Theta Functions and the Lefschetz Embedding Theorem

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Abstract

The aim of this paper is to prove that a complex torus T possessing a non-degenerate Riemann form is the manifold of complex points on an abelian variety. We will do so by making use of theta functions.

1 Theta Functions and Riemann Forms

Let V be a complex vector space of dimension n and let Λ be a lattice in V, that is to say a discrete subgroup of real dimension $2n$. Then the quotient $T = V/\Lambda$ is a complex torus. Let $L : V \times \Lambda \to \mathbb{C}$ and $J : \Lambda \to \mathbb{C}$ be maps with $L(x, \lambda)$ linear in x for all $\lambda \in \Lambda$. A *theta function* on V with respect to Λ of *type* (L, J) is a meromorphic function $F : V \to \mathbb{C}$ satisfying the relation

$$F(x + \lambda) = \mathbf{e}(L(x, \lambda) + J(\lambda))F(x) \text{ for all } x \in V,\ \lambda \in \Lambda,$$

where $\mathbf{e}(x) = e^{2\pi i x}$. Note that the theta functions form a multiplicative group and that the theta functions of same type form a vector space over $\mathbb{C}$. Computing $F(x+(\lambda_1+\lambda_2)) = F((x+\lambda_1)+\lambda_2)$ we find:

$$L(x, \lambda_1 + \lambda_2) + J(\lambda_1 + \lambda_2) \equiv L(x + \lambda_1, \lambda_2) + J(\lambda_2) + L(x, \lambda_1) + J(\lambda_1) \mod \mathbb{Z}. \tag{1}$$

Letting $x = 0$, by linearity of L in the first variable, we find:

$$J(\lambda_1 + \lambda_2) - J(\lambda_1) - J(\lambda_2) \equiv L(\lambda_1, \lambda_2) \mod \mathbb{Z}. \tag{2}$$

This expression is symmetric in λ_1 and λ_2, it follows that:

$$L(\lambda_1, \lambda_2) \equiv L(\lambda_2, \lambda_1) \mod \mathbb{Z}. \tag{3}$$

Finally, replacing $L(x + \lambda_1, \lambda_2)$ by $L(x, \lambda_2) + L(\lambda_1, \lambda_2)$ in (1) and using (2) we find:

$$L(x, \lambda_1 + \lambda_2) \equiv L(x, \lambda_1) + L(x, \lambda_2) \mod \mathbb{Z}. \tag{4}$$

The difference between the two sides of (4) is an integer, but being linear in x it must be zero, hence we can replace the congruence by an equality. We can therefore extend L to a function on $V \times V$ which is $\mathbb{C}$-linear in the first variable and $\mathbb{R}$-linear in the second variable. If we let $E(x, y) = L(x, y) - L(y, x)$ then we have the following:

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Theorem 1. $E : V \times V \to \mathbb{C}$ is $\mathbb{R}$-bilinear, alternating and real valued ($E(x,y) \in \mathbb{R}$ for all $x, y \in V$). Furthermore E takes integral values on $\Lambda \times \Lambda$.

Proof. E takes integer values on $\Lambda \times \Lambda$ by (3) and is real valued by $\mathbb{R}$-bilinearity. □

Lemma 1. $E(ix, iy) = E(x, y)$

Proof. Using $\mathbb{C}$-linearity of L in the first variable we have on one side

$$E(ix, iy) = L(ix, iy) - L(iy, ix) = i(L(x, iy) - L(y, ix))$$

and on the other side

$$E(x, y) = L(x, y) - L(y, x) = -i(L(ix, y) - L(iy, x)).$$

Thus

$$E(ix, iy) - E(x, y) = i(E(ix, y) - E(iy, x))$$

is zero since it is in $\mathbb{R} \cap i\mathbb{R}$. □

It is now easy to show that $H(x, y) = E(ix, y) + iE(x, y)$ is a hermitian form. We call H the *Riemann form associated to* F (a Riemann form on a torus $T = V/\Lambda$ is a Hermitian form H on V such that Im H is integer valued on Λ). If $H(u, u) \geq 0$ for all $u \in V$, we say H is a *positive Riemann form.* If H is positive definite, i.e. $H(u, u) > 0$ for all $u \in V$, $u \neq 0$, we say H is a *non-degenerate Riemann form* on T.

1.1 Trivial Theta Functions

Let q be a quadratic form on V, let l be a $\mathbb{C}$-linear form on V and c a complex number. The function

$$F(x) = \mathbf{e}(q(x) + l(x) + c)$$

is obviously a theta function, which we call a *trivial theta function.* If we call $B(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$ the associated symmetric, $\mathbb{C}$-bilinear form on V, then the theta function F is of type $(2B(x, \lambda), q(\lambda) + l(\lambda))$. Note that the theta function $F_1(x) = \mathbf{e}(q(x))$ contributes by $L_{F_1}(x, \lambda) = 2B(x, \lambda)$ and $J_{F_1}(\lambda) = q(\lambda)$ to the type of F, while the theta function $F_2(x) = \mathbf{e}(l(x))$ contributes by $L_{F_2} = 0$ and $J_{F_2}(\lambda) = l(\lambda)$ to the type of F.

Since for a trivial theta function F, we have $L_F(x, y) = 2B(x, y)$, a symmetric form, it follows that the associated Hermitian form is zero.

Lemma 2. An entire theta function which has no zero is a trivial theta function.

Proof. Let F be a theta function which has no zero. Then we can write $F(x) = \mathbf{e}(g(x))$, where g is an entire function. We have

$$\mathbf{e}(g(x + \lambda)) = F(x + \lambda) = \mathbf{e}(L(x, \lambda) + J(\lambda))\mathbf{e}(g(x))$$

and hence

$$g(x + \lambda) - g(x) \equiv L(x, \lambda) + J(\lambda) \mod \mathbb{Z}.$$

Since L is linear in x, all second order partial derivatives of the right hand side with respect to x vanish. Hence the second partials of g are periodic entire functions, whence constants by Liouville's theorem. Therefore g is a polynomial of degree at most 2, thus a sum of a quadratic form, a linear form and a constant. □

1.2 Normalized Theta Functions

We call two theta functions *equivalent* if their quotient is a trivial theta function. By adding some normalizing conditions, we want to be able to find a unique theta function (up to a constant factor) in each equivalence class. First observe the following:

Lemma 3. Two equivalent theta functions have the same associated Riemann form.

Proof. For theta functions F_1 and F_2, suppose their quotient is given by

$$\frac{F_1}{F_2}(x) = \mathbf{e}(q(x) + l(x) + c),$$

with q a quadratic form (defined by the bilinear, symmetric form B, i.e. $q(x) = B(x, x)$), l a linear form and c a constant. Expanding both sides in $x + \lambda$, it follows that

$$L_1(x, \lambda) - L_2(x, \lambda) = 2B(x, \lambda), \text{ and that} \tag{5}$$

$$J_1(\lambda) - J_2(\lambda) + c_1 - c_2 \equiv q(\lambda) + l(\lambda) \mod \mathbb{Z}. \tag{6}$$

By (5) we have

$$L_1(x, \lambda) = 2B(x, \lambda) + L_2(x, \lambda) \text{ and}$$

$$L_1(\lambda, x) = 2B(\lambda, x) + L_2(\lambda, x),$$

and thus by symmetry of B,

$$E_1(x, \lambda) = L_1(x, \lambda) - L_1(\lambda, x) = L_2(x, \lambda) - L_2(\lambda, x) = E_2(x, \lambda),$$

as desired. □

Conversely, if two theta functions F_1 and F_2 define the same Riemann form, i.e.,

$$L_1(x, y) - L_1(y, x) = E(x, y)$$

$$L_2(x, y) - L_2(y, x) = E(x, y),$$

then L_1 and L_2 differ by a symmetric, $\mathbb{C}$-bilinear form. To see this, it is obvious that $(L_1 - L_2)(x, y) = (L_1 - L_2)(y, x)$, hence symmetric, and $L_1 - L_2$ being $\mathbb{C}$-linear in x, it follows (together with symmetry) that $L_1 - L_2$ is $\mathbb{C}$-bilinear.

Acording to the above observation, within an equivalence class of theta functions, we can obtain all L by addition of $\mathbb{C}$-bilinear, symmetric forms, say $B_q(x, y)$, to a a known L_F (by multiplying F with $\mathbf{e}(\frac{1}{2}q(x))$). On the other hand, we have

$$E(x, y) = \frac{1}{2i}(H(x, y) - H(y, x)),$$

so that a natural choice for a "representative" in an equivalence class is

$$L(x, y) = \frac{1}{2i}H(x, y).$$

For further normalization, we still can multiply the function by the exponential of a $\mathbb{C}$-linear form. Define $K(\lambda) = J(\lambda) - \frac{1}{2}L(\lambda, \lambda)$. From (2) we get

$$K(\lambda_1 + \lambda_2) \equiv K(\lambda_1) + K(\lambda_2) + \frac{1}{2}E(\lambda_1, \lambda_2) \mod \mathbb{Z}.$$

Since E is real valued, the imaginary part of K, Im K, is additive on Λ, with values in $\mathbb{R}/\mathbb{Z}$. Since Λ is free, we can lift Im K to an $\mathbb{R}$-valued function which is additive on Λ, and then extend it to an $\mathbb{R}$-linear function on V, say $g : V \to \mathbb{R}$. If we let $\gamma(x) = g(ix) + ig(x)$, then

$$\gamma(ix) = g(-x) + ig(ix) = -g(x) + ig(x) = i(ig(x) + g(ix)) = i\gamma(x).$$

Hence γ is $\mathbb{C}$-linear, and $K - \gamma$ is real valued by construction.

Definition 1. A theta function F is *normalized* if it satisfies the conditions:

1) $L(x, y) = \frac{1}{2i} H(x, y)$.

2) The function K on V is real valued.

For a normalized theta function, the basic realation takes the form

$$F(x + \lambda) = F(x)\mathbf{e}\left(\frac{1}{2i} H(x, \lambda) + \frac{1}{4i} H(\lambda, \lambda) + K(\lambda)\right).$$

Clearly, the normalized theta functions form a subgroup of all theta functions.

Summarizing the above, we have:

Theorem 2. In any equivalence class of theta functions, there exists a normalized theta function, unique up to a non-zero constant factor.

Proof. Let F be a theta function of type (L, J). Then the product

$$\tilde{F}(x) = F(x)\mathbf{e}(\frac{1}{4i} H(x, x) - \frac{1}{2} L(x, x)))$$

is a theta function of type $(\frac{1}{2i} H(x, \lambda), \frac{1}{4i} H(\lambda, \lambda) - \frac{1}{2} L(\lambda, \lambda) + J(\lambda))$. With $K(\lambda) = J(\lambda) - \frac{1}{2} L(\lambda, \lambda)$, we can find a $\mathbb{C}$-linear function γ on V, such that $K - \gamma$ is real valued. Finally,

$$\tilde{\tilde{F}}(x) = \tilde{F}(x)\mathbf{e}(-\gamma(x))$$

is of type $(\frac{1}{2i} H(x, \lambda), \frac{1}{4i} H(\lambda, \lambda) + \tilde{K}(\lambda))$, with $\tilde{K} = K - \gamma$ real valued. By construction, $\tilde{\tilde{F}}$ and F are equivalent.

The uniqueness of the quadratic factor is obvious. For the linear factor, if two $\mathbb{C}$-linear maps γ_1 and γ_2 both turn $K - \gamma_i$ into a real valued function, then $\gamma_1 - \gamma_2$ is a $\mathbb{C}$-linear, real valued function, and hence 0 ($\forall x \in V$, $\underbrace{(\gamma_1 - \gamma_2)(x)}_{\in \mathbb{R}} = i\, \underbrace{(\gamma_1 - \gamma_2)(-ix)}_{\in \mathbb{R}}$). □

As a last result in this section we state:

Theorem 3. Suppose that F is an entire theta function. Then the associated Riemann form H is positive (not necessarily definite).

Proof. See [Lan]. □

2 Dimension of the space of Theta Functions

Let V be a complex vector space of dimension n, with a lattice Λ. Let H be a non-degenerate Riemann form on $T = V/\Lambda$, and let $E = \text{Im } H$ be the imaginary part of H. If we select a basis for Λ over $\mathbb{Z}$, then it is also a basis for V over $\mathbb{R}$. The matrix representing E with respect to such a basis has integer coefficients, and its determinant is a perfect square (the second assertion is not straightforward but will come out of the next lemma). The square root of this determinant is called the *pfaffian* of E with respect to Λ, $\text{Pf}(E)$, and is independent of the choice of such a basis (the matrix of the form E with respect to different bases changes by an integral matrix and its transpose, of determinant 1).

As a matter of notation, if $u_1, ..., u_n$ are elements of V, we denote by $[u_1, ..., u_n]$ the $\mathbb{Z}$-module generated by these elements. The following lemma is due to Frobenius.

Lemma 4. Let E be an alternating, non-degenerate bilinear form on a free $\mathbb{Z}$-module Λ of rank $2n$, having values in $\mathbb{Z}$. Then Λ is an E-orthogonal direct sum

$$\Lambda = [e_1, v_1] \oplus \cdots \oplus [e_n, v_n]$$

of 2-dimensional submodules $[e_j, v_j]$, such that $E(e_j, v_j) = d_j$ is an integer > 0, and $d_1 \mid d_2 \mid \cdots \mid d_n$. Moreover, $\{e_1, ..., e_n\}$ is a $\mathbb{C}$-basis for V.

Proof. See [Lan]. □

Such a basis is called a *Frobenius basis* for Λ with respect to E. Note that the matrix of E with respect to the Frobenius basis is of the form

$$\begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix},$$

where $D_j = \begin{pmatrix} 0 & d_j \\ -d_j & 0 \end{pmatrix}$, and thus the pfaffian of E is given by $\text{Pf}(E) = d_1 d_2 \cdots d_n$.

Let L and J be maps such that (L, J) satisfies the conditions of a type of a theta funtion (with respect to V and Λ). Denote by $\text{Th}(L, J)$ the complex vector space of theta functions of type (L, J), together with 0.

Theorem 4 (Frobenius)**.** Let (L, J) be a type, H the associated Riemann form, and $E = \text{Im } H$. Assume H is positive definite. Then the vector subspace of $\text{Th}(L, J)$ of entire theta functions has dimension $\text{Pf}(E)$ over $\mathbb{C}$.

Proof. Sketch. For a complete proof, see [Lan]. Let $\{e_1, v_1, ..., e_n, v_n\}$ be a Frobenius basis for Λ with respect to E. By multiplying by a suitable trivial theta function, the proof is reduced to the case when

$$L(z, e_j) = 0 \text{ for all } z \in V, \text{ and } J(e_j) = 0 \text{ for all } j = 1, ..., n.$$

For $z \in V$, let $\{z_1, ..., z_n\}$ be the coordinates of z with respect to the $\mathbb{C}$-basis $\{e_1, ..., e_n\}$. Then the space of entire theta functions of type (L, J) is precisely the space of entire functions on V satisfying

$$F(z + e_j) = F(z)$$

$$F(z+v_j) = F(z)\mathbf{e}(z_j d_j + c_j),$$

where $c_1, ..., c_n$ are some fixed constants. The first set of equations show that we can expand F as a Fourier series

$$F(z) = \sum_{r\in\mathbb{Z}^n} a(r)\mathbf{e}(r\cdot z).$$

The second set of equations impose recurrence relations on the set of coefficients $a(r)$ which show that all the $a(r)$ can be expressed in terms of those $a(r)$ with $0 \leq r_j < d_j$, $j = 1, .., n$. This gives an upper bound of $d_1 d_2 \cdots d_n = \mathrm{Pf}(E)$ on the dimension of the given space. Equality is achieved by showing that for a collection of $a(r)$ satisfying the recurrence relations, the corresponding formal Fourier series is in fact a holomorphic function. □

For an entire theta function F_0 of type (L, J), we denote by $\mathscr{L}(F_0)$ the space of all entire theta functions of the same type. We call $\mathscr{L}(F_0)$ the *linear system* of F_0.

Remark. Let (L, J) be a non-degenerate type with respect to (V, Λ). It may happen that there is a lattice Λ' containing Λ for which (L, J) is also a non-degenerate type with respect to (V, Λ'). In this case, let $\lambda' \in \Lambda'$, and write

$$\lambda' = \sum_{j=1}^{n} a_j e_j + \sum_{j=1}^{n} b_j v_j,$$

in terms of a Frobenius basis for Λ. Then

$$E(e_j, \lambda') = b_j d_j \text{ and } E(v_j, \lambda') = -a_j d_j.$$

Since E is integer valued, a_j, b_j can take only a finite number of values mod $\mathbb{Z}$. Hence there are only a finite number of such possible lattices Λ'.

Theorem 5. Let (L, J) be a non-degenerate type with respect to (V, Λ). Then all theta functions of this type except possibly those lying in a finite union of subspaces of dimension $< \mathrm{Pf}(E)$ are not theta functions with respect to a lattice strictly larger than Λ.

Proof. If F is a theta function of type (L, J') with respect to (V, Λ'), then the pfaffian of E with respect to Λ' is equal to

$$\frac{d}{[D' : D]} < d,$$

where d is its pfaffian with respect to Λ. Hence the space of theta functions of type (L, J') with respect to (V, Λ') has lower dimension. By the above remark, there is only a finite number of lattices Λ' containing Λ properly, thus there is only a finite number of such spaces. □

3 Projective Embedding

Let V be a complex vector space, Λ a lattice in V, and H a Hermitian form on V. Let V_H be the subset of V consisting of all z such that $H(z, z) = 0$. V_H is called the *kernel* of H, or *null space* of H. Note that if $z \in V_H$, then $H(z, w) = 0$ for all $w \in V$, hence V_H is a complex subspace.

Theorem 6. Let F be an entire, non-degenerate theta function, normalized, with associated hermitian Riemann form H. Let V_H be the null space of H. Then $V_H \mod \Lambda$ is finite in V/Λ.

Proof. See [Lan]. □

3.1 Translations of Theta Functions

Let $a \in V$, and let F be a theta function. Let F_a be the function such that

$$F_a(x) = F(x - a),$$

and call F_a the *translation* of F by a. It is obviously a theta function, and if F is of type (L, J), then F_a is of type $(L, J - L_a)$, where $L_a(\lambda) = L(a, \lambda)$. See [Lan] for the proof of the following theorem:

Theorem 7. Let F, F' be entire normalized theta functions, with associated hermitian forms H, H', respectively. Then $H = H'$ if and only if there exists $a \in V$ such that F_a and F' are of the same type.

3.2 The Lefschetz Embedding Theorem

Let F_0 be an entire theta function, let $\{F_0, F_1, ..., F_m\}$ be a basis for $\mathscr{L}(F_0)$. We can see this basis as giving a map

$$x \mapsto (F_0(x) : ... : F_m(x))$$

of V/Λ into projective space $\mathbb{P}^m$, defined at all those points x where not all F_j vanish simultaneously. We call this map the *map induced by the linear system* of F_0. Note that for two Λ-equivalent points x, x' on the torus,

$$(F_0(x'), ..., F_m(x')) = \mathbf{e}(L(x, x' - x) + J(x' - x))(F_0(x), ..., F_m(x)),$$

and thus it makes sense that the above map takes values in projective space.

Note that the map is well defined at x if and only if there exists some F in $\mathscr{L}(F_0)$ such that $F(x) \neq 0$.

Let F be an entire theta function. Let X be the set of its zeros, i.e., the set of points x such that $F(x) = 0$. We can view X either as a subset of V or as a subset of V/Λ. The union of a finite number of such zero sets $X_1 \cup \cdots \cup X_m$ is the set of zeros of the product $F_1 \cdots F_m$, of the corresponding theta functions. In particular, it is not the whole space.

We can now state and prove the main result of this paper.

Theorem 8 (Lefschetz)**.** Let F be an entire non-degenerate theta function. Then the map of V/Λ into projective space induced by the linear system $\mathscr{L}(F^3)$ is everywhere defined, and is an analytic embedding of V/Λ into projective space.

Proof. Obviously the function

$$F(x - a)F(x - b)F(x + a + b)$$

lies in $\mathscr{L}(F^3)$ for any $a, b \in V$. The map is well defined if for a given point x, there exist a, b such that the above product is not equal to 0. We first find a such that $x - a$ does not lie in the set of zeros of F. Then find b such that $x - b$ and $x + a + b$ do not lie in the zero set of F. In each case, this is just finding a point not lying in a finite union of zero sets.

Let's now prove that the map is injective. Suppose $x, y \in V$ have the same image in projective space. We will show that x and y differ by a lattice point. The assumption that x and y have the same image is equivalent to saying that there exists a complex number $\gamma \neq 0$ such that for all $b, z \in V$ and all G of the same type as F, we have

$$G(x - z)G(x - b)G(x + z + b) = \gamma G(y - z)G(y - b)G(y + z + b). \tag{7}$$

By Theorem (5) we can select G in $\mathscr{L}(F)$ such that G is not a theta function with respect to any lattice strictly larger than Λ. Let $v = x - y$, and let $\Lambda' = \Lambda + \mathbb{Z}v$. We will prove that G is a theta function with respect to Λ'. It then follows that $v \in \Lambda$.

By a similar argument as above, given any point $z_0 \in V$, we can find b such that

$$G(x-b)G(x+z_0+b)G(y-b)G(y+z_0+b) \neq 0, \tag{8}$$

and hence such that this inequality holds in a neighborhood of z_0. Combining (9) and (10), there is a non vanishing holomorphic function g_0, such that

$$G(x-z) = G(y-z)g_0(z)$$

in that neighborhood of z_0. The functions g_0 coincide on the intersection of the underlaying open sets, and we can therefore define a non vanishing entire function g, such that for all z we have

$$G(x-z) = G(y-z)g(z).$$

By changing $z \mapsto -z + y$, this formula can be written in the form

$$G(z+v) = G(z)h(z),$$

for some entire function h without zeros. By the theta relation for G (v being fixed), the function h is in fact a trivial theta function, of the form

$$h(z) = \mathbf{e}(l(z) + c),$$

where l is $\mathbb{C}$-linear. On the other hand, if (L, J) is the type of G, then $(L, J + L_v)$ is the type of G_{-v}, so that as a quotient of theta functions, h is of type $(0, L_v)$. Computing $h(z+u)$ for $u \in \Lambda$, gives

$$h(z+u) = h(z)\mathbf{e}(L(v,u)) \text{ and}$$
$$h(z+u) = \mathbf{e}(l(z+u)+c) = \mathbf{e}(l(z)+l(u)+c) = h(z)\mathbf{e}(l(u)),$$

and therefore $l(u) - L(v,u) \in \mathbb{Z}$. But

$$l(u) - L(v,u) = l(u) - L(u,v) + E(v,u),$$

hence $l(u) - L(u,v)$ is a real number, since $E(v,u)$ is so. The map $u \mapsto l(u) - L(u,v)$ is real valued on Λ, hence real valued on V, since the elements of Λ generate V over $\mathbb{R}$, and l, L are $\mathbb{R}$-linear in z. But this map is also $\mathbb{C}$-linear, and consequently we must have

$$l(z) = L(z,v).$$

It follows that G is a theta function with respect to Λ', and therefore the map induced by the linear system $\mathscr{L}(F)$ is injective.

To see that the map is an embedding, first observe that it is a homeomorphism onto its image. It is clearly continuous, it is bijective by the above, and it is a closed map, since V/Λ is compact. There remains to see that the rank of the mapping (seen as a differentiable map between complex manifolds) equals the dimension of V. In other words, we have to check that the differential of our projective mapping is injective at any point. So fix a point x on the torus. Select a function $G \in \mathscr{L}(F^3)$ such that $G(x) \neq 0$, and fix some nonzero vector v in V. The image of x by the map equals $\left(\frac{F_0}{G}(x) : ... : \frac{F_m}{G}(x)\right)$, so that for injectivity, it suffices to check that for some i,

$$d(F_i/G)_x v \neq 0.$$

This is done if we show that there exists a function $G' \in \mathscr{L}(F^3)$ such that

$$d(G'/G)_x v \neq 0.$$

Suppose that for every $G' \in \mathscr{L}(F^3)$, we have $d(G'/G)_x v = 0$. We may take a basis for V such that $v = (1, 0, ..., 0)$. Using the relation

$$d(G'/G)_x = \frac{G(x)dG'_x - G'(x)dG_x}{G(x)^2},$$

we have

$$\frac{dG'_x}{G'(x)} v = \frac{dG_x}{G(x)} v = \alpha,$$

for some fixed number α, and every G' such that $G'(x) \neq 0$. By the choice of the basis for V, we have

$$dG'_x v = \frac{\partial G'}{\partial z_1}(x), \text{ and } \frac{dG'_x}{G'(x)} v = \frac{1}{G'(x)} \frac{\partial G'}{\partial z_1}(x). \tag{9}$$

Let

$$f(z) = \frac{1}{F(z)} \frac{\partial F}{\partial z_1}, \tag{10}$$

wherever defined. We select for G' the function

$$G'(z) = F(z - a)F(z - b)F(z + a + b), \tag{11}$$

with a, b arbitrary. Combining (9), (10) and (11), we have

$$f(x - a) + f(x - b) + f(x + a + b) = \alpha,$$

for all a, b outside an exceptional set where the denominators on the left vanish. Consider the function given by

$$z \mapsto f(x - z) + f(x - b) + f(x + z + b),$$

which is constant. Differentiate with respect to each variable z_j. We get

$$\frac{\partial f}{\partial z_j}(x - z) = \frac{\partial f}{\partial z_j}(x + z + b).$$

From the right-hand side, we see that these partial derivatives are constant for some open set of z, whence it follows that in a nopen set of z where f is defined, we have

$$\frac{1}{F(z)} \frac{\partial F}{\partial z_1} = \alpha_1 z_1 + \cdots + \alpha_n z_n + \beta,$$

with constants $\alpha_1, ..., \alpha_n, \beta$. Let

$$q(z) = \frac{1}{2}\alpha_1 z_1^2 + \alpha_2 z_1 z_2 + \cdots + \alpha_n z_1 z_n + \beta z_1$$

be the sum of a quadratic form and a linear form. The theta function

$$F_1(z) = F(z)e^{-q(z)}$$

has first partial derivative equal to 0 in some open set, whence everywhere. This means that F_1 depends only on the last $n-1$ variables. Hence F which is equivalent to F_1 depends only on the last $n-1$ variable. But F is an entire non-degenerate theta function, and we have a contradiction with Theorem 6.

□

Now combining Chow's theorem with the above Lefschetz Embedding theorem, we have the desired result, i.e., that a complex torus possessing a non-degenerate Riemann form is a projective variety.

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