5 Holomorphic de Rham Complexes and Spectral Sequences

5.1 Hypercohomology

As before, much of what we now say will hold for an arbitrary abelian category, but we will generally make our statements in terms of sheaves of abelian groups over a topological space or abelian groups.

Definition 5.1. Let X be a topological space. Let $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ be a morphism of complexes of sheaves of abelian groups on X. Then $\varphi^\bullet$ is called a *quasi-isomorphism* if the induced morphisms $H^i(\varphi^\bullet) : H^i(\mathscr{F}^\bullet) \to H^i(\mathscr{G}^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$. A complex $\mathscr{F}^\bullet$ is *bounded below* or *left-bounded* if there exists some $r_0 \in \mathbb{Z}$ with $\mathscr{F}^i = 0$ for $i < r_0$. An *injective resolution of a left-bounded complex* $\mathscr{F}^\bullet$ is a quasi-isomorphism $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$ where $\mathscr{I}^\bullet$ is a left-bounded complex with each $\mathscr{I}^i$ an injective sheaf. An *acyclic resolution resolution of* $\mathscr{F}^\bullet$ is a quasi-isomorphism $\sigma^\bullet : \mathscr{F}^\bullet \to \mathscr{S}^\bullet$, where $\mathscr{S}^\bullet$ is a left-bounded complex with each $\mathscr{S}^k$ an acyclic sheaf, i.e., $H^i(X, \mathscr{S}^k) = 0$ for $i > 0$.

Remark 5.2. Given a sheaf $\mathscr{F}$, we may construct a complex $\mathscr{F}^\bullet$ with $\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq 0$. Then an injective resolution of $\mathscr{F}$ in the sense of Definition **??** is the same an injective resolution of $\mathscr{F}^\bullet$ in the above sense.

Proposition 5.3. Let $\mathscr{F}^\bullet$ be a complex left-bounded. Then there is an injective resolution $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$ such that each $\alpha^i : \mathscr{F}^i \to \mathscr{I}^i$ is an injective map of sheaves.

Proof. By relabelling the indices, we may assume that $\mathscr{F}^i = 0$ for $i < 0$. We choose an embedding $\alpha^0 : \mathscr{F}^0 \to \mathscr{I}^0$ for some injective sheaf $\mathscr{I}^0$ (this may be done by Lemma **??**). We form the fibre sum $\mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1$ (for the maps (α^0, d^0)), which fits into a commutative square

$$\begin{array}{ccc} \mathscr{F}^0 & \xrightarrow{d^0} & \mathscr{F}^1 \\ {\scriptstyle \alpha^0}\downarrow & & \downarrow{\scriptstyle \gamma^1} \\ \mathscr{I}^0 & \xrightarrow[i^0]{} & \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \end{array}$$

and we observe that γ^1 is an inclusion since α^0 is. Now, we choose an embedding $\sigma^0 \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \to \mathscr{I}^1$, with $\mathscr{I}^1$ an injective sheaf and let $D^0 := \sigma^0 \circ i^0, \alpha^1 := \sigma^0 \circ \gamma^1$; observe that α^1 is the composition of two inclusions and hence is also one. This gives a commutative square

$$\begin{array}{ccc} \mathscr{F}^0 & \xrightarrow{d^0} & \mathscr{F}^1 \\ {\scriptstyle \alpha^0}\downarrow & & \downarrow{\scriptstyle \alpha^1} \\ \mathscr{I}^0 & \xrightarrow[D^0]{} & \mathscr{I}^1. \end{array}$$

We now show that $H^0(\alpha^\bullet)$ is an isomorphism. Since the first (possibly) non-zero terms are in degree zero, this amounts to $\alpha^0(\ker d^0) = \ker D^0 = \ker i^0$, the latter equality coming from the fact that σ^0 is injective. The left side is obviously contained in the right. To see the opposite inclusion, we may to to stalks. Fix $x \in X$ and suppose $u \in \ker i^0_x \subseteq \mathscr{I}^0_x$. This means $i^0_x(u) = [u, 0] = 0$ in $\mathscr{I}^0_x \oplus_{\mathscr{F}^0_x} \mathscr{F}^1_x$ and hence there exists $a \in \mathscr{F}^0_x$ with $(u, 0) = (\alpha^0_x(a), -d^0_x(a))$, i.e., $u = \alpha^0_x(a)$ for some $a \in \ker d^0_x$.

To continue, we may extend the above diagram to

$$\begin{array}{ccccc} \mathscr{F}^0 & \xrightarrow{d^0} & \mathscr{F}^1 & \xrightarrow{d^1} & \mathscr{F}^1 \\ {\scriptstyle \alpha^0}\downarrow & & {\scriptstyle \alpha^1}\downarrow & & \downarrow{\scriptstyle \gamma^2} \\ \mathscr{I}^0 & \xrightarrow[D^0]{} & \mathscr{I}^1 & \xrightarrow[i^1]{} & \operatorname{coker} D^0 \oplus_{\mathscr{F}^1} \mathscr{F}^2 \end{array}$$

and again choose an embedding $\sigma^1 : \operatorname{coker} D^0 \oplus_{\mathscr{F}^1} \mathscr{F}^2 \to \mathscr{I}^2$ for some injective sheaf $\mathscr{I}^2$. Letting $D^1 := \sigma^1 \circ i^1, \alpha^2 := \sigma^1 \circ \gamma^2$, the above yields

$$\begin{array}{ccccc} \mathscr{F}^0 & \xrightarrow{d^0} & \mathscr{F}^1 & \xrightarrow{d^1} & \mathscr{F}^1 \\ {\scriptstyle \alpha^0}\downarrow & & {\scriptstyle \alpha^1}\downarrow & & \downarrow{\scriptstyle \alpha^2} \\ \mathscr{I}^0 & \xrightarrow[D^0]{} & \mathscr{I}^1 & \xrightarrow[D^1]{} & \mathscr{I}^2 \end{array}$$

and α^2 is an inclusion since σ^1, γ^2 are. We now wish to show that $H^1(\alpha^\bullet)$ is an isomorphism, which again we do at stalks. Again, $\ker D^1 = \ker i^1$.

First, we show injectivity. Suppose $a \in \ker d_x^1$ is such that $\alpha_x^1(a) = D_x^0(u)$ for some $u \in \mathscr{I}_x^0$. The left side is $\sigma_x^0 \circ \gamma_x^1(a)$ and the right $\sigma_x^0 \circ i_x^0(u)$. since σ^0 is injective, $\gamma_x^1(a) = i_x^0(u)$ or $[u, a] = 0$ in $\mathscr{I}_x^0 \oplus_{\mathscr{F}_x^0} \mathscr{F}_x^1$ and hence there exists $b \in \mathscr{F}_x^0$ with $u = \alpha_x(b), a = d_x^0(b)$. In particular, a represents the zero cohomology class.

Now, suppose that $v \in \ker D_x^1 = \ker i_x^1$. This means that $i_x^1(v) = [\bar{v}, 0] = 0$ and hence there exists $a \in \mathscr{F}_x^1$ with $(\bar{v}, 0) = (\overline{\alpha_x^1(a)}, d_x^1(a))$, so $\overline{\alpha_x^1(a)} = \bar{v}$ for some $a \in \ker d_x^1$, i.e. $\alpha_x^1(a)$ and v represent the same cohomology class and $H^1(\alpha^\bullet)$ is surjective. The required resolution is obtained by continuing along in this manner. □

Definition 5.4. Let $\mathscr{F}^\bullet$ be a left-bounded complex of sheaves of abelian groups on X and let $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$ be an injective resolution with $\alpha^i : \mathscr{F}^i \to \mathscr{I}^i$ an injective map of sheaves for each $i \in \mathbb{Z}$, as provided for by Proposition 5.3. Then the *kth hypercohomology group of $\mathscr{F}$* is defined to be

$$\mathbb{H}^k(X, \mathscr{F}^\bullet) := H^k\big(\Gamma(X, \mathscr{I}^\bullet)\big).$$

Of course, we need to see that this is independent of the injective resolution.

Remark 5.5. By Remark 5.2, if $\mathscr{F}^\bullet$ is a complex with $\mathscr{F}^i = 0$ for $i \neq 0$, then

$$\mathbb{H}^k(X, \mathscr{F}^\bullet) = H^k(X, \mathscr{F}^0).$$

Lemma 5.6. Let $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$ be a quasi-isomorphism with $\mathscr{I}^\bullet$ left-bounded and such that $\alpha^i : \mathscr{F}^i \to \mathscr{I}^i$ is an injective map of sheaves for each $i \in \mathbb{Z}$, let $\mathscr{G}^\bullet$ be a left-bounded complex, $\beta^\bullet : \mathscr{G}^\bullet \to \mathscr{J}^\bullet$ an injective resolution and $\varphi : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ be a morphism of complexes. Then there exists a morphism of complexes $\psi^\bullet : \mathscr{I}^\bullet \to \mathscr{J}^\bullet$ such that

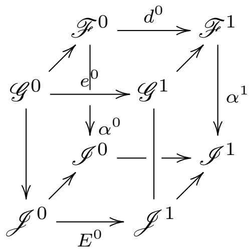
$$\begin{array}{ccc} \mathscr{F}^\bullet & \xrightarrow{\varphi^\bullet} & \mathscr{G}^\bullet \\ {\scriptstyle \alpha^\bullet}\downarrow & & \downarrow{\scriptstyle \beta^\bullet} \\ \mathscr{I}^\bullet & \xrightarrow[\psi^\bullet]{} & \mathscr{J}^\bullet \end{array}$$

commutes. Furthermore, any two such morphisms with this property are homotopic.

Proof. Again, we may assume $\mathscr{F}^i = \mathscr{G}^i = 0$ for $i < 0$. By assumption, α^0 is an embedding, so the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathscr{F}^0 & \xrightarrow{\alpha^0} & \mathscr{I}^0 \\ & & {\scriptstyle \varphi^0}\downarrow & \swarrow{\scriptstyle \psi^0} & \\ & & \mathscr{G}^0 & & \\ & & {\scriptstyle \beta^0}\downarrow & & \\ & & \mathscr{J}^0 & & \end{array}$$

gives the existence of $\psi^0 : \mathscr{I}^0 \to \mathscr{J}^0$ with $\psi^0 \circ \alpha^0 = \beta^0 \circ \varphi^0$. We now wish to construct $\psi^1 : \mathscr{I}^1 \to \mathscr{J}^1$ which makes the diagram

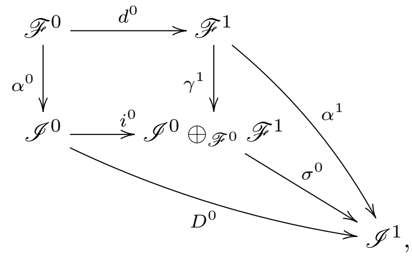


(5.1)

First, we observe that the commutative square

$$\begin{array}{ccc} \mathscr{F}^0 & \xrightarrow{d^0} & \mathscr{F}^1 \\ {\scriptstyle \alpha^0}\downarrow & & \downarrow{\scriptstyle \alpha^1} \\ \mathscr{I}^0 & \xrightarrow[D^0]{} & \mathscr{I}^1 \end{array}$$

fits into the commutative diagram



where i^0 and γ^1 are the canonical maps and σ^0 is the unique map making everything commute.

Our first claim is that σ^0 is an embedding. We may see this at stalks, so fix $x \in X$ and suppose $[u, a] \in \ker \sigma^0_x$. This means that $D^0_x(u) = -\alpha^1_x(a)$. We have $\alpha^2_x \circ d^1_x(a) = D^1_x \circ \alpha^1_x(a) = -D^1_x \circ D^0_x(u) = 0$ and since α^2 is injective by assumption, $d^1_x(a) = 0$. Hence a represents a cohomology class $\overline{a}$ and $\alpha^1_x(a) = -D^0_x(u)$ means $H^1(\alpha^\bullet)(\overline{a}) = 0$. Since $H^1(\alpha^\bullet)$ is an isomorphism by assumption, there exists $r \in \mathscr{F}^0_x$ with $a = d^0_x(r)$. With this,

$$D^0_x\big(u + \alpha^0_x(r)\big) = -\alpha^1_x(a) + \alpha^1_x\big(d^0_x(r)\big) = 0$$

and hence $u + \alpha^0_x(r)$ represents a cohomology class in the $\mathscr{I}^\bullet$ complex. Since $H^0(\alpha^\bullet)$ is surjective, there exists $s \in \mathscr{F}^0_x$ with $u + \alpha^0_x(r) = \alpha^0_x(s)$. Let $b := s - r$. Then

$$\alpha^0_x(b) = \alpha^0_x(s) - \alpha^0_x(r) = u \qquad\qquad -d^0_x(b) = d^0_x(r) - d^0_x(s) = a$$

since

$$\alpha^1_x \circ d^0_x(s) = D^0_x \circ \alpha^0_x(s) = D^0_x(u) + D^0_x \circ \alpha^0_x(r) = D^0_x(u) + \alpha^1_x \circ d^0_x(r) = D^0_x(u) + \alpha^1_x(a)$$

and since α^1 is injective, $d^0_x(s) = 0$. This shows that $(u, a) = (\alpha^0_x(b), -d^0_x(b))$ with $b \in \mathscr{F}^0_x$ and hence $[u, a] = 0$.

Our next claim is that we have a morphism $\rho^0 : \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \to \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1$. But this is the same as the existence of a commutative square

$$\begin{array}{ccc} \mathscr{F}^0 & \longrightarrow & \mathscr{F}^1 \\ \downarrow & & \downarrow \\ \mathscr{I}^0 & \longrightarrow & \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1 \end{array}$$

but this in turn comes from the larger square

$$
\begin{tikzcd}
\mathscr{F}^0 \arrow[rr, "d^0"] \arrow[dr, "\varphi^0"] \arrow[dd, "\alpha^0"'] & & \mathscr{F}^1 \arrow[d, "\varphi^1"] \\
& \mathscr{G}^0 \arrow[r, "e^0"] \arrow[d, "\beta^0"'] & \mathscr{G}^1 \arrow[d, "\delta^1"] \\
\mathscr{I}^0 \arrow[r, "\psi^0"'] & \mathscr{J}^0 \arrow[r, "j^0"'] & \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1
\end{tikzcd}
$$

[Essentially, we are considering the diagram (5.1) with $\mathscr{I}^1$ replaced by $\mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1$, α^1 by γ^1, $\mathscr{J}^1$ by $\mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1$ and β^1 by δ^1.] Furthermore, we see that ρ^0 makes the square

$$
\begin{tikzcd}
\mathscr{F}^1 \arrow[r, "\varphi^1"] \arrow[d, "\gamma^1"'] & \mathscr{G}^1 \arrow[d, "\delta^1"] \\
\mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \arrow[r, "\rho^0"'] & \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{I}^0 \arrow[r, "i^0"] \arrow[d, "\psi^0"'] & \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \arrow[d, "\rho^0"] \\
\mathscr{J}^0 \arrow[r, "j^0"'] & \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1
\end{tikzcd}
$$

commute.

Now, the commutative square

$$
\begin{tikzcd}
\mathscr{G}^0 \arrow[r, "e^0"] \arrow[d, "\beta^0"'] & \mathscr{G}^1 \arrow[d, "\beta^1"] \\
\mathscr{J}^0 \arrow[r, "E^0"'] & \mathscr{J}^1
\end{tikzcd}
$$

yields a morphism $\tau^0 : \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1 \to \mathscr{J}^1$ and hence we have a diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 \arrow[r, "\sigma^0"] \arrow[d, "\rho^0"'] & \mathscr{I}^1 \arrow[ddl, dashed, "\psi^1"] \\
& \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1 \arrow[d, "\tau^0"'] & \\
& \mathscr{J}^1 &
\end{tikzcd}
$$

and the injectivity of $\mathscr{J}^1$ yields the existence of ψ^1 which makes the above diagram commute.

Finally, we show that ψ^1 makes (5.1) commute, which is the same as the two diagrams

$$
\begin{tikzcd}
\mathscr{F}^1 \arrow[r, "\varphi^1"] \arrow[d, "\alpha^1"'] & \mathscr{G}^1 \arrow[d, "\beta^1"] \\
\mathscr{I}^1 \arrow[r, "\psi^1"'] & \mathscr{J}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{I}^0 \arrow[r, "D^0"] \arrow[d, "\psi^0"'] & \mathscr{I}^1 \arrow[d, "\psi^1"] \\
\mathscr{J}^0 \arrow[r, "E^0"'] & \mathscr{J}^1
\end{tikzcd}
$$

commuting. The first comes from the diagram

$$\begin{CD} \mathscr{F}^1 @>{\varphi^1}>> \mathscr{G}^1 \\ @V{\gamma^1}VV @VV{\delta^1}V \\ \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 @>{\rho^0}>> \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1 \\ @V{\sigma^0}VV @VV{\tau^0}V \\ \mathscr{I}^1 @>>{\psi^1}> \mathscr{J}^1 \end{CD}$$

and the second from

$$\begin{CD} \mathscr{I}^0 @>{i^0}>> \mathscr{I}^0 \oplus_{\mathscr{F}^0} \mathscr{F}^1 @>{\sigma^0}>> \mathscr{I}^1 \\ @V{\psi^0}VV @VV{\rho^0}V @VV{\psi^1}V \\ \mathscr{J}^0 @>>{j^0}> \mathscr{J}^0 \oplus_{\mathscr{G}^0} \mathscr{G}^1 @>>{\tau^0}> \mathscr{J}^1. \end{CD}$$

The morphism $\psi^\bullet$ is then constructed by continuing in this manner.

The fact that any map $\nu^\bullet$ satisfying the properties of $\psi^\bullet$ is homotopic to $\psi^\bullet$ is left as an exercise. □

Proposition 5.7. The definition of $\mathbb{H}^k(X, \mathscr{F}^\bullet)$ is independent of the injective resolution chosen.

Proof. Exercise. □

Proposition 5.8. Let $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ be a morphism of left-bounded complexes. Then for each $k \in \mathbb{Z}$, there is a canonical morphism

$$\mathbb{H}^k(\varphi^\bullet) : \mathbb{H}^k(X, \mathscr{F}^\bullet) \to \mathbb{H}^k(X, \mathscr{G}^\bullet).$$

Proof. Exercise. □

Definition 5.9. Let $\mathscr{F}^\bullet$ be a complex. For $n \in \mathbb{Z}$, the *n-shifted complex* $\mathscr{F}^\bullet[n]$ is the complex with $(\mathscr{F}^\bullet[n])^k := \mathscr{F}^{k+n}$ and whose differentials are similarly shifted and multiplied by $(-1)^n$.

Let $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ be a morphism of complexes. Then we define the *cone of* $\varphi^\bullet$ to be the complex $C(\varphi^\bullet)^\bullet := \mathscr{F}^\bullet[1] \oplus \mathscr{G}^\bullet$ with differential $D^k : \mathscr{F}^{k+1} \oplus \mathscr{G}^k \to \mathscr{F}^{k+2} \oplus \mathscr{G}^{k+1}$ given by

$$D^k := \begin{bmatrix} -d^{k+1} & \\ \varphi^{k+1} & e^k \end{bmatrix}$$

where $d^\bullet$ and $e^\bullet$ are the differentials of $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$, respectively. One easily checks that $D^{k+1} \circ D^k = 0$.

Proposition 5.10. Given a morphism $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$, one has a long exact sequence

$$\cdots \to H^k(\mathscr{F}^\bullet) \to H^k(\mathscr{G}^\bullet) \to H^k\big(C(\varphi^\bullet)^\bullet\big) \to H^{k+1}(\mathscr{F}^\bullet) \to \cdots .$$

Proof. We claim that we have a short exact sequence of complexes

$$0 \to \mathscr{G}^\bullet \to C(\varphi^\bullet)^\bullet \to \mathscr{F}^\bullet[1] \to 0.$$

We only need to check that the diagram

$$\begin{CD} \mathscr{G}^k @>>> \mathscr{F}^{k+1} \oplus \mathscr{G}^k @>>> \mathscr{F}^{k+1} \\ @V{e^k}VV @VV{D^k}V @VV{-d^{k+1}}V \\ \mathscr{G}^{k+1} @>>> \mathscr{F}^{k+2} \oplus \mathscr{G}^{k+1} @>>> \mathscr{F}^{k+2} \end{CD}$$

whose horizontal maps are the canonical inclusions and projections, commutes, but this is easy. Therefore, we get a long exact sequence

$$\cdots \to H^k(\mathscr{G}^\bullet) \to H^k\big(C(\varphi^\bullet)^\bullet\big) \to H^k(\mathscr{F}^\bullet[1]) \to H^{k+1}(\mathscr{G}^\bullet) \to \cdots .$$

We now want to show that the connecting homomorphism $H^k(\mathscr{F}^\bullet[1]) = H^{k+1}(\mathscr{F}^\bullet) \to H^{k+1}(\mathscr{G}^\bullet)$ is simply $H^{k+1}(\varphi^\bullet)$, but this is clear by the definition of the connecting homomorphism and tracing through the above diagram. □

Corollary 5.11. If $\varphi^\bullet$ is a quasi-isomorphism, then $C(\varphi^\bullet)^\bullet$ is an acyclic complex.

Lemma 5.12. If $\mathscr{S}^\bullet$ is an exact left-bounded complex of acyclic sheaves (i.e., $H^i(X, \mathscr{S}^k) = 0$ for all $i > 0, k \in \mathbb{Z}$), then $\Gamma(X, \mathscr{S}^\bullet)$ is exact.

Proof. Again, we may assume $\mathscr{S}^i = 0$ for $i < 0$. Since $\Gamma(X, \cdot)$ is left exact, we know that $H^i(\Gamma(X, \mathscr{S}^\bullet)) = 0$ for $i = 0, 1$. Now, we may regard $0 \to \mathscr{S}^1 \to \mathscr{S}^2 \to \cdots$ as a resolution of $\mathscr{S}^0$. With this, we see that for $i \geq 2$,

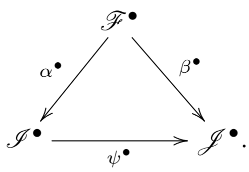
$$H^i\big(\Gamma(X, \mathscr{S}^\bullet)\big) = H^{i-1}(X, \mathscr{S}^0) = 0. \qquad \square$$

Proposition 5.13. Let $\beta^\bullet : \mathscr{F}^\bullet \to \mathscr{J}^\bullet$ be an injective resolution. Then $\beta^\bullet$ induces canonical isomorphisms

$$\mathbb{H}^k(X, \mathscr{F}^\bullet) \xrightarrow{\sim} H^k\big(\Gamma(X, \mathscr{J}^\bullet)\big).$$

[The statement is that we do not necessarily need the β^k to be injective.]

Proof. Let $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$ be an injective resolution with the α^k embeddings. Then by Lemma 5.6, there is a morphism of complexes $\psi^\bullet : \mathscr{I}^\bullet \to \mathscr{J}^\bullet$ such that $\beta^\bullet = \psi^\bullet \circ \alpha^\bullet$:



Since $\alpha^\bullet$ and $\beta^\bullet$ are quasi-isomorphisms, so is $\psi^\bullet$ and hence $C(\psi^\bullet)^\bullet$ is an exact complex of injectives by Corollary 5.11. Now, by the abelian group version of Proposition 5.10, we have a long exact sequence of groups

$$\cdots \to H^k\big(\Gamma(X, \mathscr{I}^\bullet)\big) \to H^k\big(\Gamma(X, \mathscr{J}^\bullet)\big) \to H^k\big(\Gamma(X, C(\psi^\bullet)^\bullet)\big) \to H^{k+1}\big(\Gamma(X, \mathscr{I}^\bullet)\big) \to \cdots,$$

but Lemma 5.12 tells us that $\Gamma(X, C(\psi^\bullet)^\bullet)$ is an exact complex, so since its cohomology vanishes, we get isomorphisms

$$\mathbb{H}^k(X, \mathscr{F}^\bullet) = H^k\big(\Gamma(X, \mathscr{I}^\bullet)\big) \to H^k\big(\Gamma(X, \mathscr{J}^\bullet)\big). \qquad \square$$

Corollary 5.14. Let $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ be a quasi-isomorphism. Then the maps

$$\mathbb{H}^k(\varphi^\bullet) : \mathbb{H}^k(X, \mathscr{F}^\bullet) \to \mathbb{H}^k(X, \mathscr{G}^\bullet)$$

are isomorphisms for all $k \in \mathbb{Z}$.

Proof. If $\beta^\bullet : \mathscr{G}^\bullet \to \mathscr{J}^\bullet$ is any injective resolution, then so is $\beta^\bullet \circ \varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{J}^\bullet$ and hence the hypercohomology of $\mathscr{F}$ and $\mathscr{G}$ can both be computed by the same complex. $\square$

Lemma 5.15. Let $\mathscr{S}^\bullet$ be a left-bounded complex of acyclic sheaves. Then we have canonical isomorphisms

$$H^k\big(\Gamma(X, \mathscr{S}^\bullet)\big) \xrightarrow{\sim} \mathbb{H}^k(X, \mathscr{S}^\bullet).$$

Proof. Let $\alpha^\bullet : \mathscr{S}^\bullet \to \mathscr{J}^\bullet$ be an injective resolution with each α^k an embedding. Then we have an exact sequence

$$0 \to \mathscr{S}^\bullet \to \mathscr{J}^\bullet \to \mathscr{Q}^\bullet \to 0,$$

where $\mathscr{Q}^\bullet$ is the quotient complex. Since each $\mathscr{S}^k$ is acyclic, we have an exact sequence of complexes of abelian groups

$$0 \to \Gamma(X, \mathscr{S}^\bullet) \to \Gamma(X, \mathscr{J}^\bullet) \to \Gamma(X, \mathscr{Q}^\bullet) \to 0.$$

Since $\alpha^\bullet$ is a quasi-isomorphism, $\mathscr{Q}^\bullet$ is an exact complex. Furthermore, each $\mathscr{Q}^k$ is acyclic, since $\mathscr{S}^k$, $\mathscr{J}^k$ are. So by Lemma 5.12, $\Gamma(X, \mathscr{Q}^\bullet)$ is exact. Thus, the above gives isomorphisms

$$H^k\big(\Gamma(X, \mathscr{S}^\bullet)\big) \xrightarrow{\sim} H^k\big(\Gamma(X, \mathscr{J}^\bullet)\big) = \mathbb{H}^k(X, \mathscr{S}^\bullet). \qquad \square$$

Proposition 5.16. If $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{S}^\bullet$ is an acyclic resolution then for all $k \in \mathbb{Z}$, we have canonical isomorphisms

$$H^k\big(\Gamma(X, \mathscr{S}^\bullet)\big) \xrightarrow{\sim} \mathbb{H}^k(X, \mathscr{F}^\bullet).$$

Proof. Since $\alpha^\bullet$ is a quasi-isomorphism, Corollary 5.14 gives us isomorphisms

$$\mathbb{H}^k(\alpha^\bullet) : \mathbb{H}^k(X, \mathscr{F}^\bullet) \xrightarrow{\sim} \mathbb{H}^k(X, \mathscr{S}^\bullet).$$

Then Lemma 5.15 tells us that the latter can be computed as $H^k(\Gamma(X, \mathscr{S}^\bullet))$. $\square$

5.2 Holomorphic de Rham Complexes

Definition 5.17. Let X be a complex manifold of (complex) dimension n. Let Ω^p_X denote the sheaf of holomorphic differential forms of degree p. We have $d = \partial : \Omega^p_X \to \Omega^{p+1}_X$ and this satisfies $\partial \circ \partial = 0$. Thus, we have a complex

$$0 \to \mathscr{O}_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n_X \to 0;$$

this is called the *holomorphic de Rham complex of* X. Observe that we have an inclusion $i : \mathbb{C}_X \to \mathscr{O}_X$ of the sheaf of locally constant functions into the sheaf of holomorphic functions.

Lemma 5.18 (Holomorphic Poincaré Lemma)**.** The de Rham complex is a resolution of $\mathbb{C}_X$.

Proof. It is clear that $\mathbb{C}_X$ is the kernel of $\partial : \mathscr{O}_X \to \Omega^1_X$. Thus, it is enough to show that the complex is exact at Ω^k_X for $1 \leq k \leq n$ and of course we do this at stalks. Fix $x \in X$, and suppose $A \in \Omega^1_{X,x}$ is such that $\partial_x A = 0$; this is represented by some ∂-closed $\alpha \in \Gamma(U, \Omega^1_X)$ on some open neighbourhood U of x. Since α is holomorphic, it is d-closed, and by the Poincaré lemma, there is some smooth function β in a neighbourhood V of x with $d\beta = \alpha$. But $d\beta = \partial\beta + \overline{\partial}\beta$, and since α is a $(1,0)$-form, $\overline{\partial}\beta = 0$, so $\beta \in \Gamma(V, \mathscr{O}_X)$ is holomorphic and $\partial\beta = \alpha$. This shows exactness at Ω^1_X.

Suppose $\alpha \in \Gamma(U, \Omega^2_X)$ is such that $\partial\alpha = d\alpha = 0$. Again, we use the Poincaré lemma to find $\beta \in \mathscr{A}^1(V)$ such that $d\beta = 0$. Writing $\beta = \beta^{1,0} + \beta^{0,1}$, by comparing types we see that

$$\partial\beta^{1,0} = \alpha \qquad \overline{\partial}\beta^{1,0} + \partial\beta^{0,1} = 0 \qquad \overline{\partial}\beta^{0,1} = 0.$$

By the last equation, the Dolbeault lemma tells us that there is some smooth function γ in a neighbourhood V' of x such that $\overline{\partial}\gamma = \beta^{0,1}$. Substituting this into the second equation, we see that

$$0 = \overline{\partial}\beta^{1,0} + \partial\overline{\partial}\gamma = \overline{\partial}(\beta^{1,0} - \partial\gamma),$$

so $\beta^{1,0} - \partial\gamma \in \Gamma(V', \Omega^1_X)$ is a holomorphic 1-form and

$$\partial(\beta^{1,0} - \partial\gamma) = \partial\beta^{1,0} - \partial^2\gamma = \alpha.$$

This proves exactness at Ω^2_X.

If α is a holomorphic 3-form with $d\alpha = \partial\alpha = 0$, we may find some 3-form $\beta = \beta^{2,0}+\beta^{1,1}+\beta^{0,2}$ with $d\beta = \alpha$. This gives the equations

$$\partial\beta^{2,0} = \alpha \qquad \overline{\partial}\beta^{2,0} + \partial\beta^{1,1} = 0 \qquad \overline{\partial}\beta^{1,1} + \partial\beta^{0,2} = 0 \qquad \overline{\partial}\beta^{0,2} = 0.$$

The last gives the existence of $\gamma^{0,1}$ with $\overline{\partial}\gamma^{0,1} = \beta^{0,2}$. Substituting in the equation before, we have

$$0 = \overline{\partial}\beta^{1,1} + \partial\overline{\partial}\gamma^{0,1} = \overline{\partial}(\beta^{1,1} - \partial\gamma^{0,1}),$$

and so there is $\gamma^{1,0}$ with $\overline{\partial}\gamma^{1,0} = \beta^{1,1} - \partial\gamma^{0,1}$. Substituting again, we get

$$0 = \overline{\partial}\beta^{2,0} + \partial(\overline{\partial}\gamma^{1,0} + \partial\gamma^{0,1}) = \overline{\partial}(\beta^{2,0} - \partial\gamma^{1,0}).$$

Therefore, $\beta^{2,0} - \partial\gamma^{1,0} \in \Gamma(V, \Omega^2_X)$ for some neighbourhood V of x and $\partial(\beta^{2,0} - \partial\gamma^{1,0}) = \alpha$. Exactness in any degree can be proved in exactly the same way. □

Now, Remark 5.5 and Corollary 5.14 give us the following.

Corollary 5.19. We have canonical isomorphisms

$$H^k(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^k(X, \Omega^\bullet_X).$$

5.3 Filtered Complexes and Spectral Sequences

We make the following definitions for sheaves over a fixed topological space X, but as often before, they make sense for an arbitrary abelian category. In particular, we will transpose the same definitions to the category of abelian groups.

Definition 5.20. Let $\mathscr{F}$ be a sheaf over X. A *decreasing filtration on* $\mathscr{F}$ is a family of subsheaves $F^p\mathscr{F} \subseteq \mathscr{F}$ for $p \geq 0$ such that $F^0\mathscr{F} = \mathscr{F}$ and $F^{p+1}\mathscr{F} \subseteq F^p\mathscr{F}$ for all $p \geq 0$. The *pth graded piece* is defined as

$$\mathrm{Gr}^p_F\mathscr{F} := F^p\mathscr{F}/F^{p+1}\mathscr{F}.$$

A *decreasing filtration on a complex* $(\mathscr{F}^\bullet, d^\bullet)$ is a filtration $\{F^p\mathscr{F}^k\}_{p\geq 0}$ for all $k \in \mathbb{Z}$ such that $d^k(F^p\mathscr{F}^k) \subseteq F^p\mathscr{F}^{k+1}$ for all $p \geq 0, k \in \mathbb{Z}$. The differentials are said to be *strictly compatible with the filtration* if $d^k(F^p\mathscr{F}^k) = F^p\mathscr{F}^{k+1} \cap \operatorname{im} d^k$. A *filtered complex* $(\mathscr{F}^\bullet, F)$ is simply a complex $\mathscr{F}^\bullet$ together with a filtration F. A *bi-filtered complex* $(\mathscr{F}^\bullet, W, F)$ is simply a complex $\mathscr{F}^\bullet$ together with two filtrations W and F.

A *morphism of filtered complexes* $\varphi^\bullet : (\mathscr{F}^\bullet, F) \to (\mathscr{G}^\bullet, F)$ is a morphism of complexes $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ such that $\varphi^\bullet(F^p\mathscr{F}^\bullet) \subseteq F^p\mathscr{G}^\bullet$ for all $p \geq 0$.

Let $\mathscr{F}^\bullet$ be a complex with $\mathscr{F}^k = 0$ for $k < 0$. We define the *naive filtration on* $\mathscr{F}^\bullet$ as complex with

$$F^p\mathscr{F}^k = \begin{cases} \mathscr{F}^k & \text{if } p \leq k \\ 0 & \text{if } p > k. \end{cases}$$

For a complex $\mathscr{F}^\bullet$, the (increasing) *canonical filtration* τ is defined by taking $\tau^k\mathscr{F}^\bullet$ to be the complex

$$\cdots \xrightarrow{d^{k-2}} \mathscr{F}^{k-1} \xrightarrow{d^{k-1}} \ker d^k \to 0 \to 0 \to \cdots.$$

If we have a filtration on a complex, then by definition, $F^p\mathscr{F}^\bullet$ is a complex for each $p \geq 0$ (with the restriction of the differentials on $\mathscr{F}^\bullet$), and the inclusions $F^{p+1}\mathscr{F}^\bullet F^p\mathscr{F}^\bullet$ are morphisms of complexes. In particular, $i_p : F^p\mathscr{F}^\bullet \hookrightarrow \mathscr{F}^\bullet$ are morphisms of complexes and we have maps

$$H^k(i_p) : H^k(F^p\mathscr{F}^\bullet) \to H^k(\mathscr{F}^\bullet)$$

and we obtain filtrations $\{F^p H^k(\mathscr{F}^\bullet) := \operatorname{im} H^k(i_p)\}_{p\geq 0}$ for all $k \in \mathbb{Z}$. Similarly, we have maps

$$\mathbb{H}^k(i_p) : \mathbb{H}^k(F^p\mathscr{F}^\bullet) \to \mathbb{H}^k(\mathscr{F}^\bullet)$$

and we obtain filtrations $\{F^p\mathbb{H}^k(\mathscr{F}^\bullet) := \operatorname{im} \mathbb{H}^k(i_p)\}_{p\geq 0}$ for all $k \in \mathbb{Z}$.

Definition 5.21. A *double complex of sheaves* $(\mathscr{F}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$ consists of sheaves $\mathscr{F}^{p,q}$ and differentials $d_1^{p,q} : \mathscr{F}^{p,q} \to \mathscr{F}^{p+1,q}$, $d_2^{p,q} : \mathscr{F}^{p,q} \to \mathscr{F}^{p,q+1}$ for all $(p, q) \in \mathbb{Z}^2$ such that the diagrams

$$\begin{array}{ccc} \mathscr{F}^{p,q+1} & \xrightarrow{d_1^{p,q+1}} & \mathscr{F}^{p+1,q+1} \\ {\scriptstyle d_2^{p,q}}\uparrow & & \uparrow{\scriptstyle d_2^{p+1,q}} \\ \mathscr{F}^{p,q} & \xrightarrow[d_1^{p,q}]{} & \mathscr{F}^{p+1,q} \end{array}$$

commute and for each $p \in \mathbb{Z}$ $(\mathscr{F}^{p,\bullet}, d_2^{p,\bullet})$ is a complex and for all $q \in \mathbb{Z}$, $(\mathscr{F}^{\bullet,q}, d_1^{\bullet,q})$ is a complex. We say that the double complex $(\mathscr{F}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$ is a *first quadrant double complex* if there exist $p_0, q_0 \in \mathbb{Z}$ such that $\mathscr{F}^{p,q} = 0$ if $p < p_0$ or $q < q_0$.

Given a first quadrant double complex $(\mathscr{F}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$, the *associated single complex* $(\mathscr{F}^\bullet, d^\bullet)$ is defined as the complex with

$$\mathscr{F}^k := \bigoplus_{p+q=k} \mathscr{F}^{p,q} \qquad\qquad d^k|_{\mathscr{F}^{p,q}} := d_1^{p,q} + (-1)^p d_2^{p,q}.$$

It is not hard to verify that this indeed yields a complex.

Example 5.22. Let X be a complex manifold. Then if $\mathscr{A}^{p,q}(X)$ is the group of (globally defined) (p, q)-forms, then $(\mathscr{A}^{p,q}(X), \partial, (-1)^p\overline{\partial})$ gives a double complex of abelian groups (indeed, of vector spaces). The sign $(-1)^p$ is introduced precisely to make

$$\begin{array}{ccc} \mathscr{A}^{p,q+1}(X) & \xrightarrow{\partial} & \mathscr{A}^{p+1,q+1}(X) \\ {\scriptstyle (-1)^p\overline{\partial}}\uparrow & & \uparrow{\scriptstyle (-1)^{p+1}\overline{\partial}} \\ \mathscr{A}^{p,q}(X) & \xrightarrow[\partial]{} & \mathscr{A}^{p+1,q}(X) \end{array}$$

since $\partial\overline{\partial} = -\overline{\partial}\partial$.

The kth group in the associated double complex is

$$\bigoplus_{p+q=k} \mathscr{A}^{p,q}(X),$$

but this is precisely the full group $\mathscr{A}^k(X)$ of ($\mathbb{C}$-valued) k-forms. Furthermore, the differential is given by $\partial + (-1)^{2p}\overline{\partial} = \partial + \overline{\partial} = d$; thus, the associated double complex is precisely the smooth de Rham complex.

Definition 5.23. Let $(\mathscr{F}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$ be a double complex with $\mathscr{F}^{p,q} = 0$ for $p < 0$ or $q < 0$ and let $(\mathscr{F}^\bullet, d^\bullet)$ be the associated simple complex. The *(first) standard filtration on* $(\mathscr{F}^\bullet, d^\bullet)$ is defined by taking

$$F^p\mathscr{F}^k := \bigoplus_{\substack{r+s=k \\ r\geq p}} \mathscr{F}^{r,s} = \mathscr{F}^{p,k-p} \oplus \mathscr{F}^{p+1,k-p-1} \oplus \cdots \oplus \mathscr{F}^{k,0}.$$

Since $d^k(\mathscr{F}^{r,s}) \subseteq \mathscr{F}^{r+1,s} \oplus \mathscr{F}^{r,s+1}$, we have $d^k(F^p\mathscr{F}^k) \subseteq F^p\mathscr{F}^{k+1}$, so we do indeed get a filtration on the single complex. One can similarly define the second standard filtration, by bounding the second index rather than the first, but we will have no need for this.

Example 5.24. In the case where X is a complex manifold, the standard filtration on the smooth de Rham complex gives $F^p\mathscr{A}^k(X)$ as the sum of forms of type (r,s) with $r+s=k$ and $r \geq p$.

Definition 5.25. Let $\mathscr{F}^\bullet$ be a left-bounded complex. A *resolution of $\mathscr{F}^\bullet$ by a double complex* $(\mathscr{K}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$ refers to a first-quadrant double complex with $\mathscr{K}^{p,q} = 0$ for $q < 0$ and an embedding of complexes $i^\bullet : \mathscr{F}^\bullet(\mathscr{K}^{\bullet,0}, d_1^{\bullet,0})$ such that for all $k \in \mathbb{Z}$,

$$0 \to \mathscr{F}^k \xrightarrow{i^k} \mathscr{K}^{k,0} \to \mathscr{K}^{k,1} \to \cdots$$

is a resolution of $\mathscr{F}^k$.

Example 5.26. For a complex manifold X, the Dolbeault resolution (Proposition **??**) shows that the double complex $(\mathscr{A}^{p,q}(X))$ gives a resolution of the holomorphic de Rham complex $\Omega_X^\bullet$.

Lemma 5.27. (a) Suppose a left-bounded complex $\mathscr{F}^\bullet$ has a resolution by a double complex $(\mathscr{K}^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet})$. Let $(\mathscr{K}^\bullet, d^\bullet)$ be the associated single complex. Then one has an embedding $j^\bullet : \mathscr{F}^\bullet \hookrightarrow \mathscr{K}^\bullet$ which is a quasi-isomorphism.

(b) In the situation of (a), let $\{F^p\mathscr{F}^\bullet\}$ denote the naive filtration on $\mathscr{F}^\bullet$ and $\{F^p\mathscr{K}^\bullet\}$ the standard filtration on the associated single complex. Then for all $p \geq 0$, the quasi-isomorphism $j^\bullet$ restricts to a quasi-isomorphism $F^p\mathscr{F}^\bullet \to F^p\mathscr{K}^\bullet$.

Proof. Exercise. □

Definition 5.28. A *spectral sequence (of abelian groups)* consists of an abelian group $E_r^{p,q}$ for each $(p,q) \in \mathbb{Z}^2, r \geq 0$ and differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ (i.e., $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$) such that

$$E_{r+1}^{p,q} = \frac{\ker(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{im}(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

We say that the spectral sequence *degenerates at E_r* if $d_k^{p,q} = 0$ for all $k \geq r$. In this case, $E_k^{p,q} = E_r^{p,q}$ for all $k \geq r$ and we often write $E_\infty^{p,q} := E_r^{p,q}$.

Theorem 5.29. Let $\{F^pA^\bullet\}_{p\geq 0}$ be a filtered complex of abelian groups such that for all $k \geq 0$ there exists $l \geq 0$ with $F^lA^k = 0$. Then there exists a spectral sequence $(E_r^{\bullet,\bullet}, d_r^{\bullet,\bullet})_{r\geq 0}$ such that

(a) $E_0^{p,q} = \operatorname{Gr}_p^F A^{p+q}$ and $d_0^{\bullet,\bullet}$ is induced by $d^\bullet$.

(b) $E_\infty^{p,q} = \operatorname{Gr}_p^F H^{p+q}(A^\bullet)$.

Proof. We set

$$Z_r^{p,q} := F^{p+1}A^{p+q} + \ker(d^{p+q} : F^pA^{p+q} \to F^pA^{p+q+1}/F^{p+1}A^{p+q+1}) \subseteq F^pA^{p+q},$$

$B_0^{p,q} := F^{p+1}A^{p+q}$,

$$B_r^{p,q} := F^{p+1}A^{p+q} + d^{p+q-1}(Z_{r-1}^{p-r+1,q+r-2})$$

for $r \geq 1$ and finally

$$E_r^{p,q} := Z_r^{p,q}/B_r^{p,q}.$$

With these definitions, it is a straightforward, if tedious, exercise to prove that one indeed has a spectral sequence with the above properties. □

Remark 5.30. With the above definitions, one sees that for all p, q, we have

$$F^{p+1}A^{p+q} = B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} = F^pA^{p+q}.$$

In the case where, $A^\bullet$ is a complex of vector spaces and the $d^\bullet$ are linear maps, and $\dim E_r^{p,q} < \infty$ for some $r \geq 0$, it is clear that $\dim E_\infty^{p,q} < \infty$ and that the spectral sequence degenerates at E_r if we have $\dim E_r^{p,q} = \dim E_\infty^{p,q}$.

5.3.1 The Fröhlicher Spectral Sequence

Definition 5.31. Let X be a complex manifold and let $\Omega_X^\bullet$ be its holomorphic de Rham complex. Consider the naive filtration on $\Omega_X^\bullet$; if $\{F^p\mathscr{S}^\bullet\}_{p\geq 0}$ is any filtered complex of sheaves for which there is a filtered quasi-isomorphism $\Omega_X^\bullet \to \mathscr{S}^\bullet$, for which $\mathbb{H}^k(F^p\Omega_X^\bullet)$ can be computed as $H^k(\Gamma(X, F^p\mathscr{S}^\bullet))$ for $p \geq 0$, then the *Fröhlicher spectral sequence for X* is defined to be the spectral sequence (of abelian groups) associated to the filtered complex of abelian groups $\Gamma(X, F^p\mathscr{S}^\bullet)$. In particular, Lemma **??** allows us to take the spectral sequence induced by the standard filtration on the smooth de Rham complex.

In the case that we take the standard filtration on a double complex, one has fairly explicit expressions for the E_1 term.

Lemma 5.32. Let $(A^{\bullet,\bullet}, d_1, d_1)$ be a double complex (of abelian groups) and $(A^\bullet, d)$ the associated single complex. If $A^\bullet$ is given the first standard filtration, then the associated spectral sequence satisfies

(a) $E_0^{p,q} = A^{p,q}$ and $d_0 = (-1)^p d_2$;

(b) $E_1^{p,q} = H^q(A^{p,\bullet})$ and $d_1 : H^q(A^{p,\bullet}) \to H^q(A^{p+1,\bullet})$ is induced by $d_1 : A^{p,\bullet} \to A^{p+1,\bullet}$.

Proof. This is a fairly easy exercise using Theorem **??**. □

Applying this to the case above for the double complex $\mathscr{A}^{\bullet,\bullet}(X)$, we find that the E_1 term of the Fröhlicher spectral sequence may be computed as

$$E_1^{p,q} = H^q\big(\mathscr{A}^{p,\bullet}(X), \overline{\partial}\big) = H^{p,q}_{\mathrm{Dol}}(X) \cong H^q(X, \Omega_X^p),$$

by Dolbeault's theorem (Corollary **??**(b)). Part (b) tells us that $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is induced by

$$\partial : H^q(X, \Omega_x^p) \to H^q(X, \Omega_x^{p+1}).$$

Remark 5.33. The above shows that any complex structure on a smooth manifold induces a filtration on the underlying cohomology (with coefficients in $\mathbb{C}$), and that we can say something about the E_1 term in the spectral sequence. In the case where X is compact Kähler, we can relate this filtration to the Hodge decomposition.

Proposition 5.34. If X is a compact Kähler manifold, then the filtration on the cohomology given by the Fröhlicher spectral sequence satisfies

$$F^p H^k(X, \mathbb{C}) = \bigoplus_{r=p}^{k} H^{r,k-r}(X). \tag{5.2}$$

Definition 5.35. With X compact Kähler, the filtration (**??**) on $H^k(X, \mathbb{C})$ is called the *Hodge filtration*.

Proof. Consider the map $\ker(d : F^p\mathscr{A}^k \to F^p\mathscr{A}^{k+1}) \to H^k(X, \mathbb{C})$ which takes a closed form to its cohomology class. We first show that its image is precisely $\bigoplus_{r=p}^k H^{r,k-r}(X)$. Let $\alpha \in F^p\mathscr{A}^k(X)$ be a closed form. We may write $\alpha = \sum_{r=p}^k \alpha^{r,k-r}$ with $\alpha^{r,k-r} \in \mathscr{A}^{k,k-r}(X)$. Then since Δ_d preserves type, we may write $\alpha^{r,k-r} = \beta^{r,k-r} + \Delta_d\gamma^{r,k-r}$ for some $\beta^{r,k-r}, \gamma^{r,k-r} \in \mathscr{A}^{r,k-r}(X)$ and $\beta^{r,k-r}$ harmonic. Let $\beta := \sum \beta^{r,k-r}, \gamma := \sum \gamma^{r,k-r}$. Then since α and β are both closed, by an argument we have used before, $d^*d\gamma = 0$ and hence $\Delta_d\gamma = dd^*\gamma$. Hence α and β represent the same cohomology class, but β represents a class in $\bigoplus_{r=p}^k H^{r,k-r}(X)$. This shows that the image of the above map lies in $\bigoplus_{r=0}^k H^{r,k-r}(X)$.

Conversely, by definition, any class in $\bigoplus_{r=0}^k H^{r,k-r}(X)$ can be represented by a harmonic form α and such an α is closed in $F^p\mathscr{A}^k(X)$. This proves the claim.

We now wish to show that the kernel of the above map is precisely $\mathrm{im}(d : F^p\mathscr{A}^{k-1}(X) \to \mathscr{A}^k(X))$. Assume first that $p = k$, so that

$$F^p\mathscr{A}^k(X) = F^k\mathscr{A}^k(X) = \mathscr{A}^{k,0}(X) \quad F^p\mathscr{A}^{k+1}(X) = \mathscr{A}^{k+1,0}(X) \oplus \mathscr{A}^{k,1}(X) \quad F^k\mathscr{A}^{k-1}(X) = 0.$$

In this instance, we wish to show that the map is injective. Let $\alpha \in \mathscr{A}^{k,0}(X)$ be an exact form. Then since it is ∂- and $\overline{\partial}$-closed, by Proposition **??** there exists γ such that $\alpha = \partial\overline{\partial}\gamma$, but then comparing types, we see that $\alpha = 0$.

If $p = k - 1$,

$$F^p\mathscr{A}^k(X) = \mathscr{A}^{k,0}(X) \oplus \mathscr{A}^{k-1,1}(X) \quad F^p\mathscr{A}^{k+1}(X) = \mathscr{A}^{k+1,0}(X) \oplus \mathscr{A}^{k,1}(X) \oplus \mathscr{A}^{k-1,2}(X)$$
$$F^p\mathscr{A}^{k-1}(X) = \mathscr{A}^{k-1,0}(X).$$

Suppose $\alpha = \alpha^{k,0} + \alpha^{k-1,1}$ is exact. Then comparing types in the expression

$$0 = d\alpha = \partial\alpha^{k,0} + (\overline{\partial}\alpha^{k,0} + \partial\alpha^{k-1,1}) + \overline{\partial}\alpha^{k-1,1},$$

we find that $\overline{\partial}\alpha^{k-1,1} = 0$. Since α defines the zero cohomology class, its harmonic representatives are zero, and hence we may write

$$\alpha^{k,0} = \Delta_d\gamma^{k,0} \qquad\qquad \alpha^{k-1,1} = \Delta_d\gamma^{k-1,1}$$

for some $\gamma^{k,0} \in \mathscr{A}^{k,0}(X), \gamma^{k-1,1} \in \mathscr{A}^{k-1,1}(X)$. But $\alpha^{k-1,1} = 2\Delta_{\overline{\partial}}\gamma^{k-1,1}$ and since $\overline{\partial}\alpha^{k-1,1} = 0$, by the same argument as above, $\alpha^{k-1,1} = 2\overline{\partial}\overline{\partial}^*\gamma^{k-1,1}$. Then we have

$$\alpha' := \alpha - 2d\overline{\partial}^*\gamma^{k-1,1} = \alpha^{k,0} - 2\partial\overline{\partial}^*\gamma^{k-1,1}$$

is of type $(k, 0)$ and represents the zero cohomology class. By what we showed above, $\alpha' = 0$. Therefore $\alpha = 2d\overline{\partial}^*\gamma^{k-1,1}$ and $\overline{\partial}^*\gamma^{k-1,1} \in F^{k-1}\mathscr{A}^{k-1}$, so α lies in the claimed image. The argument for any $p \geq 0$ is a straightforward induction with this same reasoning. □

Theorem 5.36. If X is a compact Kähler manifold, then the Fröhlicher spectral sequence degenerates at E_1.

Proof. By Theorem **??**(b), one has

$$E_\infty^{p,q} = \mathrm{Gr}_p^F H^{p+q}\big(\mathscr{A}^\bullet(X)\big),$$

but by Proposition **??**, this is simply $H^{p,q}(X)$. On the other hand, we saw above that $E_1^{p,q} = H^{p,q}(X)$. Thus, we can conclude by Remark **??**. □

5.3.2 Pure Hodge Structures

Definition 5.37. Let $H_\mathbb{Z}$ be a finitely generated free abelian group and let $H = (H_\mathbb{Z})_\mathbb{C} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ be its complexification. A *pure Hodge structure of weight k on $H_\mathbb{Z}$* consists of a direct sum

$$H = \bigoplus_{p+q=k} H^{p,q},$$

with the property that

$$\overline{H^{p,q}} = H^{q,p}.$$

If X is a compact Kähler manifold, then the existence of the Hodge decomposition (Definition **??**) says that $H^k(X, \mathbb{Z})$ has a pure Hodge structure of weight k.

Lemma 5.38. Let $H_\mathbb{Z}$ be a finitely generate free abelian group. Then a pure Hodge structure of weight k on $H_\mathbb{Z}$ is equivalent to the existence of a filtration $\{F^pH\}_{p\geq 0}$ such that for p, q with $p + q = k + 1$, one has

$$F^pH \cap \overline{F^qH} = 0.$$

Proof. Exercise. □

5.4 Logarithmic de Rham Complexes

Definition 5.39. Let X be a complex manifold. A *hypersurface* $D \subseteq X$ is a closed subset defined locally as the vanishing set of a single holomorphic function; it is *irreducible* if the local equation is a non-zero divisor in the ring of holomorphic functions. An *effective divisor* is a formal sum of irreducible hypersurfaces. We say that an effective divisor D is a *normal crossing divisor* if every point of X has a neighbourhood with coordinates $z_1, \dots, z_m$ such that D is defined by $z_1 \cdots z_m = 0$; a normal crossing divisor D is *strict* if $D = \bigcup_{\lambda=1}^{r} D_\lambda$ with each D_λ a smooth hypersurface.

Remark 5.40. By way of motivating this definition, a theorem of Nagata states that any abstract complex variety can be embedded as a Zariski open subset of a complete variety. Furthermore, by Hironaka's result on the resolution of singularities, one may choose the complete variety to be smooth and the complement to be a normal crossing divisor.

Definition 5.41. Let $D \subseteq X$ be an effective divisor. Then for $1 \leq k \leq n$, we may consider the sheaf $\Omega_X^k(D)$ of holomorphic k-forms with a simple pole at D. In coordinates as above, if $f = f(z_1, \dots, z_n)$ is a local defining equation for D, then $\Omega_X^k(D)$ has local frames dz_I/f, with $I \in \wp_k(N)$. We may define the sheaf $\Omega_X^k(\log D)$ to be the subsheaf of $\Omega_X^k(D)$ of forms α such that $d\alpha$ is a section of $\Omega^{k+1}(D)$, i.e., α has a simple pole at D and $d\alpha$ also has a simple pole at D. It is clear that $d(\Omega_X^k(\log D) \subseteq \Omega_X^{k+1}(\log D)$, so that we get a complex

$$\mathscr{O}_X \xrightarrow{d=\partial} \Omega_X^1(\log D) \xrightarrow{d=\partial} \Omega_X^2(\log D) \xrightarrow{d=\partial} \cdots \xrightarrow{d=\partial} \Omega_X^n(\log D)$$

which is called the *logarithmic de Rham complex of X*.

Lemma 5.42. Let D be a normal crossing divisor and suppose $z_1, \dots, z_n$ are local coordinates such that D is defined by $z_1 \cdots z_r = 0$. Then $\Omega_X^k(\log D)$ has local frames

$$\frac{dz_{\lambda_1}}{z_{\lambda_1}} \wedge \cdots \wedge \frac{dz_{\lambda_l}}{z_{\lambda_l}} \wedge dz_{\mu_1} \wedge \cdots \wedge dz_{\mu_m}$$

for $1 \leq \lambda_1 < \cdots < \lambda_l \leq r, r+1 \leq \mu_1 < \cdots < \mu_m \leq n$ and $k = l + m$. In particular, $\Omega_X^k(\log D)$ is locally free.

Proof. Exercise. □

Remark 5.43. The logarithmic de Rham complex is not locally exact. For example, suppose $\dim_{\mathbb{C}} X = 1$ and that z is a local coordinate defining D. Then dz/z is a section of $\Omega_X^1(\log D)$, but since dz/z has non-zero integral over arbitrarily small loops around the point $z = 0$, there is no holomorphic function f with $df = dz/z$ on any neighbourhood of that point.

Definition 5.44. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathscr{F}$ is a sheaf of abelian groups on X, then the *direct image of $\mathscr{F}$ under f* is the sheaf $f_*\mathscr{F}$ on Y given by

$$(f_*\mathscr{F})(V) := \mathscr{F}\big(f^{-1}(V)\big).$$

One easily verifies that this is indeed a sheaf.

Let D be a normal crossing divisor and let $U := X \setminus D$ and $j : U \hookrightarrow X$ the inclusion map. We may consider the sheaf $j_*\Omega_U^k$; its sections over an open $V \subseteq X$ are holomorphic k-forms on $U \cap V$. Any section of $\Omega_X^k(\log D)$ over V restricts to a holomorphic k-form on $U \cap V$ and hence we have a natural inclusion

$$\Omega_X^k(\log D) \subseteq j_*\Omega_U^k.$$

This is compatible with the differentials, so it extends to an inclusion of complexes

$$\Omega_X^\bullet(\log D) \subseteq j_*\Omega_U^\bullet.$$

On U, we have an inclusion $\Omega_U^\bullet \subseteq \mathscr{A}_U^\bullet$ of complexes and hence one $j_*\Omega_U^\bullet \subseteq j_*\mathscr{A}_U^\bullet$.

Proposition 5.45 (Griffiths (1969), Deligne (1971))**.** The composition of the above two morphisms of complexes

$$\Omega_X^\bullet(\log D) \hookrightarrow j_*\mathscr{A}_U^\bullet$$

is a quasi-isomorphism.

Proof. Of course, we may check that we obtain a quasi-isomorphism at stalks. Suppose $x \in U$. Then $\Omega_X^\bullet(\log D)_x = \Omega_{X,x}^\bullet$ and this is exact by the holomorphic Poincaré lemma (Lemma 5.18). On the other hand, $(j_*\mathscr{A}_U^\bullet)_x = \mathscr{A}_{U,x}^\bullet$ and this is exact by the de Rham resolution (Proposition **??**), so there is induced quasi-isomorphism (of the zero group) at the stalk.

Therefore, we suppose $x \in D$. Since the question is local, we may assume that $X = W^n$, where $W \subseteq \mathbb{C}$ is a disc of radius $\xi > 0$ centred at 0 and

$$D = \{(z_1, \dots, z_n) \in W^n \,:\, z_1 \cdots z_r = 0\} \qquad\qquad U = (W^*)^r \times W^{n-r},$$

where $W^* := W \setminus \{0\}$. Since any smaller neighbourhood of x will always contain a polydisc of this form (of smaller radius), essentially any computations in the stalk can be done in this neighbourhood.

The first point is that $(j_*\mathscr{A}_U^\bullet)(X) = \mathscr{A}_U^\bullet(U)$, but we may write down explicitly representatives for the cohomology as follows. If $z = (x, y)$ is a coordinate in W, then the cohomology of W^* is generated by

$$\frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Note that if $S^1 \subseteq W^*$ is a circle centred at 0 and θ is the angle coordinate, then the above form restricts to $d\theta$ on S^1. Since U has r copies of W^*, we know that $H^1(U, \mathbb{C})$ is generated by

$$\frac{x_1\,dy_1 - y_1\,dx_1}{x_1^2 + y_1^2}, \dots, \frac{x_r\,dy_r - y_r\,dx_r}{x_r^2 + y_r^2},$$

and that $H^k(U, \mathbb{C})$ is generated by the k-fold wedge products of these.

On the other hand, for the coordinate z on W, we have

$$\frac{dz}{z} = \frac{x - iy}{x - iy}\frac{dx + i\,dy}{x + iy} = i\frac{x\,dy - y\,dx}{x^2 + y^2} + \frac{x\,dx + y\,dy}{x^2 + y^2} = i\frac{x\,dy - y\,dx}{x^2 + y^2} + \frac{1}{2}d\log(x^2 + y^2),$$

and up to the factor of i, this is the same as the cohomology class above. Therefore, the sections

$$\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}$$

map to generating cohomology classes in the $j_*\mathscr{A}_U^\bullet$ complex, and we have a surjection in cohomology.

To see that it is an isomorphism, it is enough to see that every non-trivial cohomology class in $\Omega_X^\bullet(\log D)$ is represented by a $\mathbb{C}$-linear combination of classes of the form

$$\frac{dz_{\lambda_1}}{z_{\lambda_1}} \wedge \cdots \wedge \frac{dz_{\lambda_l}}{z_{\lambda_l}},$$

with $\{\lambda_1 < \cdots < \lambda_l\} \subseteq \{1, \dots, r\}$. To prove this, we go by induction on r. Suppose $\alpha \in \Gamma(X, \Omega_X^k(\log D))$ satisfies $d\alpha = 0$ and assume that the only pole involves z_1; say we can write

$$\alpha = \frac{dz_1}{z_1} \wedge \beta + \gamma,$$

where β and γ are holomorphic forms, and that the coefficients of β do not depend on z_1. Then

$$0 = d\alpha = -\frac{dz_1}{z_1} \wedge d\beta + d\gamma.$$

By assumption on β, $d\beta$ will not involve any term with dz_1 and since the second term is holomorphic, we must have $d\beta = 0$; hence $d\gamma = 0$ also. If $k \geq 2$, then by the holomorphic Poincaré lemma (Lemma 5.18), there are $\sigma \in \Gamma(X, \Omega_X^{k-2})$, $\tau \in \Gamma(X, \Omega_X^{k-1})$ with $d\sigma = \beta, d\tau = \gamma$, therefore

$$\alpha = \frac{dz_1}{z_1} \wedge d\sigma + d\tau = d\left(-\frac{dz_1}{z_1} \wedge \sigma + \tau\right).$$

If $k = 1$, β is a function and $d\beta = 0$ means that β is a constant and so

$$\alpha = \beta\frac{dz_1}{z_1} + d\tau.$$

In either case, α is a linear combination of the forms above plus an exact form.

By induction we may now assume that the poles that a closed form whose poles involves only $z_1, \ldots, z_{p-1}$ with $2 \leq p \leq r$ is a $\mathbb{C}$-linear combination of forms of the above type plus an exact form. We now wish to show that the statement holds for closed forms with poles involving only $z_1, \ldots, z_p$. As before, we may write

$$\alpha = \frac{dz_p}{z_p} \wedge \beta + \gamma,$$

where now β and γ have poles only involving $z_1, \ldots, z_{p-1}$ and the coefficients of β do not depend on z_p. By the same argument as above, we have $d\beta = 0, d\gamma = 0$. The induction hypothesis tells us that $\beta = \beta' + d\sigma$, $\gamma = \gamma' + d\tau$, where β', γ' are linear combinations of wedge products of the dz_λ/z_λ with $1 \leq \lambda \leq p-1$. So

$$\alpha = \frac{dz_p}{dz_p} \wedge \beta' + \gamma' + \frac{dz_p}{dz_p} \wedge d\sigma + d\tau = \frac{dz_p}{dz_p} \wedge \beta' + \gamma' + d\left(-\frac{dz_p}{dz_p} \wedge \sigma + \tau\right)$$

and this is again of the required form. Thus we see that we get isomorphisms in cohomology, and the map is a quasi-isomorphism as claimed. □

Corollary 5.46. We have a canonical isomorphism for each $k \in \mathbb{Z}$

$$\mathbb{H}^k\big(X, \Omega_X^\bullet(\log D)\big) \xrightarrow{\sim} H^k(U, \mathbb{C}).$$

Proof. Corollary 5.14 gives an isomorphism

$$\mathbb{H}^k\big(X, \Omega_X^\bullet(\log D)\big) \xrightarrow{\sim} \mathbb{H}^k(X, j_*\mathscr{A}_U^\bullet).$$

Now, $j_*\mathscr{A}_U^\bullet$ is a sheaf of $\mathscr{A}_X$-modules (since we may restrict any smooth function on X to one on U), and hence is fine by Remark **??**. This implies that it is soft and hence acyclic, by Proposition **??** and Corollary **??**, respectively. Now, Lemma 5.15 gives canonical isomorphisms

$$H^k\big(\Gamma(X, j_*\mathscr{A}_U^\bullet)\big) = H^k\big(\Gamma(U, \mathscr{A}_U^\bullet)\big) \xrightarrow{\sim} \mathbb{H}^k(X, j_*\mathscr{A}_U^\bullet).$$

But the de Rham resolution (Proposition **??**) gives

$$H^k(\Gamma(U, \mathscr{A}_U^\bullet) = H^k(X, \mathbb{C}).$$ □

5.5 The Weight Filtration

Definition 5.47. Let D be a strict normal crossing divisor, say $D = \bigcup_{\lambda=1}^{r} D_\lambda$ with each D_λ a smooth hypersurface. We defin the (increasing) *weight filtration* $\{W_m\}_{m\geq 0}$ on $\Omega_X^\bullet(\log D)$ by

$$W_m\Omega_X^k(\log D) := \begin{cases} 0 & \text{if } m < 0 \\ \Omega_X^{k-m} \wedge \Omega_X^m(\log D) & \text{if } 0 \leq m \leq k \\ \Omega_X^k(\log D) & \text{if } m \geq k. \end{cases}$$

Fix $1 \leq m \leq r$. Let $L \in \wp_m R$, where $R = \{1, \ldots, r\}$, say $L = \{\lambda_1 < \cdots < \lambda_m\}$. We let $D_L := \bigcap_{\lambda\in L} D_\lambda$. By the normal crossing condition, this is a smooth submanifold of X. We set

$$D^m := \coprod_{L\in\wp_m R} D_L \qquad\qquad D^0 := X.$$

There is a map $i^m : D^m \to X$ which is simply the inclusion on each connected component of D^m.

Let $1 \leq k \leq n = \dim X$ and suppose $1 \leq m \leq k$. The *residue map (of degree m)* $\mathrm{Res} = \mathrm{Res}_m : W_m\Omega_X^k(\log D) \to i_*^m\Omega_{D^m}^{k-m}$ is defined as follows. First, it is clear that $i_*^m\Omega_{D^m}^{k-m}$ is supported only at points lying in an m-fold intersection of the D_λ; at such a point, we may choose coordinates $z_1, \ldots, z_n$ such that z_λ defines D_λ for $1 \leq \lambda \leq r$. Assume that we are at a point of D_L for some $L \in \wp_m R$, so that D_L is defined by $z_{\lambda_1}\cdots z_{\lambda_m}$. By definition of the weight filtration, we may write any section of $W_m\Omega_X^k(\log D)$ as

$$\alpha \wedge \frac{dz_{\lambda_1}}{z_{\lambda_1}} \wedge \cdots \wedge \frac{dz_{\lambda_m}}{z_{\lambda_m}} =: \alpha \wedge \frac{dz_L}{z_L}$$

for some section α of Ω_X^{k-m}. Then we define Res by

$$\alpha \wedge \frac{dz_L}{z_L} \mapsto \alpha|_{D_L}.$$

It is left as an exercise to show that this is independent of the coordinates chosen.

Proposition 5.48. We have exact sequences

$$0 \to W_{m-1}\Omega_X^k(\log D) \to W_m\Omega_X^k(\log D) \xrightarrow{\mathrm{Res}} i_*^m\Omega_{Y^m}^{k-m} \to 0,$$

i.e., Res gives an identification $\mathrm{Gr}_m^W\Omega_X^k(\log D) \cong i_*^m\Omega_{Y^m}^{k-m}$. Furthermore, Res gives a map of complexes $W_m\Omega_X^\bullet(\log D) \to i_*^m\Omega_{Y^m}^\bullet[-m]$.

Proof. By the local definition of Res, it is clear that it is surjective (locally, and hence as a morphism of sheaves). Suppose that a local section α of Ω_X^{k-m} is such that $\alpha|_{D_L} = 0$. We may write

$$\alpha = \sum_{K\in\mathbb{P}_{k-m}(N\setminus L)} \alpha_L\, dz_L$$

for some locally defined functions α_K. If $\alpha|_{D_L} = 0$, then each $\alpha_L|_{z_{\lambda_1}\cdots z_{\lambda_m}} = 0$, so $\alpha_L = z_{\lambda_s}\beta_L$ for some $1 \leq s \leq m$ and some holomorphic function β_L. Thus

$$\alpha_L\, dz_L = \beta_L \frac{dz_{\lambda_1}}{z_{\lambda_1}} \wedge \cdots \wedge dz_{\lambda_s} \wedge \cdots \wedge \frac{dz_{\lambda_1}}{z_{\lambda_1}}$$

and this is a seciton of $W_{m-1}\Omega_X^k(\log D)$, hence so is α. ☐

5.6 Mixed Hodge Structures and Mixed Hodge Complexes

Definition 5.49. Let $H_{\mathbb{Z}}$ be a finitely generated free abelian group. A *mixed Hodge structure on* $H_{\mathbb{Z}}$ consists of a finite increasing filtration W_k on $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, called the *weight filtration*, and a finite decreasing filtration F^p on $H = H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, called the *Hodge filtration*, which are such that for each $k \geq 0$, the graded pieces

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_k^W(H),$$

where $\overline{F}$ is the complex conjugate filtration induced from F, define a pure Hodge structure of weight k on $\mathrm{Gr}_k^W(H)$.

Theorem 5.50 (Deligne, 1971)**.** Let X be a smooth projective algebraic variety, $D \subseteq X$ a strict normal crossing divisor and $U := X \setminus D$.

(a) The filtrations $W[k]$ and F define a mixed Hodge structure on

$$\mathbb{H}^k(X, \Omega_X^\bullet(\log D)) = H^k(U, \mathbb{C}).$$

(b) If X' is another complex manifold and $D' \subseteq X'$ is a normal crossing divisor such that $X' \setminus D' \cong U$, then under the identifications

$$\mathbb{H}^k(X, \Omega_X^\bullet(\log D)) = H^k(U, \mathbb{C}) = \mathbb{H}^k(X', \Omega_X^\bullet(\log D'))$$

the filtrations agree.