1/ Exposé Topics in geometric analysis

A brief introduction to singularities and their resolutions (for curves and normal surfaces).

Goal: Introduce ~~the classification of~~ normal surface singularities ~~based on~~ weighted dual graphs of resolutions of.

Future talks: [Belotto da Silva, Fantini, Pichon]
2021 Inner geometry of complex surfaces – a valuative approach

How the weighted dual graph of a normal surface singularity captures metric information.

We will work with singularities of complex analytic (often algebraic) curves and surfaces. Let us define/discuss these geometric spaces.

Analytic sets

Geometric spaces are usually defined in two steps:

1. set a local model for our category of spaces. a) topological space
2. patch these local models into a space e.g. using sheaves. b) functions

	Differential geometry (real manifolds)	Analytic geometry ($\mathbb{C}$) (complex analytic sets)
1.	a). $U \subseteq \mathbb{R}^n$ open b). $C^\infty(U)$	a). $Z \subseteq \mathbb{C}^n$ $Z = \{f_1 = \dots = f_r = 0\}$ f_i holomorphic b). $\mathcal{H}ol(\mathbb{C}^n)/\mathcal{I}(Z)$. $\mathcal{I}(Z) := \{g \in \mathcal{H}ol(\mathbb{C}^n) \mid g\|_Z = 0\}$
2.	$M = \bigcup_i U_i$ $\mathcal{O}_M(U_i) = C^\infty(U_i)$. ↳ sheaf of smooth functions	$X = \bigcup_i Z_i$ $\mathcal{O}_X(Z_i) = \mathcal{H}ol(\mathbb{C}^n)/\mathcal{I}(Z)$. ↳ sheaf of regular functions

We will work with the latter. Morphisms are morphisms of the underlying topological spaces preserving regular functions

2/ In practice: • we will (mostly) work with analytic subsets of $\mathbb{C}^n$ in coordinates

• morphisms will be described in coordinates and their components will be holomorphic (often) functions polynomials

• to check that $Z \simeq W \subseteq \mathbb{C}^n$, one ~~requires~~ needs to find a homeomorphism s.t. $\mathcal{I}(Z) \simeq \mathcal{I}(W)$.

exmp: $\underbrace{\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}}_{U_0 \cup U_1} = \mathbb{A}^2_{\mathbb{C}} \times U_0 \cup \mathbb{A}^2_{\mathbb{C}} \times U_1 \xrightarrow{pr_1} \mathbb{A}^2_{\mathbb{C}}$

$\{xy=0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ $\qquad \underbrace{\{xv - yu = 0\}}_{V_0 \cup V_1} \subseteq \mathbb{A}^2_{\mathbb{C}} \times \underset{[u:v]}{\mathbb{P}^1_{\mathbb{C}}} \xrightarrow{\pi} \mathbb{A}^2_{\mathbb{C}}$

$V_0 = \{u \neq 0\}$ $\quad y = \frac{v}{u} \cdot x$ $\quad y' = \frac{v}{u}$ $\quad \pi|_{V_0} : (x, y') \mapsto (x, xy')$.

$V_1 = \{v \neq 0\}$ $\quad x = \frac{u}{v} \cdot y$ $\quad x' = \frac{u}{v}$ $\quad \pi|_{V_1} : (x', y) \mapsto (x'y, y)$

polynomial components !

Singularities

Being singular is a local notion $\rightsquigarrow$ we work with $Z \subseteq \mathbb{C}^n$ (analytic set).

def: Let $Z = \{f_1 = \dots = f_r = 0\} \subseteq \mathbb{C}^n$ and $z_0 \in Z$, $n = \dim_{z_0} Z$ $= \max\{r \mid \exists\, k \in Z_0 \subsetneq Z_1 \dots \subsetneq Z_r,\ Z_i \text{ an. subsets irred.}\}$

Z is said to be singular at z_0 if

$$\operatorname{rank}\begin{pmatrix} \frac{\partial f_1}{\partial z_1}(z_0) & \cdots & \frac{\partial f_1}{\partial z_N}(z_0) \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial z_1}(z_0) & \cdots & \frac{\partial f_r}{\partial z_N}(z_0) \end{pmatrix} \lneq N - n.$$

exmp: $\{xy = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$. $\begin{pmatrix} y & x \end{pmatrix}$ has rank $0 < 1$ in $(0,0)$

So the origin is a singular point.

3/ We will obtain information on singularities of surfaces by resolving them.

<u>def</u>: A morphism of analytic sets $W \xrightarrow{f} Z$ is a resolution of singularities if.

- f is proper
- $f|_{W \setminus \cancel{W_{sing}}} : W \setminus W_{sing} \longrightarrow Z \setminus Z_{sing}$
  $f^{-1}(Z_{sing})$ $f^{-1}(Z_{sing})$
  is an isomorphism.
- W is non-singular.

f is called a strong resolution (or a resolution à la Hironaka) if $f^{-1}(\cancel{Z}_{Z_{sing}})$ is a closed analytic subset of codimension one with simple normal crossings (simple normal crossing divisor).

i.e. a). each irreducible component of $f^{-1}(Z_{sing})$ is non-singular

b). $f^{-1}(Z_{sing})$ is locally isomorphic to $\{x_1 \dots x_r = 0\} \subseteq \mathbb{C}^n$ around every point

<u>thm</u>: (Artin algebraization). Any isolated singularity of a complex analytic set is locally isomorphic to a singularity of an algebraic variety.

put this in a remark

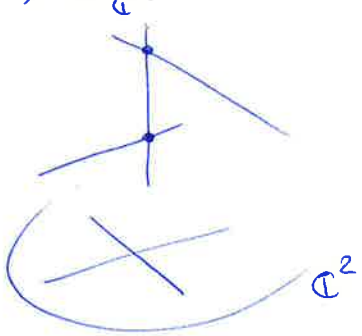
<u>thm</u> (Hironaka) Any algebraic variety over $\mathbb{C}$ admits a strong resolution of singularities.

How do we concretely resolve singularities? By blowing up.

<u>exmp</u>: $\{xy = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ under $\pi : Bl_0(\mathbb{A}^2_{\mathbb{C}}) \longrightarrow \mathbb{A}^2_{\mathbb{C}}$

$\pi^{-1}(\{xy = 0\})$ in V_0 $y' \cdot x^2 = 0$

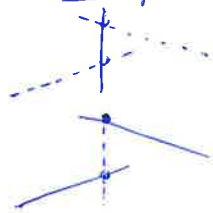
in V_1 $x' \cdot y^2 = 0$.



restrict π to the two components $\{x' = 0\}$ $\{y' = 0\}$. gives a resolution. (strict transform).

4/ **def**: Given a proper birational morphism $f: W \to \cancel{X}\, Z \supseteq Z'$ s.t. $f: W \setminus f^{-1}(Z') \to Z \setminus Z'$ is an isomorphism, given $Y \subseteq Z$ a closed | analytic subset / subvariety |.

exmp:



* the exceptional locus of f is $f^{-1}(Z')$,
* the strict transform of Y is $\overline{f^{-1}(Y \setminus Z')}$.

We will always blow up smooth | subvarieties / an. subsets | of smooth | varieties / an. sets |.

(since for any $Z \subseteq X$ smooth $Bl_F Z$ is the strict transform of Z in $Bl_F X$).

def Let $Z \subseteq X$ be a smooth analytic subset of codimension k of a complex manifold

$Bl_Z X$ ~~$\subseteq X \times \mathbb{P}^{k-1}$~~ is the ~~sub~~manifold obtained by gluing the following local models:

assume $Z = \{x_1 = \dots = x_k = 0\} \subseteq \mathbb{C}^n$

then $Bl_Z \mathbb{C}^n \subseteq \mathbb{C}^n \times \mathbb{P}^{k-1}$ is defined using the equations

$$x_i y_j - x_j y_i = 0.$$

rmk: $Bl_Z \mathbb{C}^n$ is covered by affine charts $U_i \simeq \mathbb{C}^n$.

$\pi : U_i \to \mathbb{C}^n$

$(x'_1, \dots, x'_n) \mapsto (x'_i x'_1, \dots, x'_i, \dots, x'_i x'_n)$.

Resolving planar curve singularities

Consider $C = \{f(x,y) = 0\} \subseteq \mathbb{C}^2$ a singular (algebraic) plane curve. To resolve singularities of C, one blows up the plane in singular points until:

$\pi : X \to \mathbb{C}^2$

* the strict transform of C is smooth
* the exceptional ~~di~~ locus of the resolution $\pi : X \to \mathbb{C}^2$ is a simple normal crossing divisor and the strict transform has normal crossings with it.

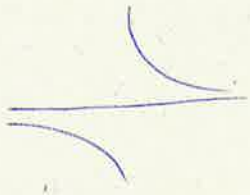
(if asked, maximal contact order decreases $\max\{\dim \mathcal{O}_p/(f,\ell)\}$.)

exmp: resolution of the cusp $f = y^2 - x^3$ * strict transform

$x = x'$, $y = x'y'$: $(x')^2 \cdot \{(y')^2 - x'\}$

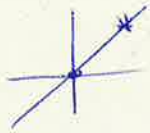
$y^2 - x^3$

$x = x''y''$, $y = y''$: $(y'')^2 \cdot \{1 - (x'')^3 \cdot y''\}$



$x^3 \cdot (x y^2 - 1)$. $x^2 y^3 \cdot (x - y)$.

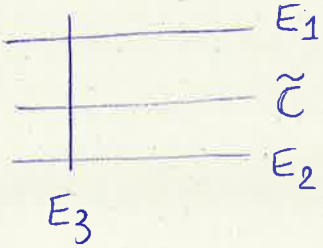
stop because we got something smooth!



$x^4 y^3 \cdot (1 - y)$. $x^2 y^5 \cdot (x - 1)$.

normal crossings, so we are done.

In total, we did 3 blowing-ups in 3 distinct points: so in total we get the strict transform and 3 exceptional divisors ($\simeq \mathbb{P}^1$). E_1, E_2, E_3.



Resolving isolated surface singularities.

Let $(X, 0) \subseteq (\mathbb{C}^n, 0)$ be an isolated surface singularity. ($\exists U \ni 0$ neighborhood in S s.t. $U \setminus \{0\}$ is smooth).

Consider a strong resolution $\pi : Y \to X$. Then $\pi^{-1}(0) = \bigcup_{i=1}^{r} E_i$.

codimension 1 irreducible smooth analytic subsets.

We need ~~to~~ information on the intersections of the E_i's to build the dual graph. $\rightsquigarrow$ work in $H^2(X)$ and use cup product as an intersection pairing.

We obtain an intersection matrix $M = (E_i \cdot E_j)_{1 \le i,j \le r}$.

NB. typically, $E_i \cdot E_i < 0$, since we get a transverse intersection by taking a homologous cycle with negative coefficients

We will also later consider multiplicities of the E_i's. These are defined algebraically

Take the ideal of equations defining $0 \in X$ $\mathcal{I} = (f_1, \dots, f_s)$

then $\pi^* \mathcal{I} = (f_1 \circ \pi, \dots, f_s \circ \pi) \subseteq \mathcal{O}_Y$ simplifies to:

$$g_1^{N_1} \cdots g_r^{N_r} \cdot \mathcal{O}_Y \text{, where } g_i \text{ is the equation of } E_i$$

N_i is well-defined and is called the multiplicity of E_i.

<u>def</u> The weighted dual graph of $(X,0)$ is built as follows:

* one vertex for each prime divisor;
* draw one edge between $E_i \neq E_j$ if they intersect;
* label each vertex by $E_i \cdot E_i$.

<u>rmk</u>: if each E_i has self-intersection -2, then the dual graph is an ADE Dynkin diagram. (known explicit equations).

~~thm~~ ~~* Any isolated surface singularity has a~~

<u>thm</u> (Mumford) M is negative definite

(rmk)

<u>thm</u> (Grauert) If a compact divisor $E = \sum E_i$ on a non-singular surface Y has negative definite intersection matrix, then there exists a unique normal singularity (X, x) and a proper morphism $f: Y \to X$ such that $f|_{Y \setminus E}: Y \setminus E \to X \setminus \{x\}$ is isomorphic.

(if asked: any isolated surface singularity has a unique minimal resolution, so the graph of this resolution is an invariant of the singularity – but the E_i need not be isomorphic if the graphs are the same)

<u>exmp</u>: $(X,0) \subseteq (\mathbb{C}^3, 0)$. $(z x^2 + y^3).(x^3 + z y^2) + z^7 = 0$.

$(X,0) \subseteq (\mathbb{C}^3, 0)$ $w^2 + x^2 + y^3 = 0$.

resolved by a single blow up ($0 \in \mathbb{C}^3$).

<u>but</u> the exceptional divisor might not be smooth.

blow up the origin in $\mathbb{C}^3$: in blowing-up charts we obtain

$w^2(1 + x^2 + w y^3)$

$x^2(1 + w^2 + x y^3)$

$y^2(y + x^2 + w^2)$

~~all~~ strict transform is smooth $\simeq Bl_0 X$.

<u>but</u> exceptional divisor is not smooth. simple but normal crossings!

(in charts $x^2 + 1 = 0$, $w^2 + 1 = 0$) $\rightsquigarrow$ additional blow ups in this singular point of the divisor (smooth in the surface) $\rightsquigarrow$