Exposé Topics in geometric analysis

A brief introduction to singularities and their resolutions
(for curves and normal surfaces).

Goal: Introduce the classification of normal surface singularities 
based on weighted dual graphs of resolutions of

Future talks: [Beletto da Silva, Fantini, Pedon]
2021: Inner geometry of complex surfaces - a valuative approach

How the weighted dual graph of a normal surface singularity captures metric information.

We will work with singularities of complex analytic (often algebraic) curves and surfaces. Let us define/discuss these geometric spaces.

Analytic sets

Geometric spaces are usually defined in two steps:
1. set a local model for our category of spaces. a) topological space, b) functions e.g. using sheaves.
2. patch these local models into a space

Differential geometry
(real manifolds)

1. a). $U \subseteq \mathbb{R}^n$ open
   b). $C^\infty(U)$

2. $M = \bigcup_i U_i$
   $\mathcal{O}_M(U_i) = C^\infty(U_i)$.
   sheaf of smooth functions

Analytic geometry
(complex analytic sets)

3. a). $Z \subseteq \mathbb{C}^n$
   $Z = \{f_1 = \ldots = f_r = 0\}$
   $f$: holomorphic
   b). $\mathcal{H}ol(\mathbb{C}^n)/\mathcal{I}(Z)$.
   $\mathcal{I}(Z) = \{g \in \mathcal{H}ol(\mathbb{C}^n) \mid g|_Z = 0\}$
   $X = \bigcup_i Z_i$
   $\mathcal{O}_X(Z_i) = \mathcal{H}ol(\mathbb{C}^n)/\mathcal{I}(Z_i)$.
   sheaf of regular functions

We will work with the latter. Mophisms are morphisms of the underlying topological spaces preserving regular functions.
In practice, we will work with analytic subsets of \( \mathbb{C}^n \) (mostly) in coordinates.

- Morphisms will be described in coordinates and their components will be holomorphic (often) functions polynomials.

- To check that \( Z \sim W \subseteq \mathbb{C}^n \), one requires to find a homeomorphism \( s.t. \ J(Z) \sim J(W) \).

Example: \( \mathbb{A}^2 \times \mathbb{D}^2 \subseteq \mathbb{C}^2 \) \( U_0 \cup U_1 \) \( \pi \) \( \mathbb{A}^2 \)

\[ \begin{align*}
\{ \nu - y_1 & = 0 \} \subseteq \mathbb{A}^2 _{\mathbb{C}} \times \mathbb{D}^2 _{\mathbb{C}} \rightarrow \mathbb{A}^2 _{\mathbb{C}} \\
& V_0 \cup V_1 \\
V_0 & = \{ \nu - 0 \} \quad y = \frac{\nu}{u} \cdot k \quad y' = \frac{\nu}{u} \pi |V_0 : (x, y') \mapsto (x, xy') \}
\]

\[ \begin{align*}
V_1 & = \{ \nu = 0 \} \quad k = \frac{\nu}{u} \quad x = \frac{\nu}{u} \pi |V_1 : (x, y) \mapsto (xy, y) \\
& \text{polynomial components!}
\end{align*} \]

**Singularities**

Being singular is a local notion \( \Rightarrow \) we work with \( Z \subseteq \mathbb{C}^n \) (analytic sets).

**Def:** Let \( Z = \{ f_1 = \ldots = f_r = 0 \} \subseteq \mathbb{C}^n \) and \( z_0 \in Z \), \( n = \dim_{\mathbb{C}} Z \)

\( Z \) is said to be singular at \( z_0 \) if \( \max \left\{ \left| \frac{\partial f_i}{\partial z_j} (z_0) \right| \right\} < \infty \)

\( \exists \; 1 \leq i \leq r \), \( \exists \; 1 \leq j \leq n \), \( z_0 \) an. Sub-set \( \in \mathbb{C}^n \) innd.

\( \operatorname{rank} \begin{pmatrix} \frac{\partial f_1}{\partial z_1} (z_0) & \ldots & \frac{\partial f_1}{\partial z_n} (z_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial z_1} (z_0) & \ldots & \frac{\partial f_r}{\partial z_n} (z_0) \end{pmatrix} < n - n \)

Example: \( \{ 2y_1 = 0 \} \subseteq \mathbb{A}^2 _{\mathbb{C}} \) \( (y, x) \) has rank 0 \( \neq 1 \) in \( (0, 0) \)

So the origin is a singular point.
We will obtain information on singularities of surfaces by resolving them.

**Def.** A morphism of analytic sets $W \to Z$ is a resolution of singularities if:

- $f$ is proper
- $f|_{W \setminus \text{Sing}(f)} : W \setminus \text{Sing}(f) \to Z \setminus \text{Sing}(f)$ is an isomorphism.
- $W$ is non-singular.

$f$ is called a strong resolution (or a resolution à la Artin) if:

1. $f^{-1}(\text{Sing}(f))$ is a closed analytic subset of codimension one with simple normal crossings (simple normal crossing division).
2. Each irreducible component of $f^{-1}(\text{Sing}(f))$ is non-singular.
3. $f^{-1}(\text{Sing}(f))$ is locally isomorphic to \( \{ x_1 \ldots x_r = 0 \} \subset \mathbb{C}^n \) around every point.

**Thm.** (Artin algebraization). Any isolated singularity of a complex analytic set is locally isomorphic to a singularity of an algebraic variety.

**Thm.** (Hironaka) Any algebraic variety over $\mathbb{C}$ admits a strong resolution of singularities.

How do we concretely resolve singularities? By blowing up.

**Exmp.:** \( \{ xy = 0 \} \subset \mathbb{A}^2_\mathbb{C} \) under \( \pi : \text{Bl}_0(\mathbb{A}^2_\mathbb{C}) \to \mathbb{A}^2_\mathbb{C} \).

- \( \pi^{-1}(\{ xy = 0 \}) \) in $V_0$ \( y', x^2 = 0 \)
- \( \pi^{-1}(\{ xy = 0 \}) \) in $V_1$ \( x^1, y^2 = 0 \).

Restrict $\pi$ to the two components \( \{ x^1 = 0 \} \) and \( \{ y' = 0 \} \), gives a resolution (strict transform).
4/ def: Given a proper birational morphism \( f : W \to \mathbb{P}^2 \) st. \( f : W \setminus f^{-1}(Z) \to \mathbb{P}^2 \setminus Z \) is an isomorphism, given \( Y \subseteq Z \) a closed analytic subset, 

- the exceptional locus of \( f \) is \( f^{-1}(Z) \), 
- the strict transform of \( Y \) is \( f^{-1}(Y) \).

We will always blow up smooth subvarieties of smooth varieties on subsets.

(Since for \( Z \subseteq X \) smooth, \( Bl_Z X \) is the strict transform of \( Z \) in \( Bl_Z X \)).

def: Let \( Z \subseteq X \) be a smooth analytic subset of a complex manifold of codimension \( k \).

\( Bl_Z X \subseteq X^{k+1} \) is the submanifold obtained by gluing the following local models:

- assume \( Z = \{ x_1 = \ldots = x_k = 0 \} \subseteq C^m \)
- then \( Bl_Z C^m \subseteq C^m \times \mathbb{P}^{k+1} \) is defined using the equations \( x_i y_j - x_j y_i = 0 \).

rmk: \( Bl_Z C^m \) is covered by affine charts \( U_i \times C^m \).

\[ \pi : U_i \to C^m \]

\[ (x_1, \ldots, x_n) \mapsto (x_1^1 x_1', \ldots, x_i^1 x_i', \ldots, x_n^1 x_n') \]

Resolving planar curve singularities

Consider \( C = \{ f(x, y) = 0 \} \subseteq \mathbb{P}^2 \) a singular (algebraic) plane curve. To resolve singularities of \( C \), one blows up singular points until:

- the strict transform of \( C \) is smooth
- the exceptional and locus of the resolution \( \pi : X \to \mathbb{P}^2 \) is a simple normal crossing division and the strict transform has normal crossings with it.

(if asked, maximal contact order decreases \( \min \{ \dim \mathcal{O}_p / (f, \ell) \} \)).
Cusp: resolution of the cusp \( f = y^2 - x^3 \) * strict transform

\[
\begin{align*}
x &= x' \\
y &= x'y' \\
y^2 - x^3 &= (y')^2 - x' \\
\end{align*}
\]

\[
\begin{align*}
x^3(y^2 - 1) &= (y')^2 \cdot (1 - (x')^3) \\
x^2 \cdot y^3(x-y) &= \text{stop because we got something smooth!} \\
\end{align*}
\]

normal crossings, so we are done.

In total, we did 3 blowing-ups in 3 distinct points: so in total we get the strict transform and 3 exceptional divisors \((\cong \mathbb{P}^1)\) \(E_1, E_2, E_3\).

Resolving isolated surface singularities.

Let \((x,0) \in (\mathbb{C}^2,0)\) be an isolated surface singularity. (exists \(\mathcal{U} \) a neighborhood in \( S \) s.t. \( \mathcal{U}\{0\} \) is smooth).

Consider a strong resolution \( \pi : Y \rightarrow X \). Then \( \pi^{-1}(0) = \bigcup_{i=1}^{\infty} E_i \).

We need information on the intersections of the \( E_i \)'s to build the dual graph \( \Rightarrow \) work in \( H^2(X) \) irreducible Smooth analytic subsets and use cup product as an intersection pairing.

We obtain an intersection matrix \( M = (E_i \cdot E_j)_{1 \leq i, j \leq n} \).

N.B. Typically, \( E_i \cdot E_j = 2 \) since we get a transverse intersection by taking a homologous cycle with negative coefficient.
We will also later consider multiplicities of the $E_i$'s. These are defined algebraically.

Take the ideal of equations defining $0 \in \mathcal{X} : I = (f_1, \ldots, f_k)$. Then
\[ x^* I = (f_{1|y}, \ldots, f_{k|y}) \subseteq \mathcal{O}_Y \] simplifies to:
\[ g_1 \ldots g_n \subseteq \mathcal{O}_Y \],
where $g_i$ is the equation of $E_i$.

$N_i$ is well-defined and is called the multiplicity of $E_i$.

**Def.** The weighted dual graph of $(X, 0)$ is built as follows:
* one vertex for each prime divisor $E_i$;
* draw one edge between $E_i \neq E_j$ if they intersect;
* label each vertex by $E_i \cdot E_i$.

**Rem.** If each $E_i$ has self-intersection $-2$, then the dual graph is an ADE Dynkin diagram. (Known explicit equations.)

**Thm.** Any isolated surface singularity has a

**Thm. (Mumford)** $M$ is negative definite

**Rem.**

**Thm. (Grammel)** If a compact divisor $E = \Sigma E_i$ on a non-singular surface $\mathcal{Y}$ has negative definite intersection matrix, then there exists a unique normal singularity $(X, \mathcal{E})$ and a proper morphism $f : \mathcal{Y} \to X$ such that $\mathcal{Y} \to \mathcal{X}$ is isomorphic.

(If asked: any surface singularity has a minimal resolution, so the graph of this resolution is an invariant of the singularity — but the $E_i$ need not be isomorphic if the graphs are the same.)

**Prop.:**
\[ (X, 0) \subseteq (\mathbb{C}^3, 0) \]
\[ (3x^2 + y^3)(x^3 + 3y^2) + z^3 = 0 \]
\[ (Y, 0) \subseteq (\mathbb{C}^3, 0) \]
\[ w^2 + x^2 + y^3 = 0 \]

 Blow up the origin in $\mathbb{C}^3$; in blowing-up charts we obtain
\[ w^2(1 + w^2 + wy^3) \]
\[ x^2(1 + w^2 + xy^3) \]
\[ y^2 (y + x^2 + w^2) \]
all strict transform is smooth.
\[ \cong \text{Bl}_X \]

Simple but normal crossings!

But exceptional divisor is not smooth
\[ (\text{in charts } x^2 + 1 = 0) \]
\[ w^2 + 1 = 0 \]

Additional blow up in this singular point of the divisor. (Smooth on the surface.)
\[ o-o \]