Automorphic representations

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Goals

To explain

- Langlands functoriality principle and applications
- spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$
- Eisenstein series
- L-functions
- Local theory
- Structure theory of algebraic groups

Plan

Second week: Global theory

- Lecture 4: Langlands functoriality principle
- Lecture 3: spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$

First week: Local theory

- Lecture 2: local representation theory
- Lecture 1: structure theory of algebraic groups

Structure theory of (affine) algebraic groups

Throughout the talk F is of characteristic zero.
Affine algebraic group: can be realized as a subgroup of GL_n (invertible matrices of size $n \times n$) given by the zero locus of (finitely many) polynomials (with coefficients in F) in the entries.
(Does not include: elliptic curves)
(Henceforth we will omit the word "affine")
It is sometimes convenient to distinguish between the group $\mathbf{G}$ as a variety and its F-points $G = \mathbf{G}(F)$.

Examples

- GL_n, $\dim = n^2$; in particular, the (one-dimensional) multiplicative group $\mathbb{G}_m = \mathrm{GL}_1$
- $\mathrm{SL}_n = \{g \in \mathrm{GL}_n : \det g = 1\}$, $\dim = n^2 - 1$
- additive group $\mathbb{G}_a = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}$
- mixture of multiplicative and additive: $\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0\}$, $\dim = 2$
- symplectic group: $\{g \in \mathrm{GL}_n : {}^t g J g = J\}$ where J is anti-symmetric invertible (and n even), $\dim = \binom{n+1}{2}$
- orthogonal group: same except that J is a symmetric invertible matrix, $\dim = \binom{n}{2}$
- spin groups – double covers of orthogonal groups
- exceptional groups: [1] G_2 (14,7), F_4 (52,26), E_6 (78, 27), E_7 (133, 56), E_8 (248, 248) (the only simple, simply-connected, simply-laced adjoint group!)

[1] in parenthesis: the dimension of the group and the smallest dimension of a representation

Important difference between algebraic and Lie groups: the groups $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^*$ and $\mathbb{G}_a(\mathbb{R}) = \mathbb{R}$ are (almost) isomorphic as Lie groups but $\mathbb{G}_m$ and $\mathbb{G}_a$ are not isomorphic as algebraic groups. Their algebraic representations are quite different. Every algebraic representation of $\mathbb{G}_m$ is diagonalizable, while evidently this is not the case for $\mathbb{G}_a$.

Differences if F not algebraically closed

The group $\mathrm{SO}(2,\mathbb{R}) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\}$ is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$. It is a form of $\mathbb{G}_m$, i.e., the $\mathbb{C}$-points are isomorphic.

A form of $\mathbb{G}_m^r$ is called a torus of rank r. A group isomorphic to $\mathbb{G}_m^r$ (over F) is a split torus. A rational representation (over F) of a split torus is diagonalizable over F.

In the list above:

- the multiplicative group of a division algebra is a form of GL_n
- unitary groups are also forms of GL_n
- different quadratic forms of a given rank give rise to forms of orthogonal groups (but there are others, even for symplectic groups)
- Over $\mathbb{R}$ every simple group has a unique compact form

Classification

The basic idea is to linearize the problem, i.e. to break the group into one-dimensional subgroups with simple commutation relations.
The closest to vector spaces: unipotent (can be realized as upper unitriangular matrices).
They are hard to classify but their structure is fairly easy (nilpotent groups).
Opposite extreme: Reductive groups: groups with no normal unipotent subgroups (i.e., unipotent radical is trivial). Equivalently: reductive group are completely reducible: every (algebraic) representation decomposes as a direct sum of irreducible ones.
Another equivalent definition: G can be embedded in GL_n in such a way that it is stable under transpose.
Levi decomposition: every group is a semidirect product $G = M \ltimes U$ of a reductive group M and the unipotent radical U of G (the maximal normal unipotent subgroup of G); M is uniquely determined up to conjugation.

Reductive groups are essentially (up to taking quotients by a finite central group) the direct product of a torus with almost simple groups (finite center). However, we need to consider reductive groups even if we are only interested in simple groups.

For simplicity we consider only split reductive groups, such as GL_n, SL_n, Sp_{2n}, $\mathrm{SO}(n, n+1)$, $\mathrm{SO}(n, n)$

They have a maximal torus (i.e., $C_G(T) = T$) which is split over F (along with many others which are not split if F is not algebraically closed). Such a torus is unique up to conjugation in $G(F)$.

The rank of G is by definition the rank of T.

The Weyl group $W = N_G(T)/C_G(T) = N_G(T)/T$, a finite group.

Examples of split maximal tori: the diagonal subgroup in GL_n (rank n) or of SL_n (rank $n-1$), $W = S_n$

Symplectic group Sp_{2n} with respect to

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

$$T = \{ \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} : t_1, \ldots, t_n \in F^* \}$$

(rank n) $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$

same for $SO(n,n)$ with respect to $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ (W is of index two in $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$). For $SO(n, n+1)$

$$T = \left\{ \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} : t_1, \ldots, t_n \in F^* \right\}$$

(again, rank n), $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$

Roots

G acts on $\mathfrak{g} = \mathrm{Lie}(G)$ by the adjoint representation Ad. (For GL_n this is conjugation on $\mathrm{Mat}_n = \mathfrak{gl}_n = \mathrm{Lie}(\mathrm{GL}_n)$; if $G \subset \mathrm{GL}_n$ this is just conjugation on $\mathfrak{g} \subset \mathfrak{gl}_n$.)
The action of T on $\mathfrak{g}$ is diagonalizable.

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi \subset X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ is the (finite) set of roots and

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = \alpha(t)X \quad \forall t \in T\}$$

are the T-eigenspaces which happen to be one-dimensional.

Example: GL_n: the roots are the characters t_i/t_j, $i \neq j$; root spaces are $e_{i,j}$; $|\Phi| = n(n-1)$ same for SL_n.
roots of Sp_{2n}: t_i/t_j, $(t_i t_j)^{\pm 1}$, $i \neq j$, $t_i^{\pm 2}$; $|\Phi| = 2n^2$
roots of $\mathrm{SO}(n, n+1)$: t_i/t_j, $(t_i t_j)^{\pm 1}$, $i \neq j$, $t_i^{\pm 1}$; $|\Phi| = 2n^2$
roots of $\mathrm{SO}(n, n)$: t_i/t_j, $(t_i t_j)^{\pm 1}$, $i \neq j$; $|\Phi| = 2n(n-1)$
We usually view the roots additively in the real vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. They form a root system, which I won't define here.
To a large extent G can be recovered from its root system.

Parabolic subgroups

They measure the various "infinities" of G

They are the "largest" subgroups of G in the sense that $P\backslash G$ is projective (equivalently $P(\mathbb{C})\backslash G(\mathbb{C})$ is compact).

One way to define them (for local fields): they are the groups of the form $P_x = \{g \in G(F) : \{x^n g x^{-n}, n = 1, 2, \dots\} \text{ is bounded}\}$ where $x \in G(F)$.

Fixing a minimal parabolic subgroup P_0 (= a maximal solvable subgroup = a Borel subgroup), there are precisely 2^r (parabolic) subgroups containing P_0. They are called standard. Any parabolic subgroup is conjugate to a unique standard one.

By definition, two parabolic subgroups are associate if their Levi parts are conjugate.

Example: GL(V)

The rank is $\dim V$ (semisimple rank $\dim V - 1$), parabolic subgroups are the stabilizers of flags $0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = V$ of V. The standard parabolic subgroups correspond to compositions $n = n_1 + \cdots + n_k$, $n_1, \dots, n_k \geq 1$. $P_{n_1,\dots,n_k}$ consists of the block upper-triangular

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,k-1} & A_{1,k} \\ 0 & A_{2,2} & \dots & A_{2,k-1} & A_{2,k} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{k-1,k-1} & A_{k-1,k} \\ 0 & 0 & 0 & 0 & A_{k,k} \end{pmatrix}$$

where $A_{i,j} \in \mathrm{Mat}_{n_i \times n_j}$, $A_{i,i} \in \mathrm{GL}_{n_i}$. The standard Levi decomposition is $P_{n_1,\dots,n_k} = M_{n_1,\dots,n_k} U_{n_1,\dots,n_k}$ where $M_{n_1,\dots,n_k} \simeq \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ consists of the block diagonal matrices $\mathrm{diag}(g_1, \dots, g_k)$, $g_i \in \mathrm{GL}_{n_i}$, $i = 1, \dots, k$ and $U_{n_1,\dots,n_k}$ consists of the block upper unitriangular matrices (where $A_{i,i} = \mathrm{Id}_{n_i}$ in the notation above).

Two standard parabolic subgroups $P_{n_1,\ldots,n_k}$ and $P_{m_1,\ldots,m_l}$ are associate if and only if the underlying partitions are the same, i.e., if $m_1, \ldots, m_l$ is a permutation of $n_1, \ldots, n_k$ (in particular $k = l$). Thus, associate classes of parabolic subgroups correspond to partitions $n = n_1 + \cdots + n_k$, $n_1 \geq \cdots \geq n_k$ of n.
Let h be a symplectic, symmetric or hermitian form on V. Consider the isometry group G of (V, h). The parabolic subgroups of G are the stabilizers of isotropic flags (i.e., h vanishes on $V_i \times V_i$).

Bruhat decomposition

Gauss elimination (for invertible matrices): Any invertible matrix can be written as $n_1 w a n_2$ where w is a permutation matrix, a is diagonal (uniquely determined), n_1, n_2 are upper unitriangular, (if w is $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ then n_1, n_2 are unique).

For general groups: let $P_0 = T_0 \ltimes N_0$ be a Borel subgroup. Bruhat decomposition

$$G = \coprod_{w \in W} P_0 w N_0.$$

There is a unique open double coset (the big cell) $\mathcal{B}_{w_0} = P_0 w_0 N_0$ and the map $P_0 \times N_0 \to \mathcal{B}_{w_0}$, $(p, n) \mapsto p w_0 n$ is an isomorphism of algebraic varieties. Thus G is birationally equivalent to an affine space. (Everything is defined over F.)

Iwasawa decomposition

Assume that $F = \mathbb{R}$.
Gram-Schmidt: every matrix can be written uniquely as a product of an upper triangular matrix with positive diagonal entries and an orthogonal matrix, i.e. $\mathrm{GL}_n(\mathbb{R}) = T(\mathbb{R})^0 N(\mathbb{R}) K$ where N – upper unitriangular matrices, $K = O(n)$ – maximal compact subgroup of $\mathrm{GL}_n(\mathbb{R})$.
More generally for any algebraic group G over $\mathbb{R}$, $G(\mathbb{R})$ contains a maximal compact subgroup K which is unique up to conjugation. It can be realized as $K = G(\mathbb{R}) \cap O(n)$ in some representation $G \subset \mathrm{GL}_n$. If $P_0 = T_0 \ltimes N_0$ is a Borel subgroup then we have

$$G(\mathbb{R}) = T_0(\mathbb{R})^0 N_0(\mathbb{R}) K$$

Moreover, $(t, n, k) \mapsto tnk$ defines a diffeomorphism $T(\mathbb{R})^0 \times N_0(\mathbb{R}) \times K \to G(\mathbb{R})$. (Iwasawa decomposition). In particular, $G(\mathbb{R})$ is homotopic to K since $T(\mathbb{R})^0$ and $N_0(\mathbb{R})$ are diffeomorphic to Euclidean spaces.

p-adic: maximal compact subgroups are not unique up to conjugation: They are unique for $\mathrm{GL}_n(\mathbb{Q}_p)$ but for $\mathrm{SL}_n(\mathbb{Q}_p)$ there are n of them, all conjugate in $\mathrm{GL}_n(\mathbb{Q}_p)$; for $\mathrm{PGL}_2(\mathbb{Q}_p)$ there are two and they are quite different.
There is a deep and elaborate theory of the maximal compact subgroups of G due to Bruhat-Tits.
Iwasawa decomposition: $G(F) = T_0(F)N_0(F)K$ (not uniquely) where K is a maximal compact. For GL_n – elementary
Cartan decomposition: F either $\mathbb{R}$ or p-adic $G(F) = KA_+K$ (uniquely) where

$$A_+ = \{t \in T_0(F) : |\alpha(t)| \leq 1 \text{ for all roots } \alpha \text{ of } T_0 \text{ in } \mathrm{Lie}(N_0)\}$$

e.g., for $\mathrm{GL}_n(F)$, F p-adic

$$\mathrm{GL}_n(F) = \mathrm{GL}_n(\mathcal{O})\{\begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix} k_1 \geq \cdots \geq k_n\}\,\mathrm{GL}_n(\mathcal{O}).$$

Thus, "from far away" $G(F)$ looks like a cone in a lattice!

Automorphic representations - lecture #2(4)

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In this talk we will review some elements of representation theory of reductive groups over local fields.
The founding fathers: Bernstein, Casselman, Harish-Chandra, Gelfand, Jacquet, Kazhdan, Langlands, Zelevinsky

From now on **G** will be a reductive group (often split) defined over a local field of characteristic 0 (i.e. $F = \mathbb{R}, \mathbb{C}$ or p-adic) and $G = \mathbf{G}(F)$ with the topology coming from embedding it in some GL_n.

Which representations (π, V) of G do we consider?

First, V is a vector space over $\mathbb{C}$ (not finite-dimensional in general).

Certainly the action map $G \times V \to V$ better be continuous; but what topology on V?

p-adic case: discrete topology on V i.e. the stabilizer of every $v \in V$ is an open subgroup of G. (We call these smooth representations.) This is reasonable since the topology of G has a basis of compact open subgroups.

There are no irreducible smooth finite-dimensional representations (except characters) unless G has anisotropic quotients

Let $\mathcal{S}(G)$ be the algebra of compactly supported locally constant functions on G (under convolution), i.e. $\mathcal{S}(G) = \bigcup_K C_c(K\backslash G/K)$ where K ranges over the compact open subgroups of G.
smooth representations of $G \longleftrightarrow$ non-degenerate modules of $\mathcal{S}(G)$ (i.e., $\mathcal{S}(G)M = M$)
Admissibility condition: for any open subgroup $K \subset G$, V^K is finite-dimensional.
It turns out: π is finitely generated and admissible iff π has finite length
We denote by $\mathcal{R}(G)$ the category of representations of G of finite length. $\Pi(g) \subset \mathcal{R}(G)$ the irreducible representations.
The contragredient $(\tilde{\pi}, \tilde{V})$ of $(\pi, V) \in \mathcal{R}(G)$ is the smooth part of V^*, i.e., $\tilde{V} = \{\ell \in V^* :$
there exists an open $K \subset G$ such that $\ell \circ \pi(g) = \ell$ for all $g \in K\}$
with $\tilde{\pi}(g)\ell = \ell \circ \pi(g)^{-1}$.

What about $F = \mathbb{R}$? (We treat groups over $\mathbb{C}$ as real groups by restriction of scalars, so we don't need to treat $F = \mathbb{C}$ separately.)
Representations of real groups are both simpler and more complicated than representations of p-adic groups!
More complicated: analysis
Simpler: arithmetic of F
(If we considered representations of p-adic groups over $\overline{\mathbb{Q}_p}$ then the analysis becomes very difficult.)

One option is to consider unitary representation on Hilbert spaces. There are two problems:

1. This class is not rich enough for some purposes.
2. The representation is too big. (For instance, not enough continuous functionals.)

To address the first problem we can consider Banach representations, i.e., (continuous) representations on Banach spaces.

Remark: by the uniform boundedness principle $\pi : G \times V \to V$ is continuous iff π is separately continuous in G and V.

Even more: embed G in SL_n and define $\|g\|$ as the maximum of the entries. Then there exist C and N such that $\|\pi(g)v\| \leq C\|g\|^N\|v\|$ for all $v \in V$.

To address the second problem (too few continuous functionals) we look at smooth vectors V^∞: the vectors v such that the map $g \mapsto \pi(g)v$ from G to V is smooth. We can get smooth vectors by taking $\pi(f)v = \int_G f(g)\pi(g)v\, dg$ for any v and $f \in C_c^\infty(G)$. In fact, any smooth vector is of this form (Dixmier-Malliavin).

On V^∞ acts not only G but also the Lie algebra $\mathfrak{g}$ (compatibly with G) and hence, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. (We denote the resulting representation by $d\pi$.) We topologize V^∞ with the Sobolev norms

$$p_k(v)^2 = \sum_{n \le k, X_1, \ldots, X_n \in \mathcal{B}} \|d\pi(X_1 \ldots X_n)v\|^2$$

where $\mathcal{B}$ is a basis of $\mathfrak{g}$. We get a smooth Fréchet representation of moderate growth, (SFMG) i.e., every vector is smooth and for any k there exist C, l and N such that $p_k(\pi(g)v) \le C\|g\|^N p_l(v)$ for all v.

Define: $\mathcal{S}(G) = \{f \in C^\infty(G) :$
$(X * f)(g)\|g\|^N$ is bounded for all $N \in \mathbb{N}, X \in \mathcal{U}(\mathfrak{g})\}$.
SFMG representations $\longleftrightarrow$ non-degenerate $\mathcal{S}(G)$-modules
Finally, we can impose the admissibility condition which is now that for any $\tau \in \hat{K}$, $\mathrm{Hom}_K(\tau, V)$ is finite-dimensional.
If we assume that $(\pi, \mathbb{H})$ is a representation on a Hilbert space and $(\mathcal{S}(G), \mathbb{H}^\infty)$ is finite length then it is admissible. However, there are many different $\mathbb{H}$'s with the same $\mathbb{H}^\infty$.
Denote by $\mathcal{R}(G)$ the category of finite length (with respect to $\mathcal{S}(G)$) admissible SFMG representations (any such can be realized as $\mathbb{H}^\infty$ for some Hilbert representation)
Casselman-Wallach Theorem: $V \mapsto K$-finite part of V defines an equivalence of categories between $\mathcal{R}(G)$ and $\mathcal{U}(\mathfrak{g})$-modules of finite length which are also K-modules in a compatible way.
contragredient: $(\tilde{\pi}, \tilde{V})$ is defined by $\tilde{V} =$ span of $\{\ell \circ \pi(f) : \ell \in V^*$ (continuous dual) $\}$ in V^*.

Infinitesimal character

The center of $\mathcal{U}(\mathfrak{g})$ is a polynomial algebra in r variables and its spectrum corresponds to $\text{Hom}(T_0(\mathbb{R})^0, \mathbb{C}^*)/W = X^*(T_0) \otimes \mathbb{C}/W$
The map $\Pi(G) \to X^*(T_0) \otimes \mathbb{C}/W$

$$\pi \mapsto \text{ character by which } \mathfrak{z}(\mathcal{U}(\mathfrak{g})) \text{ acts on } \pi$$

is finite-to-one.

Characters

How to define the character of a representation?
In the finite-dimensional case $\chi_\pi(g) = \operatorname{tr} \pi(g)$; but this won't work otherwise..
For any $f \in \mathcal{S}(G)$ $\pi(f)$ is of trace class; $f \mapsto \operatorname{tr} \pi(f)$ is a distribution on $\mathcal{S}(G)$
It is represented by a locally L^1-function χ_π which is G-invariant, i.e. $\operatorname{tr} \pi(f) = \int_G \chi_\pi(g) f(g)\ dg$ (!)

Quotient measures

Let G be a locally compact group with right Haar measure dg. Denote the modulus function by $\delta : G \to \mathbb{R}_+$.
If $H \subset G$ is a closed subgroup we can define a right-invariant functional on continuous functions on G compactly supported modulo H satisfying $f(hg) = \frac{\delta_H}{\delta_G}(h) f(g)$ for all $h \in H$, $g \in G$. We denote it by

$$\int_{H\backslash G} f(g) \, \frac{dg}{dh}.$$

It is characterized by

$$\int_{H\backslash G} \left(\int_H \varphi(hg) \frac{\delta_G}{\delta_H}(h) \, dh \right) \, \frac{dg}{dh} = \int_G \varphi(g) \, dg$$

for every $\varphi \in C_c(G)$.

Induced representations

For any subgroup H of G and a representation (σ, V_σ) of H we can construct a representation $\mathrm{Ind}_H^G \sigma$ of G on the space

$$\{\varphi : G \to V_\sigma \text{ smooth} |\, \varphi(hg) = (\frac{\delta_H}{\delta_G})^{\frac{1}{2}}(h)\sigma(h)[\varphi(g)]\, \forall h \in H, g \in G\}$$

with G acting by right translation.
If σ is unitary then we can consider

$$L^2\text{-}\,\mathrm{Ind}_H^G \sigma = \{\varphi \text{ as above} \mid \|\varphi\|^2 := \int_{H\backslash G} \|\varphi(g)\|^2 \, \frac{dg}{dh} < \infty\}.$$

It is a unitary representation of G.
If $P = M \ltimes U$ is a parabolic subgroup and σ is a representation of M then we can pullback $\sigma\delta_P^{\frac{1}{2}}$ to P and induce it to G. We write this as $I_P(\sigma)$ (parabolic induction).
We get a functor $I_P : \mathcal{R}(M) \to \mathcal{R}(G)$. (!)

Matrix coefficients

We would like to realize representations on spaces of functions. The simplest possibility: let $\ell \in \tilde{\pi}$ and consider $v \mapsto \ell(\pi(\cdot)v) \in C(G)$ (matrix coefficients). This is an intertwining operator from π to $C(G)$, with G acting by right translation. If $\pi \in \Pi(G)$, this procedure will not be too sensitive on ℓ. (The story is different if ℓ is only assumed continuous.)

Hierarchy of representations

Definitions An irreducible representation is called

1. *supercuspidal* if it has a non-zero matrix coefficient which is compactly supported in G (and then *all* its matrix coefficients are compactly supported
2. *square-integrable*: unitary central character and matrix coefficients are square-integrable modulo the center
3. *tempered* if its matrix coefficients times $(1+\log\|g\|)^{-N}$ are in $L^2(G)$ for some N.

supercuspidal $\implies$ square-integrable $\implies$ tempered

supercuspidal can only exist if $Z(G)$ is compact
If G itself is compact then every irreducible representation is supercuspidal
In the Archimedean case a supercuspidal representation exists only if G is compact
In the p-adic case any G with compact Z admits many supercuspidal representations
Basic example: let ρ be a cuspidal representation of $\mathbf{G}(\mathbb{F}_q)$ and pull it back to a representation of $K = \mathbf{G}(\mathcal{O})$ via the canonical projection $\mathcal{O} \to \mathbb{F}_q$. Then $\operatorname{Ind}_K^G \rho$ has finite length and its constituents are supercuspidal.
essentially supercuspidal: matrix coefficients compactly supported modulo the center
essentially square-integrable/tempered: tempered after twist by an unramified character (i.e. $\prod_{i=1}^k |\chi_i|^{s_i}$ where $\chi_1, \ldots, \chi_k$ is a basis for $X^*(G)$).

Every $\pi \in \Pi(G)$ is a submodule of $I_P(\sigma)$ where $\sigma \in \Pi_{ess.cusp.}(M)$. (In the real case M is the maximal torus.) Moreover in the p-adic case, if $I_{P'}(\sigma')$ contains π as a subquotient and $\sigma' \in \Pi_{ess.cusp.}(M')$ then (P, σ) and (P', σ') are associate.
The data $(M, \sigma)/\sim$ is the p-adic analogue of the infinitesimal character in the Archimedean case.
Harish-Chandra classified the square-integrable representations for real groups parameterizing them in a way very similar to the Cartan-Weyl theory for compact Lie groups.
For GL_n there exists an elegant description of square-integrable representations in terms of supercuspidal representations. (Bernstein-Zelevinsky). However, for other groups the class of square-integrable representations is more stable than the class of supercuspidal representations.
The tempered representations are the irreducible constituents of $\mathrm{Ind}_P^G \sigma$ where σ is square-integrable of M.

Langlands classification

Let $P = MU$ be a parabolic subgroup and τ an essentially tempered representation of M such that $|\omega_\tau|$ is positive with respect to P. Then $\mathrm{Ind}_P^G \tau$ admits a unique irreducible quotient called Langlands quotient or equivalently a unique maximal proper submodule (which consists of the vectors whose matrix coefficients are of smaller growth).

We write the Langlands data as (P, σ, λ) where $\sigma \in \Pi_{temp}(M)$ and $\tau = \sigma[\lambda]$.

Every $\pi \in \Pi(G)$ can be obtained as a Langlands quotient; the data (P, σ, λ) is unique up to conjugation.

λ measures the (non-)temperedness of π, i.e. the growth of matrix coefficients.

The data for $\tilde{\pi}$ is $(\bar{P}, \tilde{\sigma}, -\lambda)$ where $\bar{P}$ is the opposite parabolic to P (i.e., same Levi part M and $P \cap \bar{P} = M$)

In particular, if $\pi = LQ(P, \sigma, \lambda)$ is unitary then necessarily $\tilde{\pi} \simeq \bar{\pi}$ (where $\bar{\pi}$ is the representation $\bar{\pi}(g) = \pi(g)$ on $\overline{V_\pi}$ – the conjugate vector space (same additive group as V_π but with conjugate scalar multiplication)). Thus, $(P, \sigma, \lambda) \sim (\bar{P}, \sigma, -\bar{\lambda})$.

Also, the matrix coefficients are (at the very least) bounded – this gives a bound on λ.

spherical (unramified) representations: having fixed vector under $K = G(\mathcal{O})$

For any unramified character χ of T_0 the representation $I_{P_0}(\chi)$ admits a unique irreducible spherical representation $J(\chi)$

Every spherical representation is of this type. Moreover, $J(\chi) = J(\chi')$ if and only if $\chi' = \chi^w$ for some $w \in W$. Thus,

$$\text{spherical representations} \longleftrightarrow X^*(T_0) \otimes (\mathbb{C}/2\pi\mathrm{i}\log q)/W$$

Example: GL_n the possible (P, σ) are $(n_1, \ldots, n_k, m, n_k, \ldots, n_1)$ (possibly $m = 0$) and $(\sigma_1, \sigma_2, \ldots, \sigma_k, \sigma, \sigma_k, \ldots, \sigma_1)$, $(\lambda_1, \ldots, \lambda_k, 0, -\lambda_k, \ldots, -\lambda_1)$. $\lambda_1 > \cdots > \lambda_k > 0$.
There is a precise description of the unitary dual of GL_n due to Vogan in the Archimedean case and Tadić in the p-adic case The non-degenerate part is $0 < \lambda_1, \ldots, \lambda_k < \frac{1}{2}$.

Spectral decomposition of $L^2(G(F))$

Recall the Hilbert-Schmidt class $\mathcal{E}(\mathbb{H})$ of bounded linear operators on $\mathbb{H}$ with

$$\|A\|_{HS}^2 = \sum_{e_i \text{ orthonormal basis}} \|Ae_i\|^2 = \operatorname{tr} A^*A = \operatorname{tr} AA^* < \infty$$

(does not depend on the basis). We have

$$\mathcal{E}(\mathbb{H}) \equiv \mathbb{H}\hat{\otimes}\mathbb{H}^\vee \text{ as Hilbert spaces}$$

Peter-Weyl For any compact group G we have an equivalence of unitary representations of $G \times G$

$$L^2(G) \simeq \hat{\oplus}_{\pi\in\Pi(G)}\pi \otimes \tilde{\pi} \simeq \oplus_{\pi\in\Pi}\mathcal{E}(\mathcal{H}_\pi)$$

where on the left $M(g_1, g_2)f(x) = f(g_1^{-1}xg_2)$, given by $f \mapsto (\sqrt{d_\pi}\pi(f))_{\pi\in\Pi(G)}$ where $d_\pi = \dim \pi$.
Equivalently, for dense subspace of continuous functions (containing the smooth functions for compact Lie groups) we have

$$f(1) = \sum_{\pi\in\Pi(G)} d_\pi \operatorname{tr} \pi(f).$$

Plancherel formula in general

$$L^2(G) \simeq \hat{\oplus}_{([P])} \int_{\Pi_2(M)} \mathcal{E}(\mathcal{H}_\pi)\mu_{pl}(\pi)$$

(sum over associate classes of parabolic subgroups) In particular the support of the Plancherel measure is the tempered spectrum. For G semisimple $\mu_{pl}\big|_{\Pi_2(G)} = \sum_{\pi\in\Pi_2(G)} d_\pi \delta_\pi$ where d_π is the formal dimension of π.

Equivalent statement

$$f(1) = \sum_{[P]} \int_{\Pi_2(M)} \operatorname{tr} \pi(f)\mu_{pl}(\pi)$$

for $f \in \mathcal{S}(G)$.

Automorphic representations

Erez Lapid

Lausanne, 2011

L-functions

Roughly speaking, L-functions provide means to glue local data (of a global object) into a global analytic object. They are Euler products of the form

$$L^S(s, O) = \prod_{p \notin S} L(s, O_p)$$

where $L(s, O_p) = P_p(p^{-s})^{-1}$, P_p is a monic polynomial (depending on p) of degree n (the degree of the L-function – independent of p). Often, but not always, P_p has integer coefficients.

As a Dirichlet series

$$L(s, O) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where a_n is multiplicative and for all p the sequence a_{p^k}, $k \in \mathbb{N}$ satisfies a linear recurrence relation of order n.

Analytic information about $L(s, O)$ yields (at least potentially) valuable arithmetic information.

Functional equation

$$L^S(s, O) = \gamma_S(s, O) L^S(1 - s, \tilde{O})$$

where γ_S is again a local object (handling bad primes, including ∞).
Another way to write it

$$L(s, O) = \epsilon_0(O) Q^{\frac{1}{2} - s} L(1 - s, \tilde{O})$$

where $Q \in \mathbb{N}$ is the conductor and $\epsilon_0(O)\epsilon_0(\tilde{O}) = 1$.
The basic and most important L-function is the Riemann zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Basic problem: there is no technique to obtain meromorphic continuation of an Euler product *as such*. We need an alternative realization.

Estermann phenomenon

Consider the simplest type of Euler products $E(s) = \prod_p f(p^{-s})^{-1}$ where $f \in \mathbb{Z}[x]$ is a *fixed* monic polynomial (independent of p). There is a dichotomy: if f is a product of cyclotomic polynomials (equivalently, if we can write

$$f = \frac{\prod_{i=1}^{k}(1-x^{n_i})}{\prod_{j=1}^{l}(1-x^{m_j})}$$

for some $n_1, \ldots, n_k, m_1, \ldots, m_l \in \mathbb{N}$) then

$$E(s) = \frac{\prod_{i=1}^{k}\zeta(n_i s)}{\prod_{j=1}^{l}\zeta(m_j s)}$$

and hence, $E(s)$ admits meromorphic continuation. Otherwise, $E(s)$ has a natural boundary (i.e., a line $\operatorname{Re} s = c$ consists of accumulation points of the poles).

Cusp forms

Let $G = \mathrm{GL}_n$. Consider cuspidal automorphic forms: smooth φ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ of uniform moderate growth such that

$$\int_{M_{m\times n-m}(\mathbb{Q})\backslash M_{m\times n-m}(\mathbb{A})} \varphi(\begin{pmatrix} I_m & X \\ 0 & I_{n-m} \end{pmatrix} g)\, dX = 0$$

for all $g \in G(\mathbb{A})$ and $1 \leq m < n$. Equivalently, for any proper parabolic subgroup $P = MU$ (defined over $\mathbb{Q}$)

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\, du = 0$$

for all $g \in G(\mathbb{A})$. Cusp forms are rapidly decreasing.
Let $G(\mathbb{A})^1 = \mathrm{Ker}|\det \cdot| : G(\mathbb{A}) \to \mathbb{R}_{>0}$. Note that $G(\mathbb{A}) \simeq G(\mathbb{A})^1 \times \mathbb{R}_{>0}$, $G(\mathbb{Q}) \subset G(\mathbb{A})^1$ and $\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) < \infty$.
$L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ (the closure of the φ's above in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$) decomposes discretely $\hat{\bigoplus}\pi$ as a representation of $G(\mathbb{A})$ under right translation.

Standard L-functions

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. We can write (abstractly) $\pi = \otimes_p \pi_p$.
For $p \notin S$ we can write $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p$ where $\chi_p(\mathrm{diag}(t_1, \ldots, t_n)) = |t_1|^{s_{1,p}} \ldots |t_n|^{s_{n,p}}$ and $(s_{1,p}, \ldots, s_{n,p}) \in \mathbb{C}^n / \frac{2\pi \mathrm{i}}{\log p}\mathbb{Z}^n$ uniquely determined up to permutation. We call $A(\pi_p) = \mathrm{diag}(p^{-s_{1,p}}, \ldots, p^{-s_{n,p}})$ (or rather, its conjugacy class) the Frobenius-Hecke parameters of π_p.
Define

$$L^S(s, \pi) = \prod_{p \notin S} L(s, \pi_p) = \prod_{p \notin S} \det(1 - p^{-s} A(\pi_p))^{-1}$$

$$= \prod_{p \notin S} \prod_{i=1}^{n} (1 - p^{-(s + s_{i,p})})^{-1}$$

Integral representation

We want to use Riemann's method (or Tate's thesis). This was carried out by Godement-Jacquet. Let $\mathcal{S}(\mathbb{A}^n) = \otimes_p \mathcal{S}(\mathbb{Q}_p^n)$ (restricted tensor product with respect to $\mathbf{1}_{\mathbb{Z}_p^n}$) – the space of Schwartz-Bruhat functions. Introduce the zeta integral

$$Z(f, \Phi, s) = \int_{G(\mathbb{A})} \Phi(g) f(g) |\det g|^{s+(n-1)/2} \, dg, \quad \Phi \in \mathcal{S}(M_n(\mathbb{A}))$$

for $\operatorname{Re} s \gg 0$ where f is a matrix coefficient, i.e.

$$f(g) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \varphi(xg)\varphi'(x) \, dx.$$

This is factorizable: if φ, φ' are factorizable vectors in π and $\tilde{\pi}$ respectively then $f(g) = \prod_p f_p(g_p)$ where f_p is local matrix coefficients. If also $\Phi(x) = \prod_p \Phi_p(x_p)$ then

$$Z(f, \Phi, s) = \prod_p \int_{G(\mathbb{Q}_p)} \Phi_p(g_p) f_p(g_p) |\det g_p|_p^{s+(n-1)/2} \, dg_p.$$

How to compute these local factors?

Unramified representations

Recall $B = T_0 \ltimes U_0$ is the Borel subgroup of upper triangular matrices, T_0 – diagonal matrices, U_0 – upper unitriangular matrices. Suppose that $\Phi_p = \mathbf{1}_{M_n(\mathbb{Z}_p)}$, $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p$, $\chi_p(\mathrm{diag}(t_1, \ldots, t_n)) = |t_1|^{\lambda_1} \ldots |t_n|^{\lambda_n}$, $f_p(g) = \langle \pi_p(g)\varphi_0, \varphi_0^\vee \rangle$ where φ_0, $\varphi_0^\vee$ are the standard sections

$$\begin{cases} \varphi_0(utk) = \delta_{B(\mathbb{Q}_p)}(t)^{\frac{1}{2}} \chi_p(t) \\ \varphi_0^\vee(utk) = \delta_{B(\mathbb{Q}_p)}(t)^{\frac{1}{2}} \chi_p^{-1}(t) \end{cases} , u \in U_0(\mathbb{Q}_p),\ t \in T_0(\mathbb{Q}_p),\ k \in K.$$

(This is valid for almost all p.) Then

$$f_p(g) = \int_{B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \varphi_0(xg)\varphi_0^\vee(x)\ dx = \int_K \varphi_0(kg)\ dk$$

zonal spherical function. There exists an explicit formula for these (Macdonald, Casselman), but we will not use it.

Unramified computation

$$\begin{aligned} Z_p(f_p, \Phi_p, s) &= \int_{G(\mathbb{Q}_p)} \Phi_p(g) f_p(g) |\det g|_p^{s+(n-1)/2} \, dg \\ &= \int_{G(\mathbb{Q}_p)} \int_K \Phi_p(g) \varphi_0(kg) |\det g|_p^{s+(n-1)/2} \, dk \, dg. \end{aligned}$$

Changing variable $g \mapsto k^{-1}g$ and noting that Φ_p is bi-K-invariant and $\mathrm{vol}(K) = 1$ we get

$$\begin{aligned} &\int_{G(\mathbb{Q}_p)} \Phi_p(g) \varphi_0(g) |\det g|_p^{s+(n-1)/2} \, dg \\ &\qquad = \int_{B(\mathbb{Q}_p)} \Phi_p(b) \varphi_0(b) \delta_{B(\mathbb{Q}_p)}^{-1}(b) |\det b|_p^{s+(n-1)/2} \, db \\ &= \int_{T_0(\mathbb{Q}_p)} \left(\int_{U_0(\mathbb{Q}_p)} \Phi_p(ut) \, du \right) \chi_p(t) \delta_{B(\mathbb{Q}_p)}^{-\frac{1}{2}}(t) |\det t|_p^{s+(n-1)/2} \, dt. \end{aligned}$$

$$\int_{T_0(\mathbb{Q}_p)} \left(\int_{U_0(\mathbb{Q}_p)} \Phi_p(ut)\ du \right) \chi_p(t) \delta_{B(\mathbb{Q}_p)}^{-\frac{1}{2}}(t) |\det t|_p^{s+(n-1)/2}\ dt.$$

What is $\int_{U_0(\mathbb{Q}_p)} \Phi_p(ut)\ du$? We need to know when does ut have integral entries. If $t = \mathrm{diag}(t_1, \dots, t_n) \in T_0(\mathbb{Q}_p)$ and $u = (u_{i,j})_{1 \le i,j \le n} \in U_0(\mathbb{Q}_p)$ then ut has $t_1, \dots, t_n$ on the diagonal and $u_{i,j} t_j$ above the diagonal. So the condition is $|t_1|_p, \dots, |t_n|_p \le 1$ and $|u_{i,j}|_p \le |t_j|_p^{-1}$ for all $i < j$. Hence,

$$\int_{U_0(\mathbb{Q}_p)} \Phi_p(ut)\ du = \begin{cases} \prod_{j=1}^n |t_j|_p^{-(j-1)} & \text{if } |t_1|_p, \dots, |t_n|_p \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\delta_B^{-\frac{1}{2}}(t) = |\det t|_p^{-(n-1)/2} \prod_{j=1}^n |t_j|_p^{j-1}$. We get

$$\int_{t_1 \in \mathbb{Q}_p^* : |t_1|_p \le 1} \dots \int_{t_n \in \mathbb{Q}_p^* : |t_n|_p \le 1} |t_1|_p^{s+\lambda_1} \dots |t_n|_p^{s+\lambda_n}\ dt_1 \dots\ dt_n$$
$$= \left[(1 - p^{-(s+\lambda_1)}) \dots (1 - p^{-(s+\lambda_n)}) \right]^{-1}$$

provided that $\mathrm{Re}(s + \lambda_i) > 0$ for all i.

Analytic continuation

For a cusp form φ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ consider

$$I(\varphi, \Phi, s) = \int_{G(\mathbb{A})} \Phi(g)\varphi(g)|\det g|^{s+(n-1)/2}\, dg.$$

This is **not** factorizable because φ is not a factorizable function. However, by Fubini

$$Z(f, \Phi, s) = \int_{G(\mathbb{A})} \Phi(g) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \varphi(xg)\varphi'(x)|\det g|^{s+(n-1)/2}\, dx\, dg$$

$$= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int_{G(\mathbb{A})} \Phi(g)\varphi(xg)\varphi'(x)|\det g|^{s+(n-1)/2}\, dg\ dx.$$

By changing the variable $g \mapsto x^{-1}g$

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int_{G(\mathbb{A})} \Phi_x(g)\varphi(g)\varphi'(x)|\det g|^{s+(n-1)/2}\, dg\ dx$$

$$= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} I(\varphi, \Phi_x, s)\varphi'(x)\ dx$$

where $\Phi_x(g) = \Phi(x^{-1}g)$.

It remains to analytically continue $I(\varphi, \Phi, s)$. Let $G(\mathbb{A})_{\gtrless 1} = \{g \in G(\mathbb{A}) : |\det g| \gtrless 1\} \simeq G(\mathbb{A})^1 \times \mathbb{R}_{\gtrless 1}$. Following Riemann we write

$$
\begin{aligned}
I(\varphi, \Phi, s) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in G(\mathbb{Q})} \Phi(\gamma g)\varphi(\not\gamma g)|\det \not\gamma g|^{s+(n-1)/2}\, dg \\
&= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{>1}} \Theta^*_\Phi(g)\varphi(g)|\det g|^{s+(n-1)/2}\, dg \\
&\quad + \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{<1}} \Theta^*_\Phi(g)\varphi(g)|\det g|^{s+(n-1)/2}\, dg
\end{aligned}
$$

where $\Theta^*_\Phi(g) = \sum_{\gamma \in G(\mathbb{Q})} \Phi(\gamma g)$. Let

$$
\Theta_\Phi(g) = \sum_{\delta \in M_n(\mathbb{Q})} \Phi(\delta g) = \Theta^*_\Phi(g) + \text{boundary terms.}
$$

By Poisson summation formula we have the functional equation

$$
\Theta_\Phi(g) = |\det g|^{-n}\Theta_{\hat{\Phi}}(g^\iota)
$$

where $g^\iota = {}^t g^{-1}$ is the Cartan involution.

So up to boundary terms

$$
\begin{aligned}
I(\varphi, \Phi, s) &= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{>1}} \Theta^*_{\Phi}(g)\varphi(g)|\det g|^{s+(n-1)/2}\ dg \\
&\quad + \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{<1}} \Theta^*_{\Phi}(g)\varphi(g)|\det g|^{s+(n-1)/2}\ dg \\
&= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{>1}} \Theta^*_{\Phi}(g)\varphi(g)|\det g|^{s+(n-1)/2}\ dg \\
&\quad + \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{<1}} \Theta^*_{\hat{\Phi}}(g^{\iota})\varphi(g)|\det g|^{s-(n+1)/2}\ dg \\
&= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{>1}} \Theta^*_{\Phi}(g)\varphi(g)|\det g|^{s+(n-1)/2}\ dg \\
&\quad + \int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{>1}} \Theta^*_{\hat{\Phi}}(g)\varphi(g^{\iota})|\det g|^{1-s+(n-1)/2}\ dg \\
&= I(\varphi^{\iota}, \hat{\Phi}, 1-s)
\end{aligned}
$$

where $\varphi^{\iota}(g^{\iota}) = \varphi(g)$.

Boundary terms

To show that the boundary terms (usually) don't contribute, it suffices to show the vanishing of

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\gamma \in M_n(\mathbb{Q}) \text{ singular}} \Phi(\gamma g)\varphi(g)\ dg.$$

We write this as

$$\begin{aligned}
&\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\eta} \sum_{\delta \in G_\eta(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\eta\delta g)\varphi(g)\ dg \\
&\qquad = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\eta} \sum_{\delta \in G_\eta(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\eta\delta g)\varphi(\delta g)\ dg \\
&\qquad\qquad = \sum_{\eta} \int_{G_\eta(\mathbb{Q})\backslash G(\mathbb{A})^1} \Phi(\eta g)\varphi(g)\ dg
\end{aligned}$$

where η is a set of representatives of the singular orbits of $M_n(\mathbb{Q})$ under $G(\mathbb{Q})$ (acting by right-multiplication) and $G_\eta(\mathbb{Q})$ is the stabilizer $\{x \in G(\mathbb{Q}) : \eta x = \eta\}$.

Clearly, $G_\eta(\mathbb{Q})$ is conjugate in $G(\mathbb{Q})$ to $G_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & I_r \end{smallmatrix}\right)}(\mathbb{Q})$ which is $\{\begin{pmatrix} * & * \\ 0 & I_r \end{pmatrix}\} = L(\mathbb{Q})V(\mathbb{Q})$ where $L = \{\begin{pmatrix} * & 0 \\ 0 & I_r \end{pmatrix}\}$ and $V = \{\begin{pmatrix} I_{n-r} & * \\ 0 & I_r \end{pmatrix}\}$. Now,

$$\begin{aligned}
&\int_{G_\eta(\mathbb{Q})\backslash G(\mathbb{A})^1} \Phi(\eta g)\varphi(g)\ dg \\
&= \int_{L(\mathbb{Q})V(\mathbb{A})\backslash G(\mathbb{A})^1} \int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \Phi(\eta vg)\varphi(vg)\ dv\ dg \\
&= \int_{L(\mathbb{Q})V(\mathbb{A})\backslash G(\mathbb{A})^1} \Phi(\eta g) \left(\int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \varphi(vg)\ dv \right)\ dg = 0
\end{aligned}$$

by cuspidality for $r > 0$. OTOH $r = 0$ gives

$$\Phi(0) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \varphi(g)\ dg$$

which is non-zero only if $n = 1$ and $\pi = 1$.

Langlands's philosophy

Simply put, any L-function is a product of standard L-functions! In particular, they have meromorphic continuation with finitely many poles and a functional equation.
This can be thought of as a (higher dimensional) reciprocity law.
It is extremely deep. For most L-functions we don't have a clue how to meromorphically continue them (let alone, the finiteness of the poles).
The most striking success in this direction so far is the work of Wiles and Taylor-Wiles who showed that the Hasse-Weil zeta function of an elliptic curve over $\mathbb{Q}$ (with some assumptions, subsequently removed) is the L-function of a modular form. (The Taniyama-Shimura-Weil modularity conjecture.) By earlier work this implies (but is more fundamental than) Fermat's Last Theorem.

Another more recent result follows from the work of Khare-Wintenberger: the Artin L-function of an odd two-dimensional Artin representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C}), \quad \mathrm{tr}\,\rho(\text{complex conjugation}) = 0$$

(factors through a finite Galois extension of $\mathbb{Q}$) is modular.
Even more recently, Taylor and his collaborators have spectacular results on potential modularity (implying meromorphic continuation and functional equation but without finiteness of poles) for many higher degree L-functions (cf. Serre's minicourse last week).

Automorphic representations

Erez Lapid

Lausanne, 2011

Whittaker coefficients for GL_n

Fix a non-trivial character ψ of $\mathbb{A}$ which is trivial on $\mathbb{Q}$, i.e. $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$. (All other such characters are of the form $\psi^a = \psi(a\cdot)$ for some $a \in \mathbb{Q}^*$, i.e. the Pontryagin dual of $\mathbb{Q}\backslash\mathbb{A}$ is $\mathbb{Q}$.) Let $\psi_0 : U_0(\mathbb{A}) \to \mathbb{C}$ be the character $\psi_0(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$. Then ψ_0 is trivial on $U_0(\mathbb{Q})$. For any cusp form ϕ define the Fourier coefficient by

$$W_\phi^\psi(g) = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \phi(ug)\psi_0^{-1}(u)\, du, \quad g \in G(\mathbb{A}).$$

It satisfies

$$W_\phi^\psi(ug) = \psi_0(u) W_\phi^\psi(g).$$

Moreover, $W_\phi(t)$ is rapidly decreasing for $|t_1| \geq \cdots \geq |t_n|$. Finally,

$$W_\phi^{\psi^a}(g) = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \phi(ug)\psi_0^a(u)^{-1}\, du = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \phi(ug)$$

$$\psi_0^{-1}(t_a u t_a^{-1})\, du = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \phi(\cancel{t_a^{-1}} u t_a g)\psi_0^{-1}(u)\, du = W_\phi^\psi(t_a g)$$

where $t_a = \mathrm{diag}(a^{n-1}, a^{n-2}, \ldots, 1)$.

Recovering ϕ from W_ϕ

Let P_n be the subgroup of G consisting of matrices whose last row is $v_0 = (0, 0, \ldots, 0, 1)$. P_n is the stabilizer of v_0 for the right action of G on row vectors.
We have the following remarkable formula due to Piatetskii-Shapiro and Shalika

$$\boxed{\phi(g) = \sum_{\gamma \in U_0(\mathbb{Q}) \backslash P_n(\mathbb{Q})} W_\phi(\gamma g)}$$

Note that it is not a priori clear that the right-hand side is $G(\mathbb{Q})$-invariant!
In particular, $W_\phi \not\equiv 0$ if $\phi \not\equiv 0$. (Not true for other split groups.)

Proof of $\phi(g) = \sum_{\gamma \in U_0(\mathbb{Q}) \backslash P_n(\mathbb{Q})} W_\phi(\gamma g)$

We write Fourier expansion of $\phi(\cdot g)$ along the last column

$$U_{n-1}(R) = \{u_n(\xi) = \begin{pmatrix} I_{n-1} & \xi \\ 0 & 1 \end{pmatrix} \ \xi \in R^{n-1}\} \simeq R^{n-1}$$

Doing Fourier analysis on $\mathbb{Q}^{n-1} \backslash \mathbb{A}^{n-1}$

$$\begin{aligned} \phi(g) &= \sum_{\xi \in \mathbb{Q}^{n-1}} \int_{\mathbb{Q}^{n-1} \backslash \mathbb{A}^{n-1}} \phi(u_n(t) g) \psi^{-1}(\langle \xi, t \rangle) \, dt \\ &= \sum_{\xi \in \mathbb{Q}^{n-1} \backslash \{0\}} \int_{\mathbb{Q}^{n-1} \backslash \mathbb{A}^{n-1}} \phi(u_n(t) g) \psi^{-1}(\langle \xi, t \rangle) \, dt \end{aligned}$$

by cuspidality.

Note that $P_n(\mathbb{Q})$ acts transitively (by conjugation) on the non-trivial characters of $U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})$ (i.e., on $\mathbb{Q}^{n-1} \setminus \{0\}$).

The stabilizer of ψ_0 is $P_{n-1} = \{ \begin{pmatrix} * & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \}$. We can therefore rewrite the above as

$$\sum_{\gamma \in P_{n-1}(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})} \phi(ug)\psi_0^{-1}(\gamma u \gamma^{-1})\ du$$

$$= \sum_{\gamma \in P_{n-1}(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})} \phi(\gamma^{-1} u \gamma g)\psi_0^{-1}(u)\ du$$

$$= \sum_{\gamma \in P_{n-1}(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})} \phi(u \gamma g)\psi_0^{-1}(u)\ du.$$

We repeat this procedure for $\int_{U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})} \phi(ug)\psi_0^{-1}(u)\, du$. We expand using Fourier on $\mathbb{Q}^{n-2}\backslash\mathbb{A}^{n-2}$ in the $n-1$-column. Again, the constant term vanishes by cuspidality and we replace the sum over $\mathbb{Q}^{n-2} \setminus \{0\}$ by $P_{n-2}(\mathbb{Q})\backslash P_{n-1}(\mathbb{Q})$. We get

$$\int_{U_{n-1}(\mathbb{Q})\backslash U_{n-1}(\mathbb{A})} \phi(ug)\psi_0^{-1}(u)\, du$$

$$= \sum_{\delta\in P_{n-2}(\mathbb{Q})\backslash P_{n-1}(\mathbb{Q})} \int_{U_{n-2}(\mathbb{Q})\backslash U_{n-2}(\mathbb{A})} \phi(u\delta g)\psi_0^{-1}(u)\, du$$

where $U_{n-2} = \{ \begin{pmatrix} I_{n-2} & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \}$.

Summing over $\gamma \in P_{n-1}(\mathbb{Q})\backslash P_n(\mathbb{Q})$ we get $\phi(g) =$

$$\sum_{\delta\in P_{n-2}(\mathbb{Q})\backslash P_{n-1}(\mathbb{Q})} \sum_{\gamma\in P_{n-1}(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_{n-2}(\mathbb{Q})\backslash U_{n-2}(\mathbb{A})} \phi(u\delta\gamma g)\psi_0^{-1}(u)\, du$$

$$= \sum_{\gamma\in P_{n-2}(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_{n-2}(\mathbb{Q})\backslash U_{n-2}(\mathbb{A})} \phi(u\gamma g)\psi_0^{-1}(u)\, du.$$

Continuing this way we get by descending induction on i

$$\phi(g) = \sum_{\gamma \in P_i(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})} \phi(u\gamma g)\psi_0^{-1}(u)\, du$$

where $P_i = \{\begin{pmatrix} * & * \\ 0 & u \end{pmatrix} : u$ upper unitriangular in $\mathrm{GL}_{n+1-i}\}$ and U_i is the group of upper unitriangular matrices of the form $\begin{pmatrix} I_i & * \\ 0 & * \end{pmatrix}$, i.e., U_i is the unipotent radical of the parabolic of type $(i, 1, \ldots, 1)$. For $i = 1$, $U_1 = P_1 = U_0$ and we get

$$\phi(g) = \sum_{\gamma \in U_0(\mathbb{Q})\backslash P_n(\mathbb{Q})} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \phi(ug)\psi_0^{-1}(u)\, du$$

as required.

Local Whittaker models

For a local field F we can consider the space of Whittaker functions

$$\mathcal{W} = \{W : G(F) \to \mathbb{C} \mid W \text{ smooth of moderate growth and } W(ug) = \psi_0(u)W(g) \text{ for all } u \in U_0(F),\ g \in G(F)\}.$$

In the p-adic case smooth means locally constant (=invariant under right translation by a small open subgroup), moderate growth is redundant. $G(F)$ acts on $\mathcal{W}$ by right-translation. For any irreducible π there exists at most one intertwining operator up to a scalar $\pi \to \mathcal{W}$. Alternatively, there exists at most one continuous functional $\Lambda \in \pi^*$ such that $\Lambda(\pi(u)v) = \psi_0(u)\Lambda(v)$ for all $u \in U_0(F)$, $v \in V_\pi$. (Gelfand-Kazhdan – p-adic case, Shalika - Archimedean case). For this we used Frobenius reciprocity

$$\mathrm{Hom}_{G(F)}(\pi, \mathcal{W}) \simeq \mathrm{Hom}_{U_0(F)}(\pi, \psi_0)$$

$A \mapsto (v \mapsto Av(e))$
$\Lambda \mapsto (v \mapsto (g \mapsto \Lambda(\pi(g)v)))$
It may be that π does not have a Whittaker model (for instance, if π is the trivial representation). If it has, we denote it by $\mathcal{W}(\pi)$.

The map $\phi \mapsto W_\phi$ gives a non-zero intertwining map from π to the global Whittaker space

$$\mathcal{W} = \{W : G(\mathbb{A}) \to \mathbb{C} \,|W(ug) = \psi_0(u)W(g) \\ \forall u \in U_0(\mathbb{A}),\ g \in G(\mathbb{A})\} = \otimes \mathcal{W}_p$$

By multiplicity one we get an intertwining map $\pi \to \mathcal{W}(\pi) := \otimes \mathcal{W}(\pi_p)$ (restriced tensor product with respect to the spherical W_p normalized by $W_p(e) = 1$.
This is very useful because although cusp forms in π are not factorizable as functions, their Whittaker functions are.
In other words for a factorizable vector ϕ in $\pi = \otimes_p \pi_p$ we can write $W_\phi(g) = \prod_p W_p(g_p)$, $g = (g_p)_p \in G(\mathbb{A})$.
An immediate consequence of $\phi(g) = \sum_{\gamma \in U_0(\mathbb{Q}) \backslash P_n(\mathbb{Q})} W_\phi(\gamma g)$ and uniqueness of Whittaker model: multiplicity one for cuspidal representations on GL_n.

Rankin-Selberg integral

Let π, π' be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$ respectively. We want to study the degree nm L-function

$$L^S(s, \pi \times \pi') = \prod_{p \notin S} \det(1 - p^{-s} A(\pi_p) \otimes A(\pi'_p))^{-1}$$

$$= \prod_{p \notin S} \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - p^{-(s + s_{i,p} + s'_{j,p})})^{-1}$$

where we recall that $A(\pi_p)$ are the Frobenius-Hecke parameters. Sometime this is called Rankin-Selberg convolution.
$n = 2, m = 1$: Hecke (1920's)
$m = n = 2$: Rankin and Selberg (independently) - 1940
general case: Jacquet–Piatetskii-Shapiro–Shalika in the 1980's.
Final touches by Jacquet 2009.
We will discuss the case $m = n$ (most interesting).

Special Eisenstein series for GL_n

Let V be a vector space over $\mathbb{Q}$ of dimension n.
$A = \{tI_n : t \in \mathbb{R}_{>0}\} \simeq \mathbb{R}_{>0}$ – the subgroup of central matrices which are positive at ∞ and 1 at the p-adic places. We have $G(\mathbb{A}) \simeq G(\mathbb{A})^1 \times A$ where $G(\mathbb{A})^1 = \mathrm{Ker}|\det \cdot| : G(\mathbb{A}) \to \mathbb{R}_{>0}$.
For $\Phi \in \mathcal{S}(V(\mathbb{A}))$ define for $\mathrm{Re}\, s > 1$

$$\mathcal{E}_\Phi(g,s) = \int_A \sum_{v \in V(\mathbb{Q}) \setminus \{0\}} \Phi(vag) |\det ag|^s \; da = \int_A \theta^*_\Phi(ag) a^{ns} \; da.$$

It is the Mellin transform of the truncated theta function

$$\theta^*_\Phi(g) = \sum_{v \in V(\mathbb{Q}) \setminus \{0\}} \Phi(vg) = \theta_\Phi(g) - \Phi(0)$$

Alternatively, we can write

$$\mathcal{E}_\Phi(g,s) = \int_A \sum_{\gamma \in P_n(\mathbb{Q}) \setminus G(\mathbb{Q})} \Phi(v_0 \gamma a g) |\det ag|^s \; da.$$

Functional equation: $\theta_\Phi(g) = |\det g|^{-1} \theta_{\hat{\Phi}}(g^\iota)$ where $g^\iota = {}^t g^{-1}$.

Tate's thesis revisited

$$\mathcal{E}_\Phi(g,s) = |\det g|^s \int_{A_{>1}} \theta^*_\Phi(ag)a^{ns}\ da + |\det g|^{s-1}\int_{A_{>1}} \theta^*_{\hat\Phi}(ag^\iota)a^{-n(s-1)}\ da + \mathrm{vol}(\mathbb{Q}^*\backslash\mathbb{I}^1)\left(|\det g|^{s-1}\frac{\hat\Phi(0)}{s-1} - |\det g|^s\frac{\Phi(0)}{s}\right) = \mathcal{E}_{\hat\Phi}(g^\iota, 1-s).$$

The first two terms in the middle expression are entire. The residue at $s=1$ is the constant function $\hat\Phi(0)$.
Let F be a number field of degree n over $\mathbb{Q}$. We can embed $T := F^* \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$, i.e. $T(\mathbb{Q}) = F^*$, $T(\mathbb{A}) = \mathbb{I}_F$, $T(\mathbb{A})^1 = \mathbb{I}_F^1$.
For a Hecke character $\chi : F^*\backslash\mathbb{I}_F \to \mathbb{C}^*$ of F, $\Phi \in \mathcal{S}(\mathbb{A}_F)$

$$\int_{T(\mathbb{Q})\backslash T(\mathbb{A})^1} \mathcal{E}_\Phi(t,s)\chi(t)\ dt = \int_{T(\mathbb{Q})\backslash T(\mathbb{A})} \sum_{x\in F^*} \Phi(xt)\chi(t)|t|^s\ dt = \int_{\mathbb{I}_F} \Phi(t)\chi(t)|t|^s\ dt = \text{local factors } \times L^S(s,\chi)$$

Case $m = n$

Let ϕ, ϕ' be cusp forms in the space of π, π' resp.. The integral

$$I(\phi, \phi', \Phi, s) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(g)\phi'(g)\mathcal{E}_\Phi(g, s)\ dg$$

admits meromorphic continuation. The only possible pole is at $s_0 = 0, 1$ and it occurs if and only if $\pi' = \tilde{\pi}$. The residue is proportional to $\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(g)\phi'(g)\ dg$. We may write the integral as

$$\int_{P_n(\mathbb{Q})\backslash G(\mathbb{A})} \phi(g)\phi'(g)\Phi(v_0 g)|\det g|^s\ dg$$

where $v_0 = (0, \ldots, 0, 1)$. Using $\phi(g) = \sum_{\gamma\in U_0(\mathbb{Q})\backslash P_n(\mathbb{Q})} W_\phi(\gamma g)$ we get

$$\int_{P_n(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma\in U_0(\mathbb{Q})\backslash P_n(\mathbb{Q})} W_\phi(\gamma g)\phi'(g)\Phi(v_0 g)|\det g|^s\ dg$$

$$= \int_{P_n(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma\in U_0(\mathbb{Q})\backslash P_n(\mathbb{Q})} W_\phi(\gamma g)\phi'(\gamma g)\Phi(v_0 \gamma g)|\det \gamma g|^s\ dg$$

$$= \int_{P_n(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in U_0(\mathbb{Q})\backslash P_n(\mathbb{Q})} W_\phi(\gamma g)\phi'(\gamma g)\Phi(v_0\gamma g)|\det \gamma g|^s \ dg$$

$$= \int_{U_0(\mathbb{Q})\backslash G(\mathbb{A})} W_\phi(g)\phi'(g)\Phi(v_0 g)|\det g|^s \ dg$$

$$= \int_{U_0(\mathbb{A})\backslash G(\mathbb{A})} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} W_\phi(ug)\phi'(ug)\Phi(v_0 ug)|\det ug|^s \ dg$$

$$= \int_{U_0(\mathbb{A})\backslash G(\mathbb{A})} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \psi(u) W_\phi(g)\phi'(ug)\Phi(v_0 g)|\det g|^s \ dg$$

$$= \int_{U_0(\mathbb{A})\backslash G(\mathbb{A})} W_\phi^{\psi}(g) W_{\phi'}^{\psi^{-1}}(g)\Phi(v_0 g)|\det g|^s \ dg$$

$$= \prod_p \int_{U_0(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} W_p(g_p) W_p'(g_p)\Phi_p(v_0 g_p)|\det g_p|^s \ dg_p$$

$$= \prod_p I_p(W_p, W_p', \Phi_p, s).$$

Casselman-Shalika formula

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ set $p^\lambda = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}) \in T_0(\mathbb{Q}_p)$. The normalized spherical Whittaker function of π_p is given by

$$W_p(u\varpi^\lambda k) = \psi_0(u) \begin{cases} \delta_{B(\mathbb{Q}_p)}(\varpi^\lambda)^{\frac{1}{2}} \chi_\lambda(A(\pi_p)) & \lambda \text{ dominant,} \\ 0 & \text{otherwise} \end{cases}$$

where χ_λ denotes the character of the irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with highest weight λ. Incidently, by the Weyl character formula

$$\chi_\lambda(\mathrm{diag}(x_1, \ldots, x_n)) = \frac{\det(x_i^{\lambda_j + n - j})_{i,j=1,\ldots,n}}{\det(x_i^{n-j})_{i,j=1,\ldots,n}}$$

(The Schur polynomial s_λ – they form a basis for the symmetric polynomials in $x_1, \ldots, x_n$ for $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$)

Unramified computation

By Iwasawa decomposition

$$\int_{U_0(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} W_p(g_p)W'_p(g_p)\Phi_p(v_0 g_p)|\det g_p|^s \, dg_p$$
$$= \sum_{\lambda_1\geq\cdots\geq\lambda_n\geq 0} \chi_\lambda(A(\pi_p))\chi_\lambda(A(\pi'_p))p^{-|\lambda|s}, \quad |\lambda| = \lambda_1+\cdots+\lambda_n.$$

We claim that for $\operatorname{Re} s \gg 0$ this is equal to

$$\det(1-p^{-s}A(\pi_p)\otimes A(\pi'_p))^{-1} = \sum_{k=0}^{\infty} p^{-sk}\operatorname{tr}\operatorname{Sym}^k(A(\pi_p)\otimes A(\pi'_p)).$$

In other words, for any $A, A' \in \mathrm{GL}_n(\mathbb{C})$ such that $|\lambda| < |\lambda'|$ for all eigenvalues λ of A and λ' of A' we have

$$\sum_{\lambda_1\geq\cdots\geq\lambda_n\geq 0} \chi_\lambda(A)\chi_\lambda(A'^{-1})$$
$$= \operatorname{tr}((A,A')|\operatorname{Sym}(\mathbb{C}^n\otimes(\mathbb{C}^n)^\vee)) = \operatorname{tr}((A,A')|\operatorname{Sym}(M_n(\mathbb{C}))).$$

The unramified identity

$$\sum_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} \chi_\lambda(A)\chi_\lambda(A'^{-1}) = \mathrm{tr}((A, A')|\,\mathrm{Sym}(M_n(\mathbb{C}))).$$

Thus, we need to show that under the right and left action of $\mathrm{GL}_n(\mathbb{C}) \otimes \mathrm{GL}_n(\mathbb{C})$ on $M_n(\mathbb{C})$ we have

$$\mathrm{Sym}(M_n(\mathbb{C})) \simeq \oplus_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} V_\lambda \otimes \tilde{V}_\lambda.$$

Decomposition of the $\mathbb{C}[G]$ (regular functions on GL_n) as $G \times G$-module is $\sum V_\lambda \otimes \tilde{V}_\lambda$ (algebraic Peter-Weyl – formally equivalent to it). As a $G \times G$-submodule of $\mathbb{C}[G]$, $\mathrm{Sym}(M_n(\mathbb{C}))$ exactly corresponds to $\lambda_n \geq 0$ (non-negative power of the determinant).

Inner product in GL_n

Taking residue at $s = 1$ in the identity

$$I(\phi, \phi', \Phi, s) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(g)\phi'(g)\mathcal{E}_\Phi(g, s)\ dg = L^S(s, \pi \times \pi') \prod_{p \in S} I_p(W_p, W_p', \Phi_p, s)$$

we get

$$\hat{\Phi}(0)(\phi, \phi') = \hat{\Phi}(0) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(g)\phi'(g)\ dg = \operatorname{res}_{s=1} L^S(s, \pi \times \pi') \prod_{p \in S} I_p(W_p, W_p', \Phi_p, 1)$$

However,

$$I_p(W_p, W'_p, \Phi_p, 1) = \int_{U_0(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} W_p(g) W'_p(g) \Phi_p(v_0 g) |\det g| \; dg$$

$$= \int_{P_n(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \int_{U_0(\mathbb{Q}_p)\backslash P_n(\mathbb{Q}_p)} W_p(pg) W'_p(pg) \Phi_p(v_0 \not{p} g) |\det pg| \; \delta_{P_n}(p)^{-1} \; dp \; dg.$$

Since $\delta_{P_n} = |\det|$ we get

$$\int_{P_n(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \left(\int_{U_0(\mathbb{Q}_p)\backslash P_n(\mathbb{Q}_p)} W_p(pg) W'_p(pg) \; dp \right) \Phi_p(v_0 g) |\det g| \; dg.$$

For this to be proportional to $\hat{\Phi}_p(0)$,
$\int_{U_0(\mathbb{Q}_p)\backslash P_n(\mathbb{Q}_p)} W_p(pg) W'_p(pg) \; dp$ must be independent of g, i.e.,

$$[W_p, W'_p]_p := \int_{U_0(\mathbb{Q}_p)\backslash P_n(\mathbb{Q}_p)} W_p(p) W'_p(p) \; dp$$

is an invariant pairing!

Finally, since

$$\int_{P_n(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \Phi_p(v_0 g)|\det g| \; dg = \hat{\Phi}(0)$$

(polar coordinates) we get

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(g)\phi'(g) \; dg = \mathrm{res}_{s=1}\, L^S(s, \pi \times \pi') \prod_{p \in S} [W_p, W'_p]_p.$$

Langlands's functoriality conjecture (in the context of GL_n)

For any algebraic (finite-dimensional) representation $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ and any cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ there exists an automorphic representations σ of $\mathrm{GL}_N(\mathbb{A})$ (not necessarily cuspidal) such that $A(\sigma_p) = \rho(A(\pi_p))$ for $p \notin S$.
In particular,

$$L^S(s, \sigma) = L^S(s, \pi, \rho) := \prod_{p \notin S} \det(1 - p^{-s}\rho(A(\pi_p)))^{-1}.$$

so that $L^S(s, \pi, \rho)$ is "nice".
For GL_2 we know the principle of functoriality for Sym^2 (Gelbart-Jacquet), Sym^3, Sym^4 (Kim-Shahidi).
The principle of functoriality makes sense (and is extremely profound) for other groups as well. Recently, Arthur established the functorial transfer from classical groups to GL_n (from $\mathrm{SO}(2n+1)$ to $\mathrm{GL}(2n)$, $\mathrm{Sp}(2n)$ to $\mathrm{GL}(2n+1)$, $\mathrm{SO}(2n)$ to $\mathrm{GL}(2n)$).
A key ingredient is the Fundamental Lemma proved by Ngô.