LECTURE 2

Adeles

The adèles (or rather *idèles*) were invented by C. Chevalley in the 30's to formulate class field theory in the case infinite extensions of a number field. More generally, the adèles offer an extremelly convenient way to package information in the context of "lobal-to-global" principles in number theory.

1. Local to global principle for lattices

We illustrate, this principle with the space of lattice in $\mathbb{Q}^n$ (the space of finitely generated $\mathbb{Z}$-modules of rank n in $\mathbb{Q}^n$), $\mathcal{L}_n(\mathbb{Q})$ say. Let $L \subset \mathbb{Q}^n$ be such a lattice; for every prime p we define its *localization* at p,

$$L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset \mathbb{Q}_p^n;$$

as we will see L_p is a lattice in $\mathbb{Q}_p^n$ (ie. a free $\mathbb{Z}_p$-module of rank n in $\mathbb{Q}_p^n$). Let

$$\mathcal{P} = \{2, 3, 5, 7, \dots\}$$

be the set of prime numbers, the next proposition shows that L is completely determined by the set of all its localizations $(L_p)_{p\in\mathcal{P}}$:

Proposition 1.1. *Let $L \subset \mathbb{Q}^n$ be a lattice. For p a prime number, then $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a lattice in $\mathbb{Q}_p^n$ and for almost every $p \in \mathcal{P}$ (that is for all but finitely many),*

$$L_p = \mathbb{Z}_p^n;$$

moreover, the map

$$L \mapsto (L_p)_{p\in\mathcal{P}}$$

defines a bijection between

- *the set of lattices in $\mathbb{Q}^n$, $\mathcal{L}_n(\mathbb{Q})$ and*
- *the set of sequences (indexed by the set of primes $\mathcal{P}$)*

$$(L_p)_{p\in\mathcal{P}},\ L_p \text{ a lattice in } \mathbb{Q}_p^n,\ L_p = \mathbb{Z}_p^n \text{ for a.e. } p.$$

The converse of that map is the map

$$(L_p)_p \mapsto L = \bigcap_p \mathbb{Q}^n \cap L_p.$$

Proof. This is a consequence of the fact that $\mathbb{Z}$ is principal and of the chinese remainder theorem. □

2. Local actions on the space of lattices

The linear group $\mathrm{GL}_n(\mathbb{Q})$ acts transitively on $\mathcal{L}_n(\mathbb{Q})$ (on the *right*) by linear change of variable

$$L \mapsto L.g = \{g^{-1}x,\ x \in L\}$$

and the stabilizer of $\mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$ so that

$$\mathcal{L}_n(\mathbb{Q}) = \mathbb{Z}^n.\mathrm{GL}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{Q}).$$

Given p a prime, let $\mathcal{L}_n(\mathbb{Q}_p)$ be the set of lattices in $\mathbb{Q}_p^n$, similarly $\mathrm{GL}_n(\mathbb{Q}_p)$ acts transitively on $\mathcal{L}_n(\mathbb{Q}_p)$ and the stabilizer of $\mathbb{Z}_p^n$ is $\mathrm{GL}_n(\mathbb{Z}_p)$; it will be useful to let $\mathrm{GL}_n(\mathbb{Q}_p)$ act on $\mathcal{L}_n(\mathbb{Q}_p)$ on the *left*: for $g_p \in \mathrm{GL}_n(\mathbb{Q}_p)$, $g_p.L_p := g_pL_p$ so that

$$\mathcal{L}_n(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p).\mathbb{Z}_p^n \simeq \mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p).$$

2.1. The group of finite adeles of GL_n. From the previous discussion, the map $L \leftrightarrow (L_p)_p$ identify $\mathcal{L}_n(\mathbb{Q})$ with a subset of $\prod_p \mathcal{L}_n(\mathbb{Q}_p)$ namely the *restricted product*

$$\prod_p{}' \mathcal{L}_n(\mathbb{Q}_p) = \{(L_p)_p,\ L_p \in \mathcal{L}_n(\mathbb{Q}_p),\ L_p = \mathbb{Z}_p^n \text{ for a.e. } p\}$$

It follows from this discussion that $\mathcal{L}_n(\mathbb{Q})$ is acted on transitively by a subgroup of the product $\prod_p \mathrm{GL}_n(\mathbb{Q}_p)$ namely the *restricted product*

$$\mathrm{GL}_n(\mathbb{A}_f) = \prod_p{}' \mathrm{GL}_n(\mathbb{Q}_p) = \{(g_p)_p,\ g_p \in \mathrm{GL}_n(\mathbb{Z}_p) \text{ for a.e. } p\} \subset \prod_p \mathrm{GL}_n(\mathbb{Q}_p):$$

for $g_f = (g_p)_p \in \prod_p{}'\mathrm{GL}_n(\mathbb{Q}_p)$,

$$g_f.L = \bigcap_p \mathbb{Q}^n \cap g_pL_p;$$

thus, we see that this local description of the space rational lattices has revealed the existence of a much richer collection of actions by p-adic groups. The group $\mathrm{GL}_n(\mathbb{A}_f)$ is the group of finite adelic points of GL_n.

Notice that this action is compatible with the natural action of $\mathrm{GL}_n(\mathbb{Q})$, on $\mathcal{L}_n(\mathbb{Q})$: the group $\mathrm{GL}_n(\mathbb{Q})$ embed diagonally in $\mathrm{GL}_n(\mathbb{A}_f) = \prod_p{}'\mathrm{GL}_n(\mathbb{Q}_p)$ by considering for $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ the constant sequence

$$\gamma \mapsto (\gamma)_p = (\gamma, \gamma, \dots, \dots)_{p\in\mathcal{P}}$$

and for any lattice L,

$$L.\gamma = (\gamma)_p.L.$$

The stabilizer of $\mathbb{Z}^n$ (which correspond to the sequence $(\mathbb{Z}_p^n)_{p\in\mathcal{P}}$) is the product

$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) := \prod_p \mathrm{GL}_n(\mathbb{Z}_p),$$

so that

$$\mathcal{L}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}}).$$

Since $\mathrm{GL}_n(\mathbb{Z}) = \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\widehat{\mathbb{Z}})$ one finds that the map

$$g_{\mathbb{Q}} \in \mathrm{GL}_n(\mathbb{Q}) \to (g_{\mathbb{Q}}, Id) \in \mathrm{GL}_n(\mathbb{Q}) \times \mathrm{GL}_n(\mathbb{A}_f)$$

induces the identification

$$\mathcal{L}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{Q}) \times \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}})$$

with $\mathrm{GL}_n(\mathbb{Q})$ acting diagonally by left multiplications on $\mathrm{GL}_n(\mathbb{Q})\times\mathrm{GL}_n(\mathbb{A}_f)$. This interpretation can them be extended to real lattices: the map

$$g_{\mathbb{R}} \in \mathrm{GL}_n(\mathbb{Q}) \to (g_{\mathbb{R}}, Id) \in \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{A}_f)$$

induces the identification

$$\mathcal{L}_n(\mathbb{R}) \simeq \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R}) \simeq \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}})$$

with $\mathrm{GL}_n(\mathbb{Q})$ acting diagonally by left multiplications on $\mathrm{GL}_n(\mathbb{R})\times\mathrm{GL}_n(\mathbb{A}_f)$. The product

$$\mathrm{GL}_n(\mathbb{A}) := \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{A}_f)$$

is the group of adelic points of GL_n. In particular we see that for $\mathrm{GL}_n(\mathbb{Q})$ embedded diagonally in $\mathrm{GL}_n(\mathbb{A})$ one has

$$\mathrm{GL}_n(\mathbb{Q})\mathrm{GL}_n(\mathbb{R})\mathrm{GL}_n(\widehat{\mathbb{Z}}) = \mathrm{GL}_n(\mathbb{A}). \tag{2.1}$$

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \simeq Z(\mathbb{R})\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\widehat{\mathbb{Z}}). \tag{2.2}$$

2.2. Modular forms as functions on adelic spaces. We denote by $\mathcal{F}(\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$ the space of all functions on that quotient: this is an (abstract) representation of the group $\mathrm{GL}_2(\mathbb{A})$ acting by right multiplications

$$g.f : g' \to f(g'g)$$

and thus is endowed with an action of the subgroups $Z(\mathbb{R})$, $K_f = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $K_\infty = \mathrm{SO}_2(\mathbb{R})$. To resume what we have said so far we have identified classical modular forms of weight k with a certain set of functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ which are $Z(\mathbb{R})$-invariant, $K_f = \mathrm{GL}_2(\widehat{\mathbb{Z}})$-invariant and which transform as the character χ_k under the action of $K_\infty = \mathrm{SO}_2(\mathbb{R})$.

3. The ring of adeles

The ring of finite adeles is the restricted product

$$\mathbb{A}_f := \prod_p{}' \mathbb{Q}_p = \{(\lambda_p)_p,\ \lambda_p \in \mathbb{Q}_p,\ \lambda_p = \mathbb{Z}_p \text{ for a.e. } p\}$$

and the ring of adeles is the product

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f = \prod_v{}' \mathbb{Q}_v.$$

The field $\mathbb{Q}$ embeds diagonally into $\mathbb{A}_f$ and $\mathbb{A}$ via the constant sequences

$$\lambda \in \mathbb{Q} \to (\lambda)_p \in \mathbb{A}_f,\ (\lambda, (\lambda)_p) \in \mathbb{R} \times \mathbb{A}_f$$

making them $\mathbb{Q}$-algebras. We denote by

$$\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathbb{A}_f,$$

this is a subring of $\mathbb{A}_f$.

3.1. Ideles. The group of invertible elements $\mathbb{A}^\times$ (resp. $\mathbb{A}_f^\times$) is the restricted product

$$\mathbb{A}^\times = \mathbb{R}^\times \times \mathbb{A}_f^\times = \mathbb{R}^\times \times \prod_v{}' \mathbb{Q}_v^\times = \{(x_v)_v,\ x_v \in \mathbb{Q}_v^\times,\ x_p = \mathbb{Z}_p^\times \text{ for a.e. } p\},$$

and is called the group of ideles (resp. finite ideles).

3.2. The adelic topology. The rings $\mathbb{A}_f$ and $\mathbb{A}$ are restricted products of the locally compact rings $\mathbb{R}$ and $\mathbb{Q}_p$ with respect to the sequence of open compact subgroups $(\mathbb{Z}_p)_p$ and as such are endowed with the restricted product or *adelic* topology: a basis of open subsets is given by subsets of the shape

$$\Omega_\infty \times \prod_{p \in S} \Omega_p \times \prod_{p \notin S} \mathbb{Z}_p$$

where $S \subset \mathcal{P}$ is a finite subset of the primes and $\Omega_v \subset \mathbb{Q}_v$ are open subsets. With this topology $\mathbb{A}$ and $\mathbb{A}_f$ are locally compact separated topological rings and the embeddings $\mathbb{Q}_p \hookrightarrow \mathbb{A}_f$, $\mathbb{R} \hookrightarrow \mathbb{A}$, $\mathbb{A}_f \hookrightarrow \mathbb{A}$ are closed. Regarding the diagonal embedding of $\mathbb{Q}$ one has[1]

Lemma. *$\mathbb{Q}$ is dense in $\mathbb{A}_f$ (Strong approximation), $\mathbb{Q}$ is discrete in $\mathbb{A}$ but the quotient $\mathbb{Q}\backslash\mathbb{A}$ is compact; more precisely the map $x_\infty \in \mathbb{R} \to (x_\infty, (0)_p)$ induces an homeomorphism*

$$\mathbb{Z}\backslash\mathbb{R} \simeq \mathbb{Q}\backslash\mathbb{A}/\widehat{\mathbb{Z}}.$$

In particular, for any open compact subgroup $K_f \subset \mathbb{A}_f$ (for instance $\widehat{\mathbb{Z}}$) one has

$$\mathbb{Q} + K_f = \mathbb{A}_f,\ \mathbb{Q} + K_f + \mathbb{R} = \mathbb{A}.$$

Remark. The group of ideles is also endowed with a natural *adelic* topology making it a locally compact topological group; this topology however is not the restricted topology of the embedding $\mathbb{A}^\times \subset \mathbb{A}$.

[1] as a consequence of the chinese remainder theorem

4. Adelic points on algebraic groups

If $V_{\mathbb{Q}}$ is any variety defined over $\mathbb{Q}$, one may speak of $V(\mathbb{A})$ the set of points of V with value in the $\mathbb{Q}$-algebra $\mathbb{A}$. More concretely if $V \subset A^n_{\mathbb{Q}}$ is an affine variety defined by the vanishing of polynomial equations with coefficients in $\mathbb{Q}$, $V(\mathbb{A})$ is the subset of elements of $A^n_{\mathbb{Q}}(\mathbb{A}) = \mathbb{A}^n$ at which the polynomial defining V vanish. One has

$$V(\mathbb{A}) = \prod_v{}' V(\mathbb{Q}_v) = \{(x_v)_v,\ x_v \in V(\mathbb{Q}_v),\ x_p \in \mathbb{Z}_p^n \text{ for a.e. } p\}.$$

$V(\mathbb{A})$ is a closed subset of $\mathbb{A}^n$ and thus is equipped with the restricted adelic topology . We will use this for G a linear algebraic group: recall that GL_n is the affine subvariety of $M_n \times A^1 \simeq A^{n^2+1}$ defined as

$$\{(g,t) \in M_n \times A^1,\ \det(g)t - 1 = 0\}.$$

We obtain in that way $\mathrm{GL}_n(\mathbb{A})$ which come equipped with the adelic topology, making of it a separated locally compact group.

Remark. In particular, for $n = 1$, $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$ is the group of ideles and the topology defined here is finer that the topology obtained by restriction of the inclusion $\mathbb{A}^\times \subset \mathbb{A}$.

Similarly, for any Zariski closed subgroup of matrices $\mathrm{G} \subset \mathrm{GL}_n$ one defines $\mathrm{G}(\mathbb{A})$ as the set of elements of $\mathrm{GL}_n(\mathbb{A})$ satisfying the equations defining G; this is a closed subgroup of $\mathrm{GL}_n(\mathbb{A})$ and we equip it with the adelic topology; it maybe be shown that this topology does not depend on the realization of G as a closed subgroup of some linear group. The groups $\mathrm{G}(\mathbb{Q}_p)$ and $\mathrm{G}(\mathbb{A}_f)$ embed as closed subgroups of $\mathrm{G}(\mathbb{A})$ via:

$$g_p \in \mathrm{G}(\mathbb{Q}_p) \mapsto [g_p] := (\mathrm{Id}_{\mathbb{R}}, \dots, g_p, \dots, \mathrm{Id}_q, \dots) \in \mathrm{G}(\mathbb{A}_f),$$

$$g_f \in \mathrm{G}(\mathbb{A}_f) \mapsto (\mathrm{Id}_{\mathbb{R}}, g_f) \in \mathrm{G}(\mathbb{A}).$$

The group $\mathrm{G}(\mathbb{Q})$ embeds diagonally into $\mathrm{G}(\mathbb{A}_f)$ and $\mathrm{G}(\mathbb{A})$, via[2]

$$g_{\mathbb{Q}} \mapsto (g_{\mathbb{Q}})_p,\ g_{\mathbb{Q}} \mapsto (g_{\mathbb{Q}}, (g_{\mathbb{Q}})_p)$$

and the image of later embedding is discrete in $\mathrm{G}(\mathbb{A})$.

The group $\mathrm{G}(\mathbb{Z}_p) = \mathrm{G}(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p)$ is an open compact subgroup of $\mathrm{G}(\mathbb{Q}_p)$ and $\mathrm{G}(\widehat{\mathbb{Z}}) = \mathrm{G}(\mathbb{A}_f) \cap \mathrm{GL}_n(\widehat{\mathbb{Z}})$ is an open compact subgroup of $\mathrm{G}(\mathbb{A}_f)$; moreover one can also show that if one has a $\mathbb{Q}$-isomorphism of linear algebraic groups $\iota : \mathrm{G} \simeq \mathrm{G}'$ then for almost every p this induces an isomorphism between the open compact subgroups $\iota : \mathrm{G}(\mathbb{Z}_p) \simeq \mathrm{G}'(\mathbb{Z}_p)$ and therefore we obtain an homeomorphism between the adelic points of the two groups. In other terms, the adelic topology on $\mathrm{G}(\mathbb{A})$ is independent of the matrix realization of G.

[2] notice the the embedding of $\mathrm{G}(\mathbb{Q})$ into $\mathrm{G}(\mathbb{A})$ is NOT the composite of the embedding $\mathrm{G}(\mathbb{Q}) \hookrightarrow \mathrm{G}(\mathbb{A}_f)$ and of $\mathrm{G}(\mathbb{A}_f) \hookrightarrow \mathrm{G}(\mathbb{A})$.

Finally, if G is a reductive group then for ae. p, $\mathrm{G}(\mathbb{Z}_p)$ is a *maximal* open compact subgroup[3].

4.1. The components of an adelic group. As we have already seen in the case of the linear group and space of lattices, many interesting homogenoues spaces associated with linear algebraic group may be realized in terms of quotient of adelic groups and more precisely in terms of double cosets of the form

$$\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{G}(\widehat{\mathbb{Z}}).$$

We review this connection in the present section.

Since $\mathrm{G}(\mathbb{Q})$ is discrete in $\mathrm{G}(\mathbb{A})$, the quotient $\mathrm{G}(\mathbb{Q}) \subset \mathrm{G}(\mathbb{A})$ endowed with the quotient topology is locally homeomorphic to $\mathrm{G}(\mathbb{A})$. This quotient is also naturally identified with $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{Q})\mathrm{G}(\mathbb{A})$, that is the $\mathrm{G}(\mathbb{A})$-orbit the the identity class in $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$:

Unlike the case of the adèles, is not empty, $\mathrm{G}(\mathbb{Q})$ is usually not dense in $\mathrm{G}(\mathbb{A}_f)$. Still it is quite big. We have the following ([?borelfin, Theorem 5.1])

Theorem (Borel)**.** *The double quotient*

$$\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{G}(\mathbb{R})\mathrm{G}(\widehat{\mathbb{Z}})$$

is finite. The same conclusion holds more generally if $\mathrm{G}(\widehat{\mathbb{Z}})$ *is replaced by any open compact subgroup* $K_f \subset \mathrm{G}(\mathbb{A}_f)$.

We will admit this deep result. Notice that the second statement follows from the first. Indeed the two open compact subgroups $\mathrm{G}(\widehat{\mathbb{Z}})$ and K_f are commensurable: $K_f \cap \mathrm{G}(\widehat{\mathbb{Z}})$ is of finite index in both $\mathrm{G}(\widehat{\mathbb{Z}})$ and K_f.

Observe also that the inclusion $\mathrm{G}(\mathbb{A}_f) \subset \mathrm{G}(\mathbb{A})$ induces an identification

$$\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}_f)/K_f \simeq \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{G}(\mathbb{R})K_f.$$

Definition 4.1. *Given* $K_f \subset \mathrm{G}(\mathbb{A}_f)$ *an open compact subgroup, the double quotient*

$$\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{G}(\mathbb{R})\mathrm{G}(\widehat{\mathbb{Z}}) \simeq \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}_f)/\mathrm{G}(\widehat{\mathbb{Z}})$$

is called the set of components *of* K_f*. Its cardinality is called the* class number *of* K_f *and will be noted* $h(K_f)$*. If the embedding* $\mathrm{G} \hookrightarrow \mathrm{GL}_n$ *is given, the set of components (reps. class number) of* $\mathrm{G}(\widehat{\mathbb{Z}})$ *will be called simply the set of components (resp. class number) of* G.

Example. By (2.1) the class number of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ is 1.

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[3]This is even an hyperspecial maximal open compact subgroup

4.2. Adelic vs. classical spaces. we now show how to interpret certain classical "congruence" quotients of $\mathrm{G}(\mathbb{R})$ in terms of adelic quotients: consider a set of representatives of the components,

$$\mathrm{G}(\mathbb{A}) = \bigsqcup_i \mathrm{G}(\mathbb{Q})g_i\mathrm{G}(\mathbb{R})K_f,\ g_i \in \mathrm{G}(\mathbb{A})$$

writing $g_i = (g_{\mathbb{R},i}, g_{f,i})$ we may up to multiplying by $g_{\mathbb{R},i}^{-1}$ assume that $g_i = g_{f,i} \in \mathrm{G}(\mathbb{A}_f)$. Let

$$\Gamma_i = \mathrm{GL}_n(\mathbb{Q}) \cap g_{f,i}K_f g_{f,i}^{-1}$$

this subgroup is commensurable with $\mathrm{G}(\mathbb{Z})$ and hence discrete in $\mathrm{G}(\mathbb{R})$.

Proposition 4.1. *With the above notation, the family of maps*

$$\iota_i :\ g_{\mathbb{R}} \in \mathrm{G}(\mathbb{R}) \mapsto (g_{\mathbb{R}}, g_{f,i}),\ i = 1 \dots h(K_f)$$

induces an homeomorphism

$$\bigsqcup_i \Gamma_i\backslash\mathrm{G}(\mathbb{R}) \simeq \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/K_f$$

PROOF. For $g \in \mathrm{G}(\mathbb{A})$, set $[g] = \mathrm{G}(\mathbb{Q})gK_f$; consider the map given by

$$(i, \Gamma_i g_{\mathbb{R}}) \mapsto [(g_{\mathbb{R}}, g_{f,i})];$$

this map is well defined: if we replace $g_{\mathbb{R}}$ by $\gamma g_{\mathbb{R}}$ with $\gamma \in \Gamma_i$, then

$$[\gamma g_{\mathbb{R}}, g_{f,i}] = [(g_{\mathbb{R}}, \gamma^{-1}g_{f,i}] = [(g_{\mathbb{R}}, g_{f,i}k_f g_{f,i}^{-1}g_{f,i})] = [(g_{\mathbb{R}}, g_{f,i})].$$

That map is clearly continuous and closed; let us show it is bijective: if $[(g_{\mathbb{R}}, g_{f,i})] = [(g'_{\mathbb{R}}, g_{f,i'})]$ then

$$(g'_{\mathbb{R}}, g_{f,i'}) = (\gamma g_{\mathbb{R}}, \gamma g_{f,i}k_f)$$

impling that $i = i'$ and $g_{f,i} = g_{f,i'}$ and that $\gamma \in \Gamma_i$; this proves injectiveity. It is also surjective: given $g \in \mathrm{G}(\mathbb{A})$, $g = g_{\mathbb{R}}g_f$ and there is by definition i and $\gamma \in \mathrm{G}(\mathbb{Q})$ and $k_f \in K_f$ such that $g_f = \gamma g_{f,i}k_f$ and then

$$[g] = [g_{\mathbb{R}}, \gamma g_{f,i}k_f] = [\gamma^{-1}g_{\mathbb{R}}, g_{f,i}].$$

□

4.3. The G-genus of a lattices. As we have already seen there is a natural action of $\mathrm{GL}_n(\mathbb{A}_f)$ on the space of rational lattices

$$\mathcal{L}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}}).$$

Restricting this action to $\mathrm{G}(\mathbb{A}_f)$ we obtain the notion of G-genus:

Definition. *The* G*-genus of a rational lattice* $L \in \mathcal{L}_n(\mathbb{Q})$, $\mathrm{gen}_{\mathrm{G}}(L)$ *is the adelic orbit,*

$$\mathrm{gen}_{\mathrm{G}}(L) := \mathrm{G}(\mathbb{A}_f).L \subset \mathcal{L}_n(\mathbb{Q}) \simeq \mathrm{GL}_n(\mathbb{A}_f)/\mathrm{GL}_n(\widehat{\mathbb{Z}}).$$

The set of genus classes of L, $[\mathrm{gen}(L)]$, *is the set of* $\mathrm{G}(\mathbb{Q})$*-orbits in* $\mathrm{gen}_{\mathrm{G}}(L)$. *In particular* $[\mathrm{gen}(\mathbb{Z}^n)]$ *is identified with the set of component of* G*:*

$$[\mathrm{gen}_{\mathrm{G}}(\mathbb{Z}^n)] \simeq \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}_f)/\mathrm{G}(\widehat{\mathbb{Z}}).$$

4.4. Strong approximation. The property (2.1) does not hold for general open compact subgroups of $K_f \subset \mathrm{GL}_n(\mathbb{A}_f)$ or for general linear algebraic groups.

The strong approximation theorem which we do not state here in the greatest generality yields a sufficient condition on G so that the class number is 1 for any open compact subgroup. We have seen that strong approximation holds for $\mathbb{A} \simeq N(\mathbb{A}) < \mathrm{GL}_2(\mathbb{A})$.

Theorem 4.1 (Strong approximation)**.** *For* G *a semisimple, simply connected, algebraic group such that for each simple factor of* G, *it group of real points is not compact, then* $\mathrm{G}(\mathbb{Q})\mathrm{G}(\mathbb{R})$ *is dense in* $\mathrm{G}(\mathbb{A})$. *In particular for any open compact subgroup* $K_f < \mathrm{G}(\mathbb{A}_f)$ *one has*

$$\mathrm{G}(\mathbb{A}) = \mathrm{G}(\mathbb{Q})\mathrm{G}(\mathbb{R})K_f,\ \Gamma\backslash \mathrm{G}(\mathbb{R}) \simeq \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})/K_f$$

where $\Gamma = \mathrm{G}(\mathbb{Q}) \cap K_f$

the general proof is due basically to Kneser. A case where this is not too hard to prove is for SL_n; in particular, for any open compact $K_f^1 \subset \mathrm{SL}_n(\widehat{\mathbb{Z}})$, one has

$$\mathrm{SL}_n(\mathbb{Q})\mathrm{SL}_n(\mathbb{R})K_f^1 = \mathrm{SL}_n(\mathbb{A}).$$

This results from the fact that SL_n is generated by the one parameters subgroups (of transvections)

$$\{\mathrm{Id} + tE_{i,j}\} \simeq A^1_{|\mathbb{Q}},\ i \neq j$$

and by using strong approximation for $\mathbb{A}$.

In particular if $K_f \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is such that $\det(K_f) = \widehat{\mathbb{Z}}^\times$, then since $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}^\times\widehat{\mathbb{Z}}^\times$

$$\mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\mathbb{R})K_f = \mathrm{GL}_2(\mathbb{A}),\ \Gamma\backslash \mathrm{GL}_2(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_f$$

in particular if $\Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap K_f \subset \mathrm{GL}_2(\mathbb{Z})$, one has

$$\Gamma\backslash \mathrm{GL}_2(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_f.$$

5. Measure on adelic groups

The group of adelic points of a linear group $\mathrm{G}(\mathbb{A})$ is a locally compact topological group and as such is equipped with a (left) Haar measure. That group is the union of the following open subsets

$$\mathrm{G}(\mathbb{A}) = \bigcup_{\infty \subset S\ finite} \mathrm{G}(\mathbb{A}(S)),\ \mathrm{G}(\mathbb{A}(S)) = \prod_{v \in S} \mathrm{G}(\mathbb{Q}_v) \prod_{p \notin S} \mathrm{G}(\mathbb{Z}_p).$$

Let μ_∞ be a (left) Haar measure on $G(\mathbb{R})$ and for every p let μ_p be the left Haar measure on $\mathrm{G}(\mathbb{Q}_p)$ normalized so that

$$\mu_p(\mathrm{G}(\mathbb{Z}_p)) = 1.$$

Then the measure

$$\mu = \prod_v \mu_v$$

define a Haar measure on each $\mathrm{G}(\mathbb{A}(S)) = \prod_{v\in S} \mathrm{G}(\mathbb{Q}_v) \prod_{p\notin S} \mathrm{G}(\mathbb{Z}_p)$ compatible with the inclusion $\mathrm{G}(\mathbb{A}(S)) \subset \mathrm{G}(\mathbb{A}(S'))$, $S \subset S'$ and so defines a (left)-Haar measure on $\mathrm{G}(\mathbb{A})$. For instance, a Haar measure on $\mathbb{A}$ is given by

$$dx = dx_{\mathbb{R}} \prod_p dx_p$$

where $dx_{\mathbb{R}}$ is the Legesgue measure and dx_p is the measure on $\mathbb{Q}_p$ assigning mass 1 to the into ball $\mathbb{Z}_p$.

Remark. Suppose that $\mathrm{G}(\mathbb{A})$ is unimodular; one can verify that the homeomorphism defined in Proposition 4.1 map the measure on

$$\bigsqcup_i \Gamma_i\backslash\mathrm{G}(\mathbb{R})$$

defined by the choice of a single Haar measure on $\mathrm{G}(\mathbb{R})$ to the image of some left Haar measure on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/K_f$

5.1. The Module. To determine a right Haar measure from the left one, we need to compute the module

$$\Delta_{\mathrm{G}} = \Delta_{\mathrm{G}(\mathbb{A})}.$$

For this we need first to introduce the *adelic module*: the adelic module is the character

$$x \in \mathbb{A} \mapsto |x|_{\mathbb{A}} = \prod_v |x_v|_v \in \mathbb{R}_{\geq 0}.$$

It is zero on $\mathbb{A} - \mathbb{A}^\times$ for any $x \in \mathbb{Q}$

$$|x|_{\mathbb{A}} = |x|_{\mathbb{R}} \prod_p p^{-v_{p(x)}} = |x||x|^{-1} = 1.$$

Clearly $|xy|_{\mathbb{A}} = |x|_{\mathbb{A}}|y|_{\mathbb{A}}$ so the adelic modulus defines an $\mathbb{R}$-valued character on $\mathbb{A}^\times/\mathbb{Q}^\times$. The modulus may be defined more intrinsiquely from any Haar measure on $\mathbb{A}$, for instance For any $\alpha \in \mathbb{A}^\times$, th map

$$x \mapsto \alpha x$$

is a linear homeomorphism of $\mathbb{A}$ and one can see that

$$d\alpha x = |\alpha|_{\mathbb{A}} dx.$$

In particular

$$\mathrm{d}^\times x := |x|_{\mathbb{A}}^{-1} dx.$$

defines a Haar measure on $\mathbb{A}^\times$.

Let G be a linear algebraic group. We denote by $\mathrm{R}_u(\mathrm{G})$ its *unipotent radical* (the maximal normal unipotent subgroup of G), so that $G = \mathrm{R}_u(\mathrm{G}).\mathrm{G}'$ with G' reductive, and by $\mathfrak{u}$ the Lie algebra of $\mathrm{R}_u(\mathrm{G})$, then the modulus of $\mathrm{G}(\mathbb{A})$ is given by the usual formula

$$\Delta_{\mathrm{G}}(g) = |\det(\mathrm{Ad}(g)|\mathfrak{u}(\mathbb{A}))|_{\mathbb{A}}^{-1}$$

where $\mathrm{Ad}(g)$ denote the action of g by conjugation on $\mathfrak{u}(\mathbb{A}) = \mathfrak{u} \otimes_{\mathbb{Q}} \mathbb{A}$: $u \mapsto gug^{-1}$. In particular, if G is reductive, $\Delta_{\mathrm{G}} = 1$ and $\mathrm{G}(\mathbb{A})$ is unimodular.

5.2. The Borel–Harish-Chandra finiteness theorem. The group of rational points $\mathrm{G}(\mathbb{Q})$ is discrete in $\mathrm{G}(\mathbb{A})$ and by the formula above contained in the kernel of Δ_G; hence any left invariant Haar measure on $\mathrm{G}(\mathbb{A})$ induces a measure on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})$. That measure in not finite in general: for instance consider $\mathrm{G} = \mathrm{GL}_1 = \mathbb{G}_m$, one has

$$\mathbb{Q}^\times\backslash\mathbb{A}^\times/\widehat{\mathbb{Z}}^\times \simeq \mathbb{R}^\times/\mathbb{Z}^\times \simeq \mathbb{R}_{>0}$$

has not finite Haar measure. This type of obstruction is basically the only one to prevent the Haar measure to be finite.

Let $X^*_{\mathbb{Q}}(\mathrm{G})$ be the lattice of rational characters of G (the group of homomorphisms $\chi : \mathrm{G} \mapsto \mathbb{G}_m$ which are defined over $\mathbb{Q}$). Each such homomorphism induces a map on the adelic points

$$\chi : g \in \mathrm{G}(\mathbb{A}) \mapsto \chi(g) \in \mathbb{A}^\times = \mathbb{G}_m(\mathbb{A})$$

and composing with the modulus, one obtain an $\mathbb{R}$-valued character

$$|\chi|_{\mathbb{A}} : g \mapsto |\chi(g)|_{\mathbb{A}}.$$

We then define the following closed subgroup of $\mathrm{G}(\mathbb{A})$

$$\mathrm{G}(\mathbb{A})^1 = \bigcap_{\chi \in X^*_{\mathbb{Q}}(\mathrm{G})} \ker(|\chi|_{\mathbb{A}})$$

(this is a finite intersection $X^*_{\mathbb{Q}}(\mathrm{G})$ is a free $\mathbb{Z}$-module of rank the $\mathbb{Q}$-rank of G and it is sufficient to consider the intersection over a $\mathbb{Z}$-basis of $X^*_{\mathbb{Q}}(\mathrm{G})$). Obviously a compact subgroup of $\mathrm{G}(\mathbb{A})$ is contained in $\mathrm{G}(\mathbb{A})^1$. Also, since the characters χ restricted to $\mathrm{G}(\mathbb{Q})$ take value in $\mathbb{Q}^\times$, one has

$$\mathrm{G}(\mathbb{Q}) \subset \mathrm{G}(\mathbb{A})^1$$

and $\mathrm{G}(\mathbb{Q})$ is discrete in $\mathrm{G}(\mathbb{A})^1$. We have then the following deep finiteness theorem of Borel-Harish-Chandra which complement Borel finiteness theorem:

Theorem 5.1 (Borel-Harish-Chandra)**.** *The quotient (Haar) measure on* $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ *is finite. In particular, if* G *has no* $\mathbb{Q}$*-character the Haar measure on* $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})$ *is finite. Moreover, if every unipotent element of* $\mathrm{G}(\mathbb{Q})$ *is contained in the unipotent radical* $\mathrm{R}_u(\mathrm{G})$*, then* $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ *is even compact.*

This implies that $\mathrm{G}(\mathbb{A})^1$ is unimodular (since it has a discrete subgroup of finite covolume), hence $\mathrm{G}(\mathbb{A})$ is also unimodular. For instance $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})^1$ has finite volume and

$$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})^1/\mathrm{GL}_n(\widehat{\mathbb{Z}}) \simeq \mathrm{GL}_n(\mathbb{Z})\backslash\mathrm{GL}_n(\mathbb{R})^1 \simeq \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}) \simeq \mathcal{L}(\mathbb{R})^1$$

the space of lattices of covolume 1.

Also, notice the special cases:

– if G is reductive ($\mathrm{R}_u(\mathrm{G})$ is trivial) then $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})^1$ is compact if and only if $\mathrm{G}(\mathbb{Q})$ has no unipotent elements,

We will use the Borel-Harish-Chandra theorem in the following form: let Z be the split component of the center of G: this is the maximal split $\mathbb{Q}$-torus of the center of G.

Proposition. *The measure of* $Z(\mathbb{A})\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})$ *is finite.*