial C}{\partial x} = -m\beta C v = m\beta D\frac{\partial C}{\partial x}$$

$$\Rightarrow \boxed{D = \frac{kT}{m\beta}}$$

THIS IS EINSTEIN'S RELATION, CONNECTING KINETICS OF PARTICLES TO DIFFUSION CONSTANT.

– WE DON'T YET KNOW β, THE DRAG COEFF.

STEP 2: STOKES DRAG ON A SPHERE

WE HAVE A SMALL SPHERE IN A NEWTONIAN FLUID.

NAVIER-STOKES EQN:

$$\left.\begin{array}{l} \underline{\nabla}\left(\dfrac{p-p_0}{\mu}\right) = \nabla^2 \underline{u} = -\underline{\nabla}\wedge\underline{\omega} \\ \text{INCOMPRESS:} \quad \underline{\nabla}\cdot\underline{u} = 0 \end{array}\right\} \Rightarrow \begin{array}{c} \nabla^2 p = 0 \\ \& \\ \nabla^2 \underline{\omega} = 0 \end{array}$$

IF THE FLUID IS AT REST, AND THE PARTICLE IS MOVING AT VELOCITY $\underline{U}$,

THE BOUNDARY CONDITIONS ARE:

NO-SLIP ON SPHERE: $\underline{u} = \underline{U}$ at $r = a$ (RADIUS $= a$)

$\underline{u} \to 0$, $p \to p_0$ as $|\underline{x}| \to \infty$

BOTH EQN & B.Cs ARE LINEAR & HOMOGENEOUS IN $\underline{u}$, $\dfrac{p-p_0}{\mu}$ & $\underline{U}$

$\Rightarrow$ $\underline{u}$ & $\dfrac{p-p_0}{\mu}$ ARE ALSO LINEAR & HOMOGENEOUS IN $\underline{U}$.

MOVING INTO SPHERES FRAME OF REFERENCE, WE RECOGNIZE THAT SOLUTIONS FOR $\underline{u}$, $\dfrac{p-p_0}{\mu}$ MUST BE SYMMETRICAL ABOUT THE AXIS $\parallel \underline{U}$.

$\underline{u}$ MUST BE IN A PLANE THROUGH THAT AXIS.

THE DIFFERENTIAL OPERATORS ARE INDEPENDENT OF OUR COORDINATE SYSTEM; THIS IMPLIES THAT OUR SOLUTION DEPENDS ONLY ON $\underline{x}$, AND NOT ANY OTHER COMBINATION OF ITS COMPONENTS; ALSO $\underline{U}$, and a

$$\Rightarrow \frac{p-p_0}{\mu} = \underline{U}\cdot\underline{x}\, f; \quad a^2 f\left(\underbrace{\frac{\underline{x}\cdot\underline{x}}{a^2} = \frac{r^2}{a^2}}\right)$$ ONLY ARGUMENT OF f.

$p - p_0$ SATISFIES LAPLACE'S EQUATION & $= 0$ AS $r \to \infty$.

$\Rightarrow$ $p - p_0$ SHOULD BE A SERIES OF SPHERICAL HARMONICS THAT SCALE AS $1/r$.

THE ONLY FUNCTION THAT SATISFIES THIS CONDITION: THE 2nd DEGREE / DIPOLE TERM:

$$\frac{p - p_0}{\mu} = C \, \frac{\underline{U} \cdot \underline{x}}{r^3}$$

THE SAME REASONING APPLIES FOR VORTICITY, $\underline{\omega}$:

$$\underline{\omega} = C \cdot \frac{\underline{U} \times \underline{x}}{r^3}$$

THE AZIMUTHAL COMPONENT OF $\underline{\omega}$ IS:

$$\frac{1}{r} \cdot \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta}$$

WE USE STREAM FUNCTION ψ IN SPHERICAL POLAR COORDINATES TO EXPRESS THE VELOCITY COMPONENTS AS:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$

REWRITE AZIMUTHAL COMPONENT OF $\underline{\omega}$:

$$(*) \quad \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) = -\frac{C U \sin^2\theta}{r}.$$

BY INSPECTION, THE PARTICULAR SOLUTION FOR ψ IS PROPORTIONAL TO $\sin^2\theta$.

$\Rightarrow$ INNER B.C ON ψ – ON SPHERE'S SURFACE – MUST ALSO BE PROPORTIONAL TO $\sin^2\theta$.

$$\psi = U \sin^2\theta \; f(r)$$

THE VELOCITY VECTOR CORRESPONDING TO Ψ:

$$\underset{\sim}{u} = \underset{\sim}{U}\left(\frac{1}{r}\frac{df}{dr}\right) + \underset{\sim}{x}\ \frac{\underset{\sim}{x}\cdot\underset{\sim}{U}}{r^2}\left(\frac{2f}{r^2} - \frac{1}{r}\frac{df}{dr}\right)$$

CANCELING $U \sin^2\theta$ FROM $*$, WE FIND:

$$\frac{d^2 f}{dr^2} - \frac{2f}{r^2} = -\frac{C}{r}$$

$$\Rightarrow f(r) = \frac{Cr}{2} + \frac{L}{r} + M r^2$$

B.Cs $r \to \infty$: $\frac{f}{r^2} \to 0$;

NO SLIP: $u_r = U\cos\theta$ for $r=a \Leftrightarrow f(a) = \frac{a^2}{2}$

$$M = 0, \quad L = \frac{a^3}{2} - \frac{Ca^2}{2}$$

$$u_\theta\Big|_{r=a} = \frac{-1}{r\sin\theta}\frac{\partial\Psi}{\partial r} = -U\sin\theta\Big|_{r=a} \Rightarrow C = \frac{3a}{2}$$

TAKEN ALTOGETHER:

$$\boxed{\Psi = U r^2 \sin^2\theta\left(\frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right)}$$

DRAG FORCE: INTEGRATE STRESS TENSOR ON SPHERE'S SURFACE:

$$n_j\,\sigma_{ij}\Big|_{r=a} = n_j\left\{-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\}_{r=a}$$

AFTER SOME ALGEBRA:

$$n_j\,\sigma_{ij}\Big|_{r=a} = \Big\{-p n_i + \mu\, n_i\, \underset{\sim}{U}\cdot\underset{\sim}{n}\left(-\frac{f''}{r} + \frac{6f'}{r^2} - \frac{10f}{r^3}\right) \ldots$$

$$+\mu U_i\left(\frac{f''}{r} - \frac{2f'}{r^2} + \frac{2f}{r^3}\right)\Big\}_{r=a}$$

NOTE: $f' = \frac{df}{dr}$.

SUBSTITUTING p, F FROM OUR SOLUTION:

$$n_j \sigma_{ij}\Big|_{r=a} = n_i\left\{-p_0 + \frac{3\mu \underline{U}\cdot\underline{n}}{a}\underbrace{\left(\frac{2C}{a} - 3\right)}_{0}\right\} + \frac{3\mu U_i}{a}\underbrace{\left(1 - \frac{C}{a}\right)}_{-\frac{1}{2}}$$

$$= -p_0 n_i - \frac{3\mu U_i}{2a}.$$

CALCULATE $F_d = \int \underline{\underline{\sigma}}\cdot\underline{n}\, dA = \frac{3\mu U}{2a}\cdot\underbrace{\int dA}_{4\pi a^2} = \underline{\underline{6\pi a \mu U}}$

STOKES DRAG. $F_d = F_f$

$\Rightarrow m\beta = 6\pi a \mu.$

$\Rightarrow \boxed{D = \frac{kT}{6\pi a \mu}}$