The world's easiest proof of a version of Selberg's 3/16 thm

Refs 1) "Frobenius ought to be useful"
private discussion of Bernstein & Sarnak, many years ago

2) Sarnak-Xue

3) Gamburd's thesis [Israel 2002]
mostly section 5-6

Thm [Selberg]

let $\lambda_1(N)$ denote the smallest non-trivial eigenvalue of Δ on $\Gamma(N)\backslash\mathbb{H}$ where
$\Gamma(N)=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\equiv I \pmod N\}$.
Then $\lambda_1(N)\geq\frac{3}{16}$ for all N.

We only prove the case $N=p$ (argument extends to general case) & $\lambda_1(p)\geq\frac{5}{36}-\varepsilon$ (with finitely many possible exceptions).

- it fits into this course b/c of the way we prove it.

What we need from $\Gamma(p)$:

Prop. 1 $|\Gamma(p) \cap B_R| \ll_\varepsilon \frac{R^{2+\varepsilon}}{p^3} + \frac{R^{1+\varepsilon}}{p} + 1$

$$B_R = \{\gamma \mid a^2+b^2+c^2+d^2 < R^2\}$$

Pf
$$\begin{cases} |a|,|b|,|c|,|d| \le R \\ ad-bc=1 \\ a \equiv d \equiv 1 \pmod p \\ b \equiv c \equiv 0 \pmod p \end{cases}$$

$\Rightarrow$ • $O\left(\frac{R}{p}\right)+1$ choices of a

Claim $a+d \equiv 2 \pmod{p^2}$

$$a = 1+\tilde{a},\ d = 1+\tilde{d}$$

$$1 = ad-bc = (1+\tilde{a})(1+\tilde{d}) - bc = 1 + \underbrace{\tilde{a}+\tilde{d}}_{\substack{p^2 \mid \\ \text{also}}} + \underbrace{\tilde{a}\tilde{d}-bc}_{\in p^2 \mid}$$

$\Rightarrow$ • $O\left(\frac{R}{p^2}\right)+1$ choices of d once a is given

$\Rightarrow$ • $O(R^\varepsilon)$ choices of b & c once a, d and so $\underbrace{ad-1}_{\text{of size } R^2 \text{, divisor fct.}} = bc$ is known.

in total $\ll \left(\frac{R}{p}+1\right)\left(\frac{R}{p^2}+1\right)R^\varepsilon \ll \frac{R^{2+\varepsilon}}{p^3} + \frac{R^{1+\varepsilon}}{p} + 1$

last case: $ad=1 \Rightarrow b=0$ or $c=0$ & in each case $\frac{R}{p}$ possibilities for the other $\square$

What we need about multiplicities:

Frobenius: The smallest non-trivial rep of $Sl_2(\mathbb{F}_p)$ (of size $\approx p^3$) is of dimension $\frac{p-1}{2}$.

Corollary 2

Suppose
$$\begin{array}{c} M_p = \Gamma'(p) \backslash \mathbb{H} \\ \downarrow \\ M = \Gamma' \backslash \mathbb{H} \end{array}$$
is the cover of M corresp. to $\Gamma'(p)$. If Δ has a new eigenvalue on M_p, say λ, then $Sl_2(\mathbb{F}_p)$ acts nontrivially on the eigenspace and so the eigenspace has dimension $\geq \frac{p-1}{2}$.

What we need from spectral theory:

Everything in between to establish the connection between the lattice point count and eigenvalues. – This will include some black boxes.

Point-pair-invariant & Convolution

$k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$

is a point-pair-invariant if

$$k(gz, gw) = k(z,w)$$

i.e. if it only depends on their distance

If k allows we can associate to it the "automorphic kernel"

$$K(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w)$$

now a function on $(\Gamma \backslash \mathbb{H}) \times (\Gamma \backslash \mathbb{H})$

Def. k_1, k_2 as above

$$k_1 * k_2 (z,w) = \int_{\mathbb{H}} k_1(z,x) k_2(x,w) \, dm(x)$$

is the convolution

Recall $\cosh d(z,w) = 1 + \underbrace{\frac{|z-w|^2}{4 \operatorname{Im} z \operatorname{Im} w}}_{=: u(z,w)}$

Set $\cosh T = X/2$

Define $k_1(z,w) = \begin{cases} 1 & d(z,w) \leq T \\ 0 & \text{otherwise} \end{cases}$

$\boxed{\text{Fix } p}$ $\qquad = \begin{cases} 1 & u(z,w) \leq \frac{X-2}{4} \\ 0 & \text{otherwise} \end{cases}$

$$K_1(z,w) = \sum_{\gamma \in \Gamma^1(p)} k_1(z, \gamma w) = \#\{\gamma \in \Gamma^1(p) : 4u(z,\gamma w) + 2 \leq X\}$$

for $z = w = i$

$$K_1(i,i) = \#\{\gamma \in \Gamma^1(p) : 4u(i, \gamma(i)) + 2 \leq X\}$$

$$\frac{\left|i - \frac{ai+b}{ci+d}\right|^2}{\frac{1}{|ci+d|^2}} + 2 = a^2+b^2+c^2+d^2$$

$$= \#\{\gamma \in \Gamma^1(p) : a^2+b^2+c^2+d^2 \leq X\}$$

To have positivity on the spectral side, we have to work with

$$k = k_1 * k_1$$

and $K = \sum_\gamma k(\cdot, \gamma \cdot)$

instead of with k_1 / K_1.

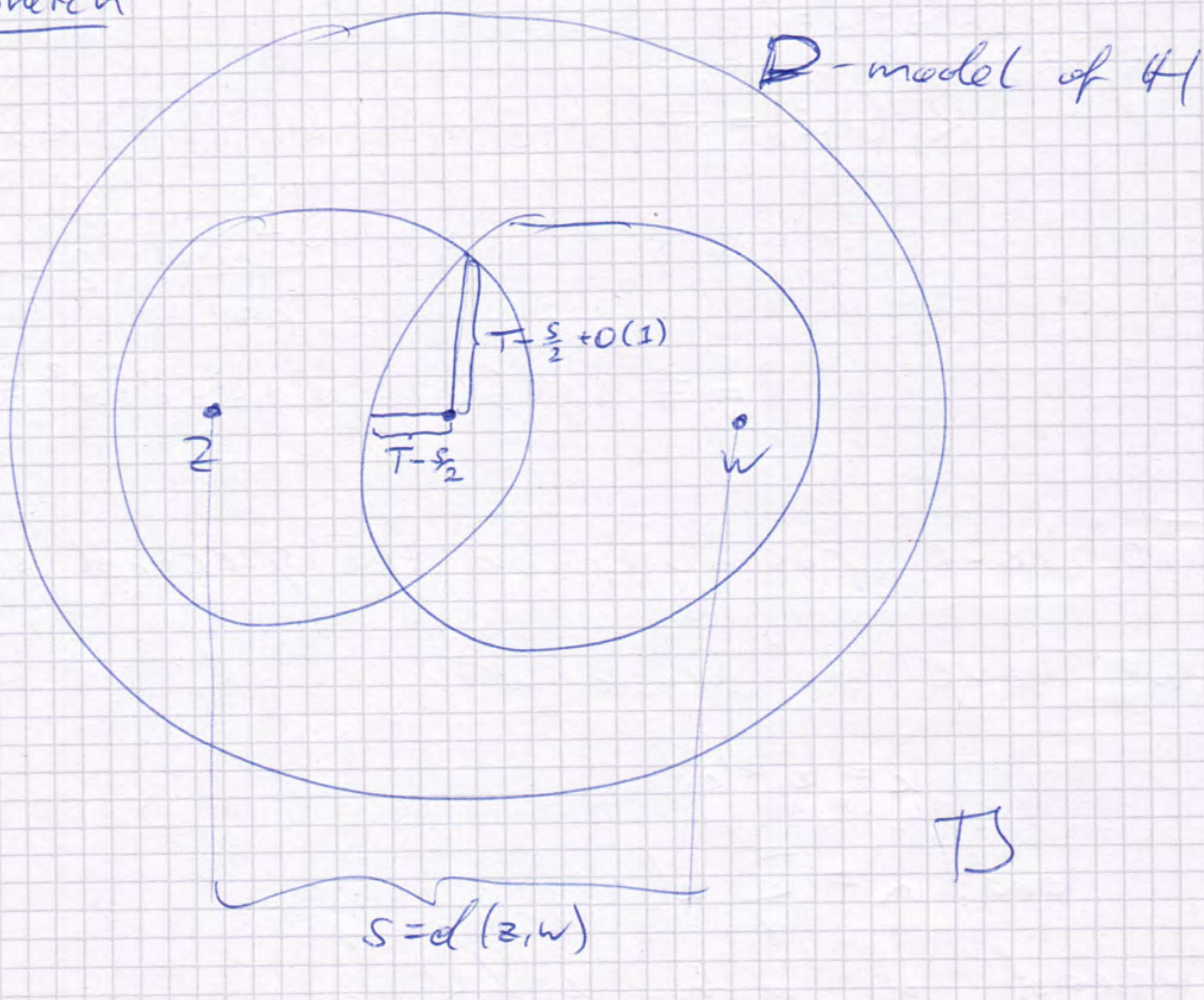
<u>Prop 3</u> $k(z,w)$ = measure of intersection of T-balls if their centers are z and w

$$= \begin{cases} O(e^{T-d(z,w)/2}) & \text{if } d(z,w) < 2T \\ 0 & \text{if } d(z,w) \geq 2T \end{cases}$$

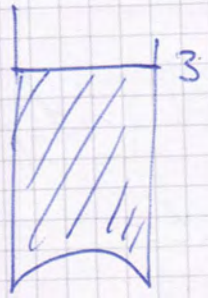
Also $k(z,w) \gg e^{T-d(z,w)/2}$ if $d(z,w) \leq 2T-1$

Heuristically this is expected if one compares $\mathbb{H}$ to a regular tree.

<u>Pf-Sketch</u>



$\square$

Let $\Omega =$  be the compact part of $\Gamma' \backslash \mathbb{H} = M_0$

And let $\Omega(p) = \bigsqcup_{\gamma \in \Gamma(p) \backslash SL_2(\mathbb{Z})} \gamma \Omega$ be the "compact" part of M_p

<u>Main estimate, part I</u>

$$\int_{\Omega(p)} K(z,z)\,dm = \sum_{\gamma \in \Gamma'(p)} \int_{\Omega(p)} k(z,\gamma z)\,dm$$

$$= \sum_{\gamma \in \Gamma'(p)} \sum_{\delta \in \Gamma'/\Gamma'_p} \int_{\Omega} \underbrace{k(\delta^{-1} z, \gamma \delta^{-1} z)}_{=\,k(z,\,\delta\gamma\delta^{-1} z)}\,dm$$

$$\ll \underset{\substack{\uparrow \\ \text{size of} \\ SL_2(\mathbb{F}_p)}}{p^3} \underbrace{\sum_{\gamma \in \Gamma'(p)} \int_{\Omega} k(z,\gamma z)\,dm}_{\int_{\Omega} K(z,z)\,dm}$$

$\Rightarrow$ need upper bd of $\int_{\Omega} K(z,z)\,dm$ coming from our upper bd of lattice pt. count.

Note Ω is compact (indep. of p) and so $\exists\ \sigma > 1$ s.t.

$$K_{1,x/\sigma}(i,i) \le K_{1,x}(z,z) \le K_{1,\sigma x}(i,i)$$

Ω is bd. – a γ that is included in the count of $K_{1,x/\sigma}(i,i)$ will also be included in $K_{1,x}(z,z)$, σ needs to take into account the diameter of Ω. (Recall $\cosh T = x/2$)

<u>Prop 4</u> $$\int_{\Omega(p)} K_x(z,z)\,dm \ll p^3 K_{\sigma x}(i,i)$$

<u>Pf</u> need $K(z,z) \ll K_{\sigma x}(i,i)$ for $z \in \Omega$ only

$$K_x(z,z) = \sum_{\substack{\gamma \in \Gamma_p \\ d(z,\gamma z) < 2T}} k_x(z,\gamma z)$$

let $d = \operatorname{diam} \Omega$, then $d(i,\gamma i) \le d(z,\gamma z) + 2d$
$\le d(i,\gamma i) + 4d$

Also $k_x(z,\gamma z) \ll k_{\sigma x}(i,\gamma i)$
b/c of Prop. 3. $\square$

Upper bound conclusion

Prop. 5 $\displaystyle\int_{\Omega(p)} K_X(z,z)\, dm \ll_\varepsilon X^{2+\varepsilon} + p^3 X^{1+\varepsilon}$

Pf. $$K_X(i,i) = \sum_{\gamma\in\Gamma'(p)} k(i,\gamma i)$$

$$\ll \sum_{n\le 2T} \sum_{d(i,\gamma i)\in[n-1,n]} k(i,\gamma i)$$

Prop 3 — lattice ct./Prop 1 — $R=\sqrt{X}\approx e^{n/2}$

$$\ll \sum_{n\le 2T} e^{T-\frac{n}{2}}\left(\frac{e^{n(1+\varepsilon)}}{p^3} + \frac{e^{\frac{n}{2}(1+\varepsilon)}}{p} + 1\right)$$

$$\ll e^T \sum_{n\le 2T}\left(\frac{e^{\frac{n}{2}(1+\varepsilon)}}{p^3} + \underbrace{\frac{e^{\frac{n}{2}\varepsilon}}{p} + e^{-\frac{n}{2}}}_{\ll e^{\varepsilon n}}\right)$$

$$\ll e^T\left(\frac{e^{\frac{2T}{2}(1+\varepsilon)}}{p^3} + e^{2\varepsilon T}\right)$$

$$\ll \frac{X^{2+\varepsilon}}{p^3} + X^{1+2\varepsilon}$$

Now combine with Prop 4 to get the proposition

$\square$

Towards lower bd.

Thm6 (Selberg)

Suppose $\Delta\phi = \lambda\phi$

$\Rightarrow$ ϕ is also an eigenfct. of the integral op. def. by a point-pair-invariant & the eigenvalue $h(\lambda)$ only depends on λ, i.e.

$$\int_{\mathbb{H}} k(x,y)\,\phi(y)\,dy = h(\lambda)\,\phi(x)$$

spherical / Selberg, Harish-Chandra transform

for k_1 this function is

$$h_1(\lambda) = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\,X^s + O(X^{1/2})$$

for $\lambda = s(s-1)$

$\frac{1}{2} < s \leq 1$

$\lambda \in [0, \frac{1}{4})$

Another fact

<u>Prop 8</u> ϕ a cusp form with eigenv. λ

$\lambda = s(1-s)$
$s \in (s_0, 1)$
$s_0 > \frac{1}{2}$

$$\int_Y^{2Y} \int_0^q |\phi(z)|^2 \, dm(x,y) \geq C_{s_0} \int_Y^{\infty} \int_0^q |\phi(z)|^2 \, dm$$

If you believe all of that, then we get ~~an upper~~ a lower bound

<u>Cor</u>

$$\int_{\Omega(p)} K(z,z) \, dm \geq \sum_{\lambda_j < \frac{1}{4}} |h_1(\lambda_j)|^2 \int_{\Omega(p)} |\phi_j|^2(z) \, dm$$

$$\gg \sum_{\lambda_j < s_0(1-s_0)} X^{2s_j} \underbrace{\int_{M_p} |\phi_j|^2 \, dm}_{=1}$$

for $k = k_1 * k_1$ the transform is

$$h(\lambda) = |h_1(\lambda)|^2$$

<u>Prop.</u> 7 (Pretrace Inequality)

$$K(z,z) \geq \sum_{\lambda_j < \frac{1}{4}} |h_1(\lambda_j)|^2 \, |\phi_j(z)|^2$$

L^2-normalized

<u>Pf.</u> The operator def. by

$$B(z,w) = K(z,w) - \sum_{\lambda_j < \frac{1}{4}} h(\lambda_j)\, \phi_j(z)\, \overline{\phi_j(z)}$$

is positive. In fact $K(z,w)$ is positive since it is defined by a self-convolution & we only subtract part of the eigenspaces to get $B(z,w)$.

$\Rightarrow \langle Bf, f \rangle \geq 0$

now take $f_n = \chi_{\text{neighborhood of } z}$

to get $B(z,z) = \lim_{n \to \infty} \frac{1}{\text{measure}^2} \boxed{\iint B(x,w)} \geq 0$

$\square$

Conclusion

Let $s = s(p)$ be the parameter corresp. to the smallest eigenvalue

$$\lambda < \frac{7}{4}$$

$\Rightarrow$ it is new & so occurs with multiplicity $m \gg p$

We get

$$mX^{2s} \ll X^{2+\varepsilon} + p^3 X^{1+\varepsilon}$$

or

$$m \ll X^{2(1-s)+\varepsilon} + p^3 X^{1-2s+\varepsilon}.$$

Now set $X = p^3$, then

$$p \ll m \ll p^{6(1-s)+\varepsilon}.$$

So for large enough p we have

$$6(1-s) + \varepsilon \geq 1$$

$$\Updownarrow$$

$$s \leq \frac{5}{6} + \varepsilon'$$

$$\Updownarrow$$

$$\lambda > \frac{5}{36} - \varepsilon''$$

as desired. $\square$