LECTURE 1

From classical modular forms to functions on groups

These lectures are intended to non-specialist of the theory of automorphic forms; their goal is to explain how the classical theory of automorphic forms (for instance modular fonctions on the upper-half plane) and some of its important features (Modular forms, Eisenstein series, Peterson inner product, spectral decomposition and Hecke operators) translate into the framework of representation theory of adelic algebraic groups (especially for GL_2).

1. Classical modular forms

An holomorphic modular form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ is usually defined as:

a function on the upper-half plane $\mathbb{H}$

(1) which satisfies an *automorphy relation*: for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\gamma.z) = j(\gamma, z)^k f(z), \tag{1.1}$$

(2) is holomorphic on $\mathbb{H}$,
(3) is "holomorphic at the cusp".

Here $j(\gamma, z) = cz+d$ is the automorphy factor; it satisfies the cocycle relation

$$j(g_1g_2, z) = j(g_1, g_2z)j(g_2, z).$$

The "holomorphic at the cusp" condition is usually defined via the Fourier expansion at ∞-cusp: f is holomorphic and 1-periodic

$$f(z) = f(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z) = f(z+1)$$

one has

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(f)\exp(2\pi i n z). \tag{1.2}$$

and one requiers that

$$n < 0 \Rightarrow a_n(f) = 0.$$

1.1. Holomorphy at the cusp and growth. This condition can equivalently be stated in terms of growth which is kind of "softer":

- The ($\mathrm{SL}_2(\mathbb{Z})$-invariant) function $z = x + iy \to y^{k/2}|f(z)|$ as at most polynomial growth in y as $y \to \infty$.
- For any $\gamma \in \mathrm{SL}_2(\mathbb{Q})$, the function $z = x + iy \to \Im(\gamma z)^{k/2}|f(\gamma.z)|$ as at most polynomial growth in y as $y \to \infty$.

f is of *moderate growth*

1.2. Maass forms. Another very important class of modular forms are *Maass forms*: a Maass form (for $\mathrm{SL}_2(\mathbb{Z})$) is a function on $\mathbb{H}$ which

(1) is of weigth 0 or in other terms $\mathrm{SL}_2(\mathbb{Z})$-invariant: for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\gamma.z) = f(z).$$

(2) is an eigenfunction of the *hyperbolic Laplace operator*

$$\Delta f = \lambda f,\ \Delta := y^2(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}) :$$

(3) of moderate growth.

As for the holomorphic case, a Maass form is 1-periodic

$$f(z+1) = f(z)$$

and the moderate growth condition can be understood in terms of the vanishing of some suitable "Fourier coefficients".

1.3. Cusp forms. Inside these spaces of modular forms, cuspforms are the function whose *constant term* defined as

$$a_0(f) :\ z \in \mathbb{H} \mapsto \int_0^1 f(z+x)dx.$$

vanishes identically.

One can show that this latter condition is equivalent to the rapid decay of the function $z = x + iy \to \Im(\gamma z)^{k/2}|f(\gamma.z)|$ as $y \to \infty$.

1.4. The Petersson inner product. Space of modular forms are equipped with the Petersson inner product defined for f_1, f_2 of the same weigth by

$$\langle f_1, f_2\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} y^k f_1(z)\overline{f_2}(z)\frac{dxdy}{y^2}.$$

here $\frac{dxdy}{y^2}$ is the hyperbolic measure.

2. Functions on groups

2.1. $\mathbb{H}$ as an homogeneous space. Now $\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ via the map

$$g \in \mathrm{SL}_2(\mathbb{R}) \to g.i \in \mathbb{H}.$$

Given $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$ a smooth function of weight k (satisfying (1.1)), let

$$F(g) := j(g,i)^{-k} f(g.i);$$

this is a smooth function on $\mathrm{SL}_2(\mathbb{R})$ which is $\mathrm{SL}_2(\mathbb{Z})$-invariant on the left:

$$\begin{aligned}\forall \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ F(\gamma.g) &= j(\gamma g, i)^{-k} f(\gamma g.i) \\ &= j(\gamma, g.i)^{-k} j(g,i)^{-k} j(\gamma, g.i)^k f(\gamma g.i) = F(g).\end{aligned}$$

2.2. The weight. Moreover the condition on f being of weight k translates into

$$\forall \kappa = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}) = \mathrm{Stab}_i(\mathrm{SL}_2(\mathbb{R}))$$

$$F(g\kappa) = j(g\kappa, i)^{-k} f(g.i) = j(\kappa, i)^{-k} F(g) = \chi_k(\kappa) F(g)$$

where χ_k is the character of $\mathrm{SO}_2(\mathbb{R})$ given by

$$\chi_k : \kappa = \kappa(t) = \exp(-2\pi i k t).$$

Thus we can identify functions on $\mathbb{H}$ of weight k with functions on the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ which transform like the one dimensional representation χ_k of $\mathrm{SO}_2(\mathbb{R})$, under right multiplication by elements of $\mathrm{SO}_2(\mathbb{R})$ (such functions will be said of type χ_k). In the sequel we will write $\mathrm{SO}_2(\mathbb{R}) =: K_\infty$.

2.3. The moderate growth condition. Since K_∞ is compact we see that the moderate growth condition on $f(z)$ is equivalent to the existence of some $A \geq 0$ such that

$$F(g) \ll \|g\|^A$$

where $\|g\|$ is for instance the operator norm for the standart action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ (or on any faithful representation).

2.4. The Lie algebra action. We now interpret either the holomorphy or the Laplace eigenform condition in group theoretic terms.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ acts (via right multiplication) on the space of smooth functions on $\mathrm{SL}_2(\mathbb{R})$ as left invariant differential operators of order 1 : given $X \in \mathfrak{sl}_2(\mathbb{R})$

$$X.F : g \mapsto \lim_{t\to 0} \frac{F(g\exp(tX)) - F(g)}{t}.$$

In particular if f is of weight k (or F of type χ_k) one has

$$W.F = -ikF \textit{ with } W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and this is in fact equivalent to being of weight k.

Let

$$C = \frac{1}{2}(H \circ H + Z \circ Z - W \circ W)$$

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Z := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be the Casimir element. C generate the center of the complexified enveloping algebra $U(\mathfrak{sl}_2(\mathbb{R}))$of $\mathfrak{sl}_2(\mathbb{R})$ and defines a left invariant differential operator of order 2.

Then f being holomorphic of weight k is then equivalent to

– F is of type χ_k and $C.F = (k^2/2 - k)F$.

and f being aMaass form with Laplace eigenvalue λ is equivalent to

– F is of type χ_0 and $C.F = \lambda F$.

2.5. The cuspidality condition. Recall the Iwasawa decomposition

$$\mathrm{SL}_2(\mathbb{R}) = N(\mathbb{R})A^1(\mathbb{R})K_\infty$$

In particular F is left $N(\mathbb{Z})$-invariant and f being cuspidal is equivalent to

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} F(ng)dn = \int_{\mathbb{Z}\backslash\mathbb{R}} F(n(x)g)dx = 0 \ \forall g \in \mathrm{SL}_2(\mathbb{R})$$

2.6. The inner product. Consider the following measure on $\mathrm{SL}_2(\mathbb{R})$ defined in terms of the coordinates associated with Iwasawa decomposition

$$\mathrm{SL}_2(\mathbb{R}) = N(\mathbb{R})A\,\mathrm{SO}_2(\mathbb{R}),$$

$$dg := dx\frac{dy}{y^2}dk$$

where

$$g = n(x)a(y)k, \ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \ k \in \mathrm{SO}_d(\mathbb{R})$$

(dk the Haar measure on $\mathrm{SO}_2(\mathbb{R})$). Then dg is a Haar measure on $\mathrm{SL}_2(\mathbb{R})$ (which is unimodular) and we find that

$$\langle f_1, f_2\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})} F_1(g)\overline{F_2}(g)dg =: \langle F_1, F_2\rangle. \tag{2.1}$$

Notice that for such inner product, the action of $\mathrm{SL}_2(\mathbb{R})$ by right multiplications is an isometry:

$$\langle g'.F_1, g'.F_2\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})} F_1(gg')\overline{F_2}(gg')dg = \langle F_1, F_2\rangle \tag{2.2}$$

and that it is defined even if F_1 and F_2 have different types: but then $\langle F_1, F_2\rangle = 0$. Moreover the unitarity of the $\mathrm{SL}_2(\mathbb{R})$ action, (**??**) with respect to this inner product implies essentially immediately that the Casimir element is self-adjoint

$$\langle C.F_1, F_2\rangle = \langle F_1, C.F_2\rangle.$$

2.7. Conclusion. As we will see, for the theory it will be much better is enlarge further the group $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{GL}_2(\mathbb{R})$: one has obviously

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \simeq Z(\mathbb{R})\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})$$

where $Z(\mathbb{R}) < \mathrm{GL}_2(\mathbb{R})$ denote the group of scalar matrices.

To resume what has been done so far: several spaces of modular forms have been identified with spaces of smooth functions on $\mathrm{GL}_2(\mathbb{R})$ which are

- left $\mathrm{GL}_2(\mathbb{Z})$-invariant: $F(\gamma g) = F(g),\ \gamma \in \mathrm{GL}_2(\mathbb{Z})$
- $Z(\mathbb{R})$-invariant: $F(zg) = F(g)$, $z \in Z(\mathbb{R})$,
- of some type χ_k: $F(g\kappa) = \chi_k(\kappa)F(g)$, $\kappa \in K_\infty = \mathrm{SO}_2(\mathbb{R})$.
- of moderate growth,
- eigenfunction of the Casimir operator.

The key point is that the space of function $\mathcal{F}(Z(\mathbb{R})\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R}))$ is a representation of $\mathrm{GL}_2(\mathbb{R})$ for the multiplication on the right

$$g.F :\ g' \mapsto F(g'g)$$

moreover (cf **??**) this representation is unitary with respect to the inner product (**??**). This will allow to use systematically methods from representation theory.

In the sequel we will also interpret in similar terms other standart features of the theory modular forms of more arithmetical aspect; for instance Hecke operators by considering representation of adelic groups. To do so we note that the quotient $Z(\mathbb{R})\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{GL}_2(\mathbb{R})$ is naturally identified with an homogeneous space, namely

$$[\mathcal{L}_2(\mathbb{R})] \text{ the space of lattices in } \mathbb{R}^2 \text{ up to homothety.}$$

THe adelic interpretation will come by considering the subspace

$$[\mathcal{L}_2(\mathbb{Q})] \text{ the space of lattices in } \mathbb{Q}^2 \text{ up to homothety.}$$