2 Sheaves and Cohomology

2.1 Sheaves and Presheaves

We fix a topological space $X$. Later we will include assumptions that are satisfied by smooth manifolds.

2.1.1 Definitions and Examples

Definition 2.1. A presheaf of abelian groups $\mathcal{F}$ on $X$ assigns to each open $U \subseteq X$ an abelian group $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ and for every inclusion of open sets $V \subseteq U$ a homomorphism of abelian groups $\rho_{UV}^\mathcal{F} : \mathcal{F}(U) \to \mathcal{F}(V)$, often called the restriction map, satisfying

[P1] $\rho_{UU}^\mathcal{F} = 1_{\mathcal{F}(U)}$

[P2] for $W \subseteq V \subseteq U$, we have $\rho_{VW}^\mathcal{F} \circ \rho_{UV}^\mathcal{F} = \rho_{UW}^\mathcal{F}$.

If $\mathcal{F}$ and $\mathcal{G}$ are two presheaves (of abelian groups) on $X$, then a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of the data of a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subseteq X$ such that if $V \subseteq U$ is an inclusion, then we have commutative diagrams

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\
\rho_{UV}^\mathcal{F} \downarrow & & \downarrow \rho_{UV}^\mathcal{G} \\
\mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V).
\end{array}
\]

Remark 2.2. We may form a category $\textbf{Top}_X$ whose objects are open sets in $X$ and whose morphisms are simply inclusions of open sets. Then the above definition says that a presheaf is a contravariant functor $\textbf{Top}_X \to \textbf{Ab}$, and that a morphism of presheaves is a natural transformation of the associated functors.

Definition 2.3. A sheaf $\mathcal{F}$ of abelian groups on $X$ is a presheaf which, for any open set $U \subseteq X$ and any open covering $\{U_i\}_{i \in I}$ of $U$, satisfies the two additional properties:

[S1] if $s \in \mathcal{F}(U)$ is such that $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$;

[S2] if $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each $i$. Note that such an element is unique by [S1].

A morphism of sheaves is a morphism of the underlying presheaves.

Let $\mathcal{A}$ be a sheaf of rings on $X$, i.e. one for which $\mathcal{A}(U)$ is a ring for each open $U \subseteq X$ and for which the restriction maps are ring homomorphisms. If $\mathcal{F}$ is a sheaf of abelian groups such that for every open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{A}(U)$-module, then $\mathcal{F}$ is called a sheaf of $\mathcal{A}$-modules.

Examples 2.4. (a) The constant presheaf Let $X$ be a topological space and let $A$ be an abelian group. We define the constant presheaf with values in $A$ to be the presheaf $A^p_X$ to be the presheaf such that

\[ A^p_X(U) = A \]

for all non-empty $U \subseteq X$, and $\rho_{UV}^{A^p_X} = 1_A$ for all $V \subseteq U$.

(b) The constant sheaf With $X$ and $A$ as above, we define the constant sheaf with values in $A$ to be the sheaf $A_X$ whose sections $A_X(U)$ over $U$ are locally constant functions $U \to A$; this is the same as the set of continuous functions $U \to A$, where $A$ is given the discrete topology.
(c) **Functions** Let \( C^0 = C^0_X \) be the sheaf for which \( C^0(U) \) is the set of continuous \( \mathbb{R} \)-valued functions on \( U \). Then \( C^0 \) is a sheaf. Similarly, if \( X \) is a smooth (respectively, complex) manifold, then \( C^\infty_X \) (respectively, \( \Theta_X \)) is a sheaf, where \( C^\infty_X(U) \) (respectively, \( \Theta_X(U) \)) is the ring of smooth (respectively, holomorphic) functions on \( U \). These are all sheaves of rings.

(d) **Skyscraper sheaves** Let \( X \) and \( A \) be as above, and fix \( x \in X \). Then we can define a sheaf \( A_x \) by

\[
A_x(U) := \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise}. \end{cases}
\]

In the case that that \( A = \mathbb{C} \), then \( C_x \) is a sheaf of \( C^0_X \)-modules, where a function acts on a section simply by multiplying by its value at \( x \).

(e) **Vector bundles** Let \( X \) be as above and let \( \pi : E \to X \) be a vector bundle over \( X \). Then for any \( U \subseteq X \),

\[
U \mapsto \Gamma(U, E)
\]

defines a sheaf. For a vector bundle \( E \), we will typically also denote its sheaf of sections by \( E \), as no confusion is likely to occur. By Remark, \( E \) is a sheaf of \( C^0 \)-modules; it is a sheaf of \( C^\infty \)-modules or \( \Theta \)-modules if \( E \) is smooth or holomorphic, respectively.

### 2.1.2 Stalks

We first recall the notion of a colimit.

**Definition 2.5.** Let \( \mathcal{I} \) be a small category (i.e., \( \text{Ob} \mathcal{I} \) is a set) and \( \mathcal{C} \) any category. Let \( \alpha : \mathcal{I} \to \mathcal{C} \) be a functor; we thus, think of this data as a set of objects of \( \mathcal{C} \) indexed by \( \mathcal{I} \), with morphisms between these objects indexed by morphisms in \( \mathcal{I} \). Any object \( X \in \text{Ob} \mathcal{C} \) determines such a functor \( \Delta_X : \mathcal{I} \to \mathcal{C} \), namely, the constant functor, i.e., \( \Delta_X \) takes all objects of \( \mathcal{I} \) to \( X \) and all morphisms to \( \text{id}_X \). If \( Y \in \text{Ob} \mathcal{C} \) is another object and \( f : X \to Y \) a morphism, then there is an induced natural transformation \( \Phi = \Phi_f : \Delta_X \to \Delta_Y \) which is simply \( f \) at each object of \( \mathcal{I} \).

We say that \( \alpha \) has a colimit \( X \in \text{Ob} \mathcal{C} \) if there is a natural transformation \( \lambda : \alpha \to \Delta_X \) such that if \( Y \in \text{Ob} \mathcal{C} \) and \( \tau : \alpha \to \Delta_Y \) is a natural transformation, then there is a unique \( f : X \to Y \) such that

\[
\begin{array}{ccc}
\lambda & \Downarrow \Phi_f \\
\Delta_X & \longrightarrow & \Delta_Y \\
\tau & \Downarrow \end{array}
\]

commutes. [The colimit is often called the “direct limit,” but for the sake of standardizing language, “colimit” is preferable.] It is not hard to see that if a colimit exists, then it is unique up to a unique isomorphism.

**Lemma 2.6.** If \( \mathcal{I} \) is a small category and \( \alpha : \mathcal{I} \to \text{Ab} \) is a functor, then \( \text{colim} \alpha \) exists.

**Proof.** For \( i \in \text{Ob} \mathcal{I} \), we will write \( X_i \) for \( \alpha(i) \). Then let \( \hat{X} := \bigoplus_{i \in \text{Ob} \mathcal{I}} X_i \) be the direct sum and let \( N \subseteq \hat{X} \) be the subgroup generated by elements of the form \((\ldots, 0, x_i, \ldots, -\alpha(f)(x_i), \ldots)\) for morphisms \( f : i \to j \). It is left as an exercise to show that \( X := \hat{X}/N \) is the colimit. \( \square \)

**Remark 2.7.** From the construction above, any element of \( \text{colim} \alpha \) is a tuple in \( \bigoplus X_i \) with finitely many non-zero components, corresponding to \( i_1, \ldots, i_r \in \text{Ob} \mathcal{I} \). Suppose that for any such \( i_1, \ldots, i_r \), there is \( j \in \text{Ob} \mathcal{I} \) with arrows \( i_p \to j \) for \( 1 \leq p \leq r \). Then by using the relation in the proof above, one can represent any element of \( \text{colim} \alpha \) by one in the image of \( X_j \to X \).
Let $X$ be a topological space and fix a point $x \in X$. Recall that $\text{Top}_X$ was the category whose objects are open sets in $X$ and whose morphisms are inclusions. Consider the (full) subcategory $\text{Top}_{X,x}$ whose objects are open neighbourhoods of $x$. It is easy to see that $\text{Top}_{X,x}$ has the property mentioned in Remark 2.7, since the intersection of any finite set of open neighbourhoods of $x$ is also one.

Now, if $\mathcal{F}$ is a presheaf, which we may think of as a functor $\mathcal{F} : \text{Top}_{X,x} \to \text{Ab}$, then we can restrict it to $\mathcal{F} : \text{Top}_{X,x} \to \text{Ab}$. By Lemma 2.6,

$$\mathcal{F}_x := \colim_{x \in U} \mathcal{F} = \colim_{x \in U} \mathcal{F}(U)$$

exists; we call it the stalk of $\mathcal{F}$ at $x$. By properties of the colimit, for each open neighbourhood $U$ of $x$, we have a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ and these “are compatible” with the restriction maps. Remark 2.7 says that any element of $\mathcal{F}_x$ is the image of some $s \in \mathcal{F}(U)$ for some neighbourhood $U$ of $x$.

We may repeat the above definition for an arbitrary set $S \subseteq X$, instead of $\{x\}$, by defining the subcategory $\text{Top}_{X,S}$ of $\text{Top}_X$ with objects open sets $U \subseteq X$ with $S \subseteq U$. In this way, we can extend the presheaf $\mathcal{F}$ to all subsets of $X$ by

$$\mathcal{F}(S) := \colim_{S \subseteq U} \mathcal{F}(U).$$

### 2.2 Sheafification

**Proposition 2.8.** Let $\mathcal{F}$ be a presheaf on $X$. Then there exists a sheaf $\mathcal{F}^+$ together with a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that if $\varphi : \mathcal{F} \to \mathcal{G}$ is any morphism and $\mathcal{G}$ is a sheaf, then there is a unique morphism $\Phi : \mathcal{F}^+ \to \mathcal{G}$ such that

$$\varphi = \Phi \circ \theta$$

commutes.

**Definition 2.9.** The sheaf $\mathcal{F}^+$ is called the sheafification of $\mathcal{F}$.

**Remark 2.10.** If $\mathcal{F}$ is a sheaf, then $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

**Proof.** We define $\mathcal{F}^+$ as follows. Let $U \subseteq X$ be open. We let $\mathcal{F}^+(U)$ be the set of elements

$$\sigma \in \prod_{x \in U} \mathcal{F}_x$$

such that for each $x \in U$, there is an open neighbourhood $V$ of $x$ contained in $U$ and a section $s \in \mathcal{F}(V)$ such that $\text{pr}_y(\sigma) = s_y$ for all $y \in V$, where $\text{pr}_y$ is the obvious projection map. We leave it as an exercise to show that this is a sheaf.

Given $s \in \mathcal{F}(U)$, there is a unique $\sigma \in \prod_{x \in U} \mathcal{F}_x$, where $\text{pr}_x(\sigma) = s_x$; clearly, we have $\sigma \in \mathcal{F}^+(U)$ and so we set

$$\theta_U(s) := \sigma.$$

It is an exercise to verify that this defines $\theta$. Finally, we also leave as an exercise the proof of the mapping property. \[\square\]
2.3 Kernels and Images

Definition 2.11. If $\mathcal{F}$ is a presheaf, then a subpresheaf $\mathcal{G}$ of $\mathcal{F}$ is a presheaf for which $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$ for all open $U \subseteq X$ and for which the restriction morphisms are induced from those of $\mathcal{F}$. A subsheaf of a sheaf is simply a subpresheaf which is also a sheaf.

Lemma 2.12. If $\mathcal{G}$ is a subpresheaf of a sheaf $\mathcal{F}$ (which is not necessarily a sheaf itself), then $\mathcal{G}^+$ is (or may be realized as) a subsheaf of $\mathcal{F}$.

Proof. It is an exercise to show that since $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, for each $x \in X$, $\mathcal{G}_x \subseteq \mathcal{F}_x$. For each open $U \subseteq X$, let

$$\mathcal{G}^p(U) := \{ s \in \mathcal{F}(U) : \forall x \in X, s_x \in \mathcal{G}_x \} \subseteq \mathcal{F}(U).$$

Then $\mathcal{G}^p$ is a subsheaf of $\mathcal{F}$. There is a natural map $\theta : \mathcal{G} \rightarrow \mathcal{G}^p$ and with this map, $\mathcal{G}^p$ satisfies the universal mapping property of $\mathcal{G}^+$, and so $\mathcal{G}^p \cong \mathcal{G}^+$.

Definition 2.13. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. Then we may define the kernel presheaf $\ker^p \varphi$ and the image presheaf $\text{im}^p \varphi$ by

$$\ker^p \varphi(U) := \ker \varphi_U \subseteq \mathcal{F}(U) \quad \text{and} \quad \text{im}^p \varphi(U) := \text{im} \varphi_U \subseteq \mathcal{G}(U).$$

Lemma 2.14. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then $\ker^p \varphi$ is a sheaf.

Proof. This is left as an exercise.

Example 2.15. Let $X = \mathbb{C}^*$, and $\mathcal{O}_X$, $\mathcal{O}^*_X$ the sheaves of holomorphic functions and non-vanishing holomorphic functions, respectively. Consider the exponential map $\exp : \mathcal{O}_X \rightarrow \mathcal{O}^*_X$.

We claim the image presheaf in $\mathcal{O}^*_X$ is not a sheaf. Let $\{U_i\}_{i \in I}$ be a covering of $X$ by simply connected open sets. Then it is clear that $1 \in \mathcal{O}^*_X(U_i)$ for all $i \in I$. Since one can choose a logarithm on each simply connected open set, each $1 \in U_i$ lies in the image presheaf. These obviously agree on overlaps $U_i \cap U_j$, however, there is no holomorphic function $f \in \mathcal{O}_X(X)$ with $\exp(f) = 1$.

Definition 2.16. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then the kernel of $\varphi$ is simply defined as the sheaf $\ker \varphi := \ker \varphi^p$. We define the image of $\varphi$ as the sheafification $\text{im} \varphi := (\text{im}^p \varphi)^\#$. By Lemma 2.12, this is a subsheaf of $\mathcal{G}$. We say that $\varphi$ is injective if $\ker \varphi = 0$ and $\varphi$ is surjective if $\text{im} \varphi = \mathcal{G}$.

Proposition 2.17. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then $\varphi$ is injective (respectively, surjective, bijective) if and only if $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective (respectively, surjective, bijective) for all $x \in X$.

Proof. Exercise.

2.4 Complexes

Virtually all of the definitions we now make in the category of sheaves of abelian groups over a topological space $X$ make sense for a general abelian category, but we will not go into this generality in the interest of saving time.

Definition 2.18. A complex $\mathcal{F}^\bullet$ of sheaves (of abelian groups) over $X$ is a sequence of morphisms of sheaves of abelian groups

$$\cdots \rightarrow \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \rightarrow \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$; the maps $d^i$ are often called differentials. Clearly, $\text{im} d^{i-1}$ is a subsheaf of $\ker d^i$ for all $i \in \mathbb{Z}$; if they are equal, then we say that the complex is exact. If $\mathcal{F}^\bullet$ and
Proof. Exercise.

Definition 2.20. Let $\varphi^i, \psi^i : \mathcal{F}^i \to \mathcal{G}^i$ be two morphisms of complexes. A homotopy from $\varphi^i$ to $\psi^i$ is a collection of morphisms $k^i : \mathcal{F}^i \to \mathcal{G}^i$ such that

\[
\varphi^i - \psi^i = k^i+1 \circ d^i_{\mathcal{F}} + d^{i-1}_{\mathcal{G}} \circ k^i
\]

for all $i \in \mathbb{Z}$. It is not hard to show that the relation “there exists a homotopy from $f$ to $g$” is an equivalence relation on the set of morphisms from $\mathcal{F}^\bullet$ to $\mathcal{G}^\bullet$. We say that the morphisms $\varphi^\bullet$ and $\psi^\bullet$ are homotopic if there is a homotopy from $\varphi^\bullet$ to $\psi^\bullet$.

Proof. Exercise.

2.5 Resolutions

Definition 2.21. Let $\mathcal{F}$ be a sheaf on $X$. A resolution of $\mathcal{F}$ is a complex of sheaves $\mathcal{A}^\bullet$, non-zero only for indices in $\mathbb{Z}_{\geq 0}$, such that

\[
0 \to \mathcal{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots
\]

is exact.

Proposition 2.22 (de Rham resolution). Let $M$ be a smooth manifold of (real) dimension $r$ and $\mathbb{R}_M$ the constant sheaf with coefficients in $\mathbb{R}$. Then if $\mathcal{A}^k_{M, \mathbb{R}}$ is the sheaf of $\mathbb{R}$-valued differential $k$-forms on $M$, one has a resolution

\[
0 \to \mathbb{R}_M \to \mathcal{A}^0_{M, \mathbb{R}} \to \mathcal{A}^1_{M, \mathbb{R}} \to \cdots \to \mathcal{A}^r_{M, \mathbb{R}} \to 0
\]

of $\mathbb{R}_M$. The statement also holds with $\mathbb{R}$ replaced by $\mathbb{C}$.

Proof. $\mathcal{A}^0_{M, \mathbb{R}}$ is the sheaf of smooth functions on $M$ and since locally constant functions are certainly smooth, we have an inclusion $\mathbb{R}_M \hookrightarrow \mathcal{A}^0_{M, \mathbb{R}}$. Furthermore, on any open set $U \subseteq M$, the kernel of $d : \mathcal{A}^0_{M, \mathbb{R}}(U) \to \mathcal{A}^1_{M, \mathbb{R}}(U)$ is precisely the subgroup of locally constant functions. To see that it is exact at $\mathcal{A}^k_{M, \mathbb{R}}$ for any $k \geq 1$, we may pass to the stalks: let $x \in M$; an element $\nu \in (\mathcal{A}^k_{M, \mathbb{R}})_x$ with $d\nu = 0$ is represented by some $\alpha \in \mathcal{A}^k_{M, \mathbb{R}}(U)$ for some open neighbourhood $U$ of $x$ with $d\alpha = 0$. If $U$ is an open ball, then the Poincaré lemma states that there is some $\beta \in \mathcal{A}^{k-1}_{M, \mathbb{R}}(U)$ with $d\beta = \alpha$; if $\xi \in (\mathcal{A}^{k-1}_{M, \mathbb{R}})_x$ is the image of $\beta$ in the stalk, then $d\xi = \nu$, and so we have exactness at stalks. The proof is exactly the same for $\mathbb{C}$. 


2.5.1 The Dolbeault Complex of a Holomorphic Bundle

Let \( E \) be a holomorphic vector bundle of rank \( r \) over a complex manifold \( X \). For an open set \( U \subseteq X \), let \( \mathcal{A}^{0,q}(U,E) \) be the space of smooth sections of \( E \otimes \Omega^q \). If \( \varphi : E_U \cong U \times \mathbb{C}^r \) is a trivialization, then a section \( b \in \mathcal{A}^{0,q}(U,E) \) corresponds to a \( r \)-tuple \( \beta = (\beta_1, \ldots, \beta_r) \) with \( \beta_i \in \mathcal{A}^{0,q}(U), 1 \leq i \leq r \). Now, we define an operator \( \overline{\partial}_E : \mathcal{A}^{0,q}(U,E) \to \mathcal{A}^{0,q+1}(U,E) \) by taking

\[
\overline{\partial}_E b = \varphi^{-1}(\overline{\partial}\beta) = \varphi^{-1}(\overline{\partial}\beta_1, \ldots, \overline{\partial}\beta_r)^t.
\]

Lemma 2.23. The definition of \( \overline{\partial}_E \) is independent of the choice of trivialization and so gives a well-defined operator \( \mathcal{A}^{0,q}(U,E) \to \mathcal{A}^{0,q+1}(U,E) \) for any open set \( U \subseteq X \).

Proof. Suppose \( \varphi_i, \varphi_j \) are trivializations of \( E \) on \( U_i, U_j \), respectively, and suppose \( b \in \mathcal{A}^{0,q}(U_i \cap U_j) \) corresponds to \( \beta_i, \beta_j \) with respect to \( \varphi_i, \varphi_j \). If the transition function is \( g_{ij} \), then we know that \( \beta_i = g_{ij} \beta_j \). To see that \( \overline{\partial}_E \) is well-defined, we want to see that

\[
\varphi_i^{-1}(\overline{\partial}\beta_i) = \varphi_j^{-1}(\overline{\partial}\beta_j).
\]

But since \( g_{ij} \) is holomorphic

\[
\overline{\partial}\beta_i = \overline{\partial}(g_{ij} \beta_j) = g_{ij} \wedge \beta_j + g_{ij} \overline{\partial}\beta_j = g_{ij} \overline{\partial}\beta_j = \varphi_i \circ \varphi_i^{-1}(\overline{\partial}\beta_j).
\]

Proposition 2.24. In the following \( U \subseteq X \) is open.

(a) We have \( \overline{\partial}_E^2 = 0 \).

(b) We have

\[
\Gamma(U, E) = \ker(\overline{\partial}_E : \mathcal{A}^{0,0}(U,E) \to \mathcal{A}^{0,1}(U,E)).
\]

(c) If \( \alpha \in \mathcal{A}^{0,q}(U), \beta \in \mathcal{A}^{0,q'}(U,E) \), then \( \alpha \wedge \beta \in \mathcal{A}^{0,q+q'}(U,E) \) and

\[
\overline{\partial}_E(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^q \alpha \wedge \overline{\partial}_E\beta.
\]

(d) Let \( \alpha \in \mathcal{A}^{0,q}(U,E), q > 0 \) be such that \( \overline{\partial}_E \alpha = 0 \). Then given \( x \in U \), there is a neighbourhood \( V \) of \( x \) and \( \beta \in \mathcal{A}^{0,q-1}(U,E) \) with

\[
\alpha|_V = \overline{\partial}_E \beta.
\]

Proof. These are all clear from the local definition above and the relevant statements for \((p,q)\)-forms: Lemma ?? for (a), Remark ?? for (b), Lemma ?? for (c) and Proposition ?? for (d).

Proposition 2.25 (Dolbeault resolution). Let \( X \) be a complex manifold of dimension \( n \) and let \( E \) be a holomorphic vector bundle on \( X \). Then the complex

\[
0 \to E \to \mathcal{A}^0(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,n}(E)
\]

gives a resolution of the sheaf of sections of \( E \). In the particular case where \( E = \Omega^p_X \), then one has the resolution

\[
0 \to \Omega^p_X \to \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}_E} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}_E} \cdots \xrightarrow{\overline{\partial}_E} \mathcal{A}^{p,n}.
\]

Proof. The proof is as for the de Rham resolution, except that we use Proposition 2.24 (d) in the place of the Poincaré lemma.
2.6 Injective Sheaves

**Definition 2.26.** Let $X$ be a topological space. A sheaf $\mathcal{I}$ over $X$ is called *injective* if given an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G}$$

and a morphism $\mathcal{F} \to \mathcal{I}$, there exists a (not necessarily unique) morphism $\mathcal{G} \to \mathcal{I}$ making the diagram commute. (Thus, any morphism from a subsheaf to an injective can be extended to the whole sheaf.)

A sheaf $\mathcal{F}$ is called *flasque* or *flabby* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

**Lemma 2.27.** A sheaf $\mathcal{F}$ is injective if and only if the functor $\text{Hom}_{\mathcal{Sh}(X)}(\cdot, \mathcal{F}) : \mathcal{Sh}(X) \to \text{Ab}$ is exact.

**Proof.** Exercise.

**Lemma 2.28.** Every injective sheaf is flasque.

**Proof.** Let $\mathcal{I}$ be an injective sheaf and let $V \subseteq U$ be an inclusion of open sets. We define a sheaf $\mathbb{Z}_{U,X}$ on $X$ by

$$\mathbb{Z}_{U,X}(W) := \begin{cases} \mathbb{Z}_X(W) & \text{if } W \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

We can similarly define $\mathbb{Z}_{V,X}$ and we have $\mathbb{Z}_{U,X}(W) = \mathbb{Z}_{V,X}(W)$ unless $W \subseteq U$ and $W \not\subseteq V$ in which case the right side is zero. In all cases, the sections of $\mathbb{Z}_{U,X}$ are contained in those of $\mathbb{Z}_{V,X}$ and hence we have an exact sequence of sheaves on $X$

$$0 \to \mathbb{Z}_{V,X} \to \mathbb{Z}_{U,X}.$$ 

Applying the functor $\text{Hom}_{\mathcal{Sh}(X)}(\cdot, \mathcal{F})$, which is exact by Lemma 2.27, we get an exact sequence

$$\text{Hom}_{\mathcal{Sh}(X)}(\mathbb{Z}_{U,X}, \mathcal{F}) \to \text{Hom}_{\mathcal{Sh}(X)}(\mathbb{Z}_{V,X}, \mathcal{F}) \to 0.$$ 

But now suppose that $\varphi : \mathbb{Z}_{U,X} \to \mathcal{F}$ is a sheaf homomorphism. Then if $W \not\subseteq U$, $\varphi_W$ must be zero. If $W = U$, then the group of sections of $\mathbb{Z}_{U,X}(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi_U$ on this connected component is determined by the image of 1 the space of sections of $\mathcal{F}$ on this connected component; thus, $\varphi_U$ can be thought of an element of $\mathcal{F}(U)$. Now, on any proper open subset of $U$, $\varphi$ is determined by the restriction maps. Hence $\text{Hom}_{\mathcal{Sh}(X)}(\mathbb{Z}_{U,X}, \mathcal{F}) = \mathcal{F}(U)$. Doing the same thing above for $V$, we get an exact sequence $\mathcal{F}(U) \to \mathcal{F}(V) \to 0$, which was our claim.

**Lemma 2.29.** Every abelian group can be embedded into an injective abelian group.

**Proof.** Exercise: give a proof or find a precise reference for this statement.

**Lemma 2.30.** Let $X$ be a topological space. Every sheaf of abelian groups on $X$ can be embedded in an injective sheaf.
Remark 2.31. This statement does not hold for an arbitrary abelian category. This is the assumption that an abelian category “has enough injectives.”

Proof. For each $x \in X$, Lemma 2.29 tells us that there is an injective abelian group $I_x$ with $\mathcal{F}_x \subseteq I_x$. Let

$$\mathcal{I}(U) := \prod_{x \in U} I_x.$$  

Then this defines a sheaf $\mathcal{I}$ on $X$ with the restriction maps coming from the projections.

We claim that for any sheaf $\mathcal{G}$ we have

$$\text{Hom}_{\mathcal{O}(X)}(\mathcal{G}, \mathcal{I}) \cong \prod_{x \in X} \text{Hom}_{\mathbb{A}(x)}(\mathcal{G}_x, I_x).$$

Given $\varphi : \mathcal{G} \to \mathcal{I}$ and $x \in X$, for any open neighbourhood $U$ of $x$, we have maps

$$\mathcal{G}(U) \xrightarrow{\varphi_U} \mathcal{I}(U) = \prod_{x \in U} I_x \xrightarrow{\text{pr}_x} I_x,$$

and since $\mathcal{G}_x$ is defined as a colimit, this defines a unique map $\mathcal{G}_x \to I_x$. As we have one such for each $x \in X$, we get a map from the left to the right. In the other direction, suppose we are given $f \in \prod_{x \in X} \text{Hom}_{\mathbb{A}(x)}(\mathcal{G}_x, I_x)$. Then we have a map in the opposite direction via the composition

$$\mathcal{G}(U) \to \prod_{x \in U} \mathcal{G}_x \xrightarrow{f|_U} \prod_{x \in U} I_x,$$

and one can check that these are inverse to each other.

Now, given an inclusion

$$0 \to \mathcal{F} \to \mathcal{G},$$

we apply $\text{Hom}_{\mathcal{O}(X)}(\cdot, \mathcal{I})$ and use the bijection above to obtain a diagram

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{O}(X)}(\mathcal{G}, \mathcal{I}) & \xrightarrow{\text{incl}} & \text{Hom}_{\mathcal{O}(X)}(\mathcal{F}, \mathcal{I}) \\
\prod_{x \in X} \text{Hom}_{\mathbb{A}(x)}(\mathcal{G}_x, I_x) & \xrightarrow{\text{incl}} & \prod_{x \in X} \text{Hom}_{\mathbb{A}(x)}(\mathcal{F}_x, I_x)
\end{array}$$

whose vertical maps are bijections; we want to show that the top row is surjective, but this is equivalent to the bottom row being surjective. But since we have inclusions $0 \to \mathcal{F}_x \to \mathcal{G}_x$ for each $x \in X$,

$$\text{Hom}_{\mathbb{A}(x)}(\mathcal{G}_x, I_x) \to \text{Hom}_{\mathbb{A}(x)}(\mathcal{F}_x, I_x)$$

is surjective since each $I_x$ is an injective abelian group.

Remark 2.32. One should be careful to note that in the above proof, it is not true that $\mathcal{I}_x = I_x$; the stalk $\mathcal{I}_x$ depends very much on the topology of $X$.

Definition 2.33. A resolution $\mathcal{F} \to \mathcal{F}^\bullet$ of a sheaf $\mathcal{F}$ is called injective if the $\mathcal{F}^i$ are injective sheaves for all $i \geq 0$.

Corollary 2.34. Every sheaf of abelian groups has an injective resolution.
**Proof.** Let \( \mathcal{F} \in \text{Ob} \mathcal{Sh}(X) \). By Lemma 2.30, there is an injective sheaf \( \mathcal{I}^0 \) and an exact sequence 

\[
0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1
\]

is exact. The resolution is obtained by continuing along in this manner. 

**Lemma 2.35.** Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a morphism of sheaves and \( \mathcal{G} \to \mathcal{F}^\bullet \) an injective resolution. 

(a) If \( \mathcal{F} \to \mathcal{F}^\bullet \) is any resolution, then there is a morphism of complexes \( \varphi^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet \) such that 

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{F}^\bullet & \xrightarrow{\varphi^\bullet} & \mathcal{G}^\bullet
\end{array}
\]

commutes.

(b) If \( \psi^\bullet : \mathcal{G}^\bullet \to \mathcal{F}^\bullet \) is any other morphism of complexes making the above diagram commute, then \( \varphi^\bullet \) and \( \psi^\bullet \) are homotopic.

**Proof.** Exercise. 

**Lemma 2.36.** Let 

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

be an exact sequence of sheaves on \( X \). Then there exist injective resolutions \( \mathcal{F}^\bullet, \mathcal{G}^\bullet, \mathcal{H}^\bullet \) of \( \mathcal{F}, \mathcal{G}, \mathcal{H} \), respectively and a commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \mathcal{F} & \mathcal{G} & \mathcal{H} & 0 \\
0 & \mathcal{F}^\bullet & \mathcal{G}^\bullet & \mathcal{H}^\bullet & 0
\end{array}
\]

where the bottom row is an exact sequence of complexes.

**Proof.** Exercise. 

### 2.7 Sheaf Cohomology

Given a sheaf \( \mathcal{F} \), we may take its group of global sections \( \Gamma(X, \mathcal{F}) \), and this defines the global section functor \( \Gamma(X, \cdot) : \mathcal{Sh}(X) \to \mathbb{Ab} \).

**Lemma 2.37.** The functor \( \Gamma(X, \cdot) \) is left exact in the sense that if 

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}
\]

is an exact sequence of sheaves, then 

\[
0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})
\]

is exact as a sequence of abelian groups.
Proof. The exact sequence says that \( \mathcal{F} \) is the kernel of the map \( \mathcal{G} \to \mathcal{H} \). Exactness at \( \Gamma(X, \mathcal{F}) \) simply says that the global sections of \( \mathcal{F} \) are a subgroup of those of \( \mathcal{G} \); exactness at \( \Gamma(X, \mathcal{G}) \) says that \( \Gamma(X, \mathcal{F}) \) is the kernel of \( \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \), but the kernel of this is the group of global sections of the kernel presheaf, which is precisely \( \mathcal{F} \), by Lemma 2.14.

**Definition 2.38.** Let \( X \) be a topological space and let \( \mathcal{F} \in \text{Ob} \mathcal{S}\text{h}(X) \) be a sheaf of abelian groups on \( X \). Let \( 0 \to \mathcal{F} \to \mathcal{F}^* \) be an injective resolution of \( \mathcal{F} \). The \( i \)th (sheaf) cohomology group of \( \mathcal{F} \) is defined as

\[
H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{F}^*)).
\]

**Lemma 2.39.** The definition above is independent (up to isomorphism) of the choice of injective resolution of \( \mathcal{F} \).

**Proof.** Exercise.

**Lemma 2.40.** Let

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

be an exact sequence of sheaves with \( \mathcal{F} \) flasque. Then

\[
0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to 0
\]

is exact.

**Proof.** Exercise.

**Theorem 2.41.** The sheaf cohomology groups satisfy the following properties.

(a) We have \( H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) \).

(b) If \( \mathcal{F} \) is an injective sheaf, then \( H^i(X, \mathcal{F}) = 0 \) if \( i > 0 \).

(c) If \( \varphi : \mathcal{F} \to \mathcal{G} \) is a sheaf morphism, then for \( i \geq 0 \), there is a group homomorphism

\[
H^i(\varphi) : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})
\]

such that

(i) \( \varphi^0 = \varphi_X : \mathcal{F}(X) \to \mathcal{G}(X) \);

(ii) \( H^i(1_\mathcal{F}) = 1_{H^i(X, \mathcal{F})} \) for all \( i \geq 0 \);

(iii) if \( \psi : \mathcal{G} \to \mathcal{H} \) is a sheaf morphism, then

\[
H^i(\psi \circ \varphi) = H^i(\psi) \circ H^i(\varphi).
\]

(d) If

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

is a short exact sequence of sheaves, then there is a long exact sequence

\[
0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to \cdots
\]

such that if

\[
\begin{array}{cccccccccc}
0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{H} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{H} & \to & 0
\end{array}
\]
is a commutative diagram with exact rows, then

\[
\begin{array}{ccccccc}
0 & \rightarrow & \Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) & \rightarrow & H^1(X, \mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) & \rightarrow & H^1(X, \mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(X, \mathcal{F}) & \rightarrow & \Gamma(X, \mathcal{G}) & \rightarrow & \Gamma(X, \mathcal{H}) & \rightarrow & H^1(X, \mathcal{F}) \\
\end{array}
\]

commutes.

**Proof.** If \( \mathcal{F} \rightarrow \mathcal{F}^\bullet \) is an injective resolution, then

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1
\]

is exact, so by left exactness of \( \Gamma(X, \cdot) \) (Lemma 2.37), we have

\[
0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{F}^1),
\]

but \( H^0(X, \mathcal{F}) \) is by definition \( \ker d^0 \) (since there is no \(-1\)-term), but the diagram says that this is precisely \( \Gamma(X, \mathcal{F}) \). This proves (a).

If \( \mathcal{F} \) is an injective sheaf, then one can take an injective resolution with \( \mathcal{F}^0 = \mathcal{F} \) and \( \mathcal{F}^i = 0 \) for \( i > 0 \); then it is obvious that \( H^i(X, \mathcal{F}) = 0 \) for \( i > 0 \). This proves (b).

Given injective resolutions \( \mathcal{F} \rightarrow \mathcal{F}^\bullet, \mathcal{G} \rightarrow \mathcal{G}^\bullet \), Lemma 2.35 gives a morphism of complexes \( \varphi^\bullet : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \) making

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{F}^\bullet & \xrightarrow{\varphi^\bullet} & \mathcal{G}^\bullet
\end{array}
\]

commute, unique up to homotopy. We then take global sections and get a map of complexes of abelian groups \( \varphi^*_X : \Gamma(X, \mathcal{F}^\bullet) \rightarrow \Gamma(X, \mathcal{G}^\bullet) \), and we define

\[
H^i(\varphi) := H^i(\varphi^*_X) : H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{F})) \rightarrow H^i(X, \mathcal{G}) = H^i(\Gamma(X, \mathcal{G})).
\]

One sees that property (i) holds by since \( \varphi^0_X \) will take \( \ker d^0_{\mathcal{F}} \) to \( \ker d^0_{\mathcal{G}} \) and that (ii) and (iii) hold by uniqueness (i.e., the fact that the relevant maps of complexes are homotopic). This proves (c).

Suppose we have an exact sequence as given. Then we may use Lemma 2.40 to obtain an exact sequence of complexes

\[
0 \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0,
\]

where \( \mathcal{F}^\bullet, \mathcal{G}^\bullet, \mathcal{H}^\bullet \) are injective resolutions of \( \mathcal{F}, \mathcal{G}, \mathcal{H} \), respectively. If we apply \( \Gamma(X, \cdot) \), then the rows remain exact, since each \( \mathcal{F}^i \) is injective and hence flasque, by Lemma 2.28. Therefore we obtain a diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \rightarrow & \Gamma(X, \mathcal{F}^0) & \xrightarrow{q^0} & \Gamma(X, \mathcal{F}^0) & \xrightarrow{p^0} & \Gamma(X, \mathcal{H}^0) & \rightarrow & 0 \\
\downarrow d^0_{\mathcal{F}} & & \downarrow d^0_{\mathcal{F}} & & \downarrow d^0_{\mathcal{H}} & & \downarrow d^0_{\mathcal{H}} \\
0 & \rightarrow & \Gamma(X, \mathcal{F}^1) & \xrightarrow{q^1} & \Gamma(X, \mathcal{F}^1) & \xrightarrow{p^1} & \Gamma(X, \mathcal{H}^1) & \rightarrow & 0 \\
\downarrow d^1_{\mathcal{F}} & & \downarrow d^1_{\mathcal{F}} & & \downarrow d^1_{\mathcal{H}} & & \downarrow d^1_{\mathcal{H}} \\
0 & \rightarrow & \Gamma(X, \mathcal{F}^2) & \xrightarrow{q^2} & \Gamma(X, \mathcal{F}^2) & \xrightarrow{p^2} & \Gamma(X, \mathcal{H}^2) & \rightarrow & 0
\end{array}
\]
so we can apply the snake lemma, which gives an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to \frac{\Gamma(X, \mathcal{I})}{\text{im} d_{\mathcal{I}}^{1}} \to \frac{\Gamma(X, \mathcal{J})}{\text{im} d_{\mathcal{J}}^{1}} \to \frac{\Gamma(X, \mathcal{K})}{\text{im} d_{\mathcal{K}}^{1}}.$$\[\text{Note that } H^1(X, \mathcal{F}) = \ker d_{\mathcal{F}}^{1}/\text{im} d_{\mathcal{F}}^{0} \subseteq \Gamma(X, \mathcal{I})/\text{im} d_{\mathcal{I}}^{0}, \text{ so we need to show that the image of the connecting homomorphism lies in } H^1(X, \mathcal{F}). \text{ Recall that if } z \in \Gamma(X, \mathcal{H}) \text{ then to obtain its image in } \text{coker } d_{\mathcal{H}}^{0}, \text{ we choose } y \in \Gamma(X, \mathcal{J}) \text{ with } p^0(y) = z \text{ and then observe that } d_{\mathcal{H}}^{0}(y) = q^1(x) \text{ for some } x \in \Gamma(X, \mathcal{I}), \text{ and } x \in \text{coker } d_{\mathcal{I}}^{0} \text{ is the image of } z. \text{ If } x \text{ is obtained like this, then we have } q^2(d_{\mathcal{I}}^{1}x) = d_{\mathcal{G}}^{1}(q^1x) = d_{\mathcal{G}}^{1}(q^1x) = 0, \text{ and since } q^2 \text{ is injective, we see that indeed } \tilde{x} \in H^1(X, \mathcal{F}). \text{ Then it is not hard to see that } q^1, p^1 \text{ map } \ker d_{\mathcal{F}}^{1}, \ker d_{\mathcal{G}}^{1} \text{ to } \ker d_{\mathcal{H}}^{0}, \ker d_{\mathcal{K}}^{0}, \text{ respectively, so we may replace the last three terms by } H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}). \text{ One must also check that this preserves exactness: for example, if } y \in \ker d_{\mathcal{F}}^{1} \text{ represents a class } \bar{y} \in H^1(X, \mathcal{F}) \text{ in the kernel of } H^1(X, \mathcal{F}) \to H^1(X, \mathcal{H}), \text{ then the snake lemma tells us that there is some } x \in \Gamma(X, \mathcal{I}) \text{ with } q^1x = y + d_{\mathcal{H}}^{0}y' \text{ for some } y' \in \Gamma(X, \mathcal{J}). \text{ But then } 0 = d_{\mathcal{G}}^{1}(y + d_{\mathcal{H}}^{0}y') = d_{\mathcal{G}}^{1}q^1x = q^2d_{\mathcal{I}}^{1}x \text{ and since } q^2 \text{ is injective, } d_{\mathcal{I}}^{1}x = 0, \text{ and so } \bar{x} \in H^1(X, \mathcal{F}), \text{ which proves that exactness is maintained. One continues in this manner to obtain the existence of the long exact sequence. The last statement is left as an exercise.}\]

\[\square\]

### 2.8 Acyclic Sheaves

**Definition 2.42.** A sheaf $\mathcal{F}$ is called *acyclic* if

$$H^i(X, \mathcal{F}) = 0$$

for $i > 0$. If $\mathcal{F}$ is any sheaf, then a resolution $0 \to \mathcal{F} \to \mathcal{P}^\bullet$ is called an *acyclic resolution* if each of the $\mathcal{P}^i$ is acyclic.

**Proposition 2.43.** If $\mathcal{F} \to \mathcal{P}^\bullet$ is any resolution of $\mathcal{F}$, then there are canonical maps

$$H^i(\Gamma(X, \mathcal{P}^\bullet)) \to H^i(X, \mathcal{F})$$

for $i \geq 0$, which are isomorphisms if $\mathcal{P}^\bullet$ is acyclic.

**Proof.** Lemma 2.35 gives us the existence of the a morphism of complexes $\varphi^\bullet : \mathcal{P}^\bullet \to \mathcal{I}^\bullet$ such that

$$\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{I}^\bullet \\
\varphi^\bullet & \simeq & \varphi^\bullet \\
\mathcal{F} & \longrightarrow & \mathcal{I}^\bullet
\end{array}$$

commutes and also states that any other such morphism is homotopic to $\varphi^\bullet$, so we get a well-defined map as claimed.

Let $\mathcal{K}^i := \ker d_{\mathcal{I}}^{i} = \text{im } d_{\mathcal{J}}^{i-1}$ and $\mathcal{L}^i := \ker d_{\mathcal{K}}^{i} = \text{im } d_{\mathcal{L}}^{i-1}$ and note that $\mathcal{K}^0 = \mathcal{L}^0 = \mathcal{F}$. For $i \geq 0$, we have commutative diagrams

$$\begin{array}{cccc}
0 & \longrightarrow & \mathcal{K}^i & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{K}^{i+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \varphi^i & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{L}^i & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{L}^{i+1} & \longrightarrow & 0
\end{array}$$

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with exact rows. Let us take \( i = 0 \) and take global sections to get:

\[
0 \to \mathcal{F}(X) \to \mathcal{F}^0(X) \to \mathcal{F}^1(X) \to H^1(X, \mathcal{F}) \to H^2(X, \mathcal{F}) \to 0
\]

Noting that \( \Gamma(X, \mathcal{F}^1) = \ker d^1_{\mathcal{F}} \), assuming that \( \mathcal{F}^0 \) is acyclic, we get an isomorphism

\[
H^1(\Gamma(X, \mathcal{F}^1)) = \ker d^1_{\mathcal{F}} / \operatorname{im} d^0_{\mathcal{F}} \cong H^1(X, \mathcal{F})
\]

compatible with the map to \( \mathcal{F}^0 \). Furthermore, for \( i \geq 1 \), we have isomorphisms \( H^i(X, \mathcal{F}^1) \cong H^{i+1}(X, \mathcal{F}) \) compatible with isomorphisms to \( H^{i+1}(X, \mathcal{F}) \).

Taking \( i = 1 \) and global sections, we get

\[
0 \to \mathcal{F}(X) \to \mathcal{F}^0(X) \to \mathcal{F}^1(X) \to H^1(X, \mathcal{F}) \to H^2(X, \mathcal{F}) \to 0
\]

and repeating the same arguments proves the statement.

\[ \square \]

### 2.9 Soft and Fine Sheaves

From now on, we will assume that \( X \) is a paracompact Hausdorff space. Note that topological (and hence smooth and complex) manifolds are such.

#### 2.9.1 Soft Sheaves

**Definition 2.44.** A sheaf \( \mathcal{F} \) is soft if for any closed set \( S \subseteq X \), the restriction map

\[
\mathcal{F}(X) \to \mathcal{F}(S)
\]

is surjective, i.e., any section over \( S \) can be extended to a global section.

**Lemma 2.45.** Flasque sheaves are soft.

**Proof.** Let \( \mathcal{F} \) be a flasque sheaf, \( S \subseteq X \) a closed set and let \( s \in \mathcal{F}(S) = \colim_{U \supseteq S} \mathcal{F}(U) \). Then \( s \) is represented by some \( \tilde{s} \in \mathcal{F}(U) \) for some \( U \supseteq S \). Since \( \mathcal{F} \) is flasque, there is \( \tilde{s} \in \mathcal{F}(X) \) that restricts to \( \tilde{s} \) and hence to \( s \).

**Theorem 2.46.** If

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

is an exact sequence of sheaves and \( \mathcal{F} \) is soft, then the induced sequence of global sections

\[
0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0
\]

is exact.

**Proof.** Exercise.

**Corollary 2.47.** If \( \mathcal{F} \) and \( \mathcal{G} \) are soft and

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

is exact, then \( \mathcal{H} \) is soft.
Proof. Let \( S \subseteq X \) be closed and \( s \in \mathcal{H}(s) \). Consider the above as a sequence of sheaves on \( S \): then \( \mathcal{F}|_S \) remains soft (since any closed set in \( S \) is also closed in \( X \)). Then applying the Theorem for the sequence over \( S \), there is a section of \( t \in \mathcal{H}(S) \) which maps to \( s \). Since \( \mathcal{G} \) is soft, there is a section \( \tilde{t} \in \mathcal{G}(X) \) restricting to \( t \). Then the image of \( \tilde{t} \in \mathcal{H}(X) \) will map to \( s \). \( \square \)

**Corollary 2.48.** If

\[
0 \to \mathcal{F}_0 \xrightarrow{d^0} \mathcal{F}_1 \xrightarrow{d^1} \cdots
\]

is an exact sequence of soft sheaves, then

\[
0 \to \Gamma(X, \mathcal{F}_0) \to \Gamma(X, \mathcal{F}_1) \to \cdots
\]

is also exact.

**Proof.** Let \( \mathcal{K}^i := \ker d^i = \text{im} d^{i+1} \). We first show by induction that \( \mathcal{K}^i \) is soft for \( i \geq 1 \). We have \( \mathcal{K}^1 = \mathcal{F}_0 \) and this is soft by assumption. Now, we have exact sequences

\[
0 \to \mathcal{K}^i \to \mathcal{F}^i \to \mathcal{K}^{i+1} \to 0,
\]

and by Corollary 2.47, \( \mathcal{K}^{i+1} \) is soft if \( \mathcal{K}^i \) is. Therefore, Theorem 2.46 gives exact sequences

\[
0 \to \Gamma(X, \mathcal{K}^i) \to \Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{K}^{i+1}) \to 0,
\]

and it follows that the exact sequence of global sections is exact. \( \square \)

**Corollary 2.49.** Soft sheaves are acyclic.

**Proof.** Let \( \mathcal{K} \) be a soft sheaf, \( \mathcal{K} \to \mathcal{I}^\bullet \) an injective resolution. Then since \( \mathcal{K} = \ker(\mathcal{I}^0 \to \mathcal{I}^1) \), we have an exact sequence

\[
0 \to \mathcal{I}^0 \xrightarrow{\mathcal{K}} \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots
\]

and since injective sheaves are flasque (Lemma 2.28) and flasque sheaves are soft (Lemma 2.45), applying Corollary 2.48, we obtain the conclusion. \( \square \)

### 2.9.2 Fine Sheaves

**Definition 2.50.** Let \( \mathcal{U} = \{U_i\}_{i \in I} \) be a locally finite open cover of \( X \) (i.e., every point of \( X \) has a neighbourhood which intersects only finitely many \( U_i \)). Recall that a partition of unity subordinate to \( \mathcal{U} \) is a collection of continuous functions \( \{f_i : X \to [0, 1]\}_{i \in I} \) such that for each \( i \in I \),

\[
\text{supp } f_i := \{x \in X : f_i(x) \neq 0\} \subseteq U_i
\]

and for any \( x \in X \),

\[
\sum_{i \in I} f_i(x) = 1.
\]

[Note that since the cover is locally finite, this is a finite sum for any \( x \in X \).] It is a fact that a Hausdorff space is compact if and only if every open cover admits a locally finite refinement with a subordinate partition of unity; in the case that \( X \) is a smooth manifold, then the partition of unity may be taken to be smooth.

A sheaf \( \mathcal{F} \) is **fine** if for any locally finite open cover \( \{U_i\}_{i \in I} \) of \( X \), there exists a family of sheaf morphisms \( \{\eta_i : \mathcal{F} \to \mathcal{F}\}_{i \in I} \) such that

(a) \( \sum_{i \in I} \eta_i = 1 \);  
(b) \( \text{supp } \eta_i \subseteq U_i \), where \( \text{supp } \eta_i = \{x \in X : \eta_{i,x} \neq 0\} \).
The family \( \{ \eta_i \}_{i \in I} \) is called a partition of unity of \( \mathcal{F} \) subordinate to the covering \( \{ U_i \}_{i \in I} \).

The following is then clear.

**Lemma 2.51.** Let \( \mathcal{C}^0 = \mathcal{C}^0_X \) be the sheaf of continuous \( \mathbb{R} \)-valued functions on \( X \) (so that \( \mathcal{C}^0(U) \) is a ring for every open \( U \subseteq X \)). If \( \mathcal{F} \) is a sheaf such that \( \mathcal{F}(U) \) is a \( \mathcal{C}^0(U) \)-modules for all open \( U \) in such a way scalar multiplication is compatible with the restriction maps (in which case we call \( \mathcal{F} \) a sheaf of \( \mathcal{C}^0 \)-modules), then \( \mathcal{F} \) is fine.

**Remark 2.52.** It is clear that in the above, if \( X \) is a smooth manifold, then we may replace \( \mathcal{C}^0 \) with \( \mathcal{A}^0_X \), \( \mathcal{A}^0_X \), or \( \mathcal{A}^0_X \), the sheaf of rings of \( \mathbb{R} \)- or \( \mathbb{C} \)-valued functions. However, even if \( X \) is a complex manifold, we cannot replace \( \mathcal{C}^0 \) by \( \mathcal{O}_X \), since partitions of unity will never be holomorphic functions.

**Example 2.53.** The following sheaves are fine

(a) \( \wedge^k T^* X \), or more generally, any locally free sheaf of \( \mathcal{A}_X \)-modules, when \( X \) is a smooth manifold.

(b) \( \mathcal{A}^{p,q}_X \) when \( X \) is a complex manifold.

**Proposition 2.54.** Fine sheaves are soft.

**Proof.** Let \( \mathcal{F} \) be a fine sheaf, \( S \subseteq X \) closed and \( s \in \mathcal{F}(S) \). Let \( \{ U_i \} \) be an open covering of \( S \) and let \( s_i \in \mathcal{F}(U_i) \) be such that \( s_i|_{S \cap U_i} = s|_{S \cap U_i} \).

Let \( U_0 = X \setminus S \) and \( s_0 := 0 \). Then \( \{ U_i \} \coprod \{ U_0 \} \) is an open covering of \( X \); since \( X \) is paracompact, we may assume that this covering is locally finite, and that there is a partition of unity \( \{ \eta_i \} \) subordinate to it. Now, for each \( i \), \( \eta_i(s_i) \in \mathcal{F}(U_i) \) and this vanishes on an open neighbourhood of \( X \setminus U_i \), so it can be extended to a section on all of \( X \), which we will also denote by \( \eta_i(s_i) \). If we set

\[ \tilde{s} := \sum_i \eta_i(s_i) \in \mathcal{F}(X), \]

then \( \tilde{s} \) maps to \( s \).

**Corollary 2.55.** (a) If \( X \) be a smooth manifold, then

\[ H^i(X, \mathcal{R}_X) = H^i_{dR}(X, \mathcal{R}), \]

where the left side is the sheaf cohomology groups of the constant sheaf \( \mathcal{R}_X \) (for \( \mathcal{R} = \mathbb{R} \) or \( \mathbb{C} \)) and the right are the de Rham cohomology groups.

(b) If \( X \) is a complex manifold and \( E \) is a holomorphic vector bundle over \( X \), then

\[ H^i(X, E) \cong H^i(\mathcal{O}^{0,\bullet}(X, E)), \]

and in particular, one has Dolbeault’s theorem:

\[ H^q(X, \Omega^p_X) \cong H^{p,q}(X), \]

where the left side is the sheaf cohomology of the holomorphic vector bundle \( \Omega^p_X \) and the right side is the Dolbeault cohomology group.

**Corollary 2.56.** (a) If \( X \) is a smooth manifold, then \( H^i(X, \mathcal{R}_X) = 0 \) for \( i > \dim_{\mathbb{R}} X \).

(b) If \( X \) is a complex manifold and \( E \) a holomorphic vector bundle on \( X \), then \( H^i(X, E) = 0 \) for \( i > \dim_{\mathbb{C}} X \).