Localization Error Analysis in Wireless Sensor Networks

Uday Kiran Pulleti
What is a Wireless Sensor Network?

- **Entities**
  - Sensor Nodes
  - Beacon Nodes
  - Gateway Nodes

- **Function**
  - Sample
  - Process
  - Communicate
Wireless Sensor Networks

- Why WSNs?
  Small size, Low cost, High Reliability and Accessibility

- Unique challenges
  Power, Random Deployment, Unreliable Communication

- Applications
  Habitat monitoring, Battle fields, Surveillance, Nuclear power plants, etc.

- Functions
  Parameter measurement, Target localization and tracking.
Localization

- Node Localization
- Source Localization

Differences
- Cooperative/Non-cooperative
- Node Density
- Computational Complexity
Localization – A generic definition

- Given a set of entities (nodes) with known locations and a source entity, the problem is to estimate the location of the source.
- Location aware --- Sensor node
- Location unaware --- Source
Localization

- **Measurements Modalities**
  - Received Signal Strength (RSS)
  - Time of Flight (TOF)
  - Time of Arrival (TOA)
  - Time Difference of Arrival (TDOA)
  - Direction of Arrival (DOA)

- **Measurements**
  - Range
  - Range Difference
Localization Algorithms

- Mostly non iterative

Range Difference
  - Locus is a hyperbola
  - At least four nodes are required
  - Least Square Estimation

Range
  - Locus is a circle
  - At least three nodes are required
  - Also Least Square Estimation
Localization Error

- **Network Parameters**
  - Node density
  - Available energy resources
  - Circuit noise
  - Location errors

- **Environmental Parameters**
  - Sensing modality and its propagation model
  - Terrain’s geographical topology
  - Ambient noise levels
Problem Formulation

- To characterize the localization error with respect to the network and environmental parameters in an algorithm independent manner
**Notation**

- \((x_s, y_s)\) --- source location
- \((\hat{x}, \hat{y})\) --- estimated source location
- \((x_i, y_i)\) --- \(i_{th}\) sensor node
- \(r_i\) --- distance between the source and \(i_{th}\) node
- \(m_i\) --- range measurement at the \(i_{th}\) sensor node.
- \(m_{ij}\) --- range-difference measurement between \(i_{th}\) and the \(j_{th}\) sensor nodes
Error models

Range Measurements
  - Gaussian error

\[ f(m(x); \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{m - \sqrt{(x \cdot \hat{x}^2 + (y \cdot \hat{y})^2)}}{2\sigma^2}\right) \]
Error models

- Range Difference Measurements
  - Joint Gaussian error

\[
f(x, y, z) = \frac{1}{(\sqrt{2\pi})^n} \exp\left( -\frac{1}{2} \begin{bmatrix} x - m \end{bmatrix}^T C \begin{bmatrix} x - m \end{bmatrix} \right)
\]

\[
\begin{bmatrix} x \\
end{bmatrix} \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T
\]

\[
\begin{bmatrix} m \\
end{bmatrix} \equiv \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}^T
\]

\[
\begin{bmatrix} r \\
end{bmatrix} \equiv \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}^T
\]
Error models

■ Range Difference Measurements
  • Derived Gaussian error

\[
f(m_{21}, m_{31}, \ldots, m_{N1}|(x_1, y_1), \ldots, (x_N, y_N); (x_s, y_s))
\]

\[
= c \exp \left\{ \left( \frac{(r_1 - (m_2 - r_2) - (m_3 - r_3) - \ldots - (m_N - r_N))^2}{2N\sigma^2} \right) - \left( \frac{r_1^2 + (m_2 - r_2)^2 + (m_3 - r_3)^2 + \ldots + (m_N - r_N)^2}{2\sigma^2} \right) \right\}
\]
Data Collection Techniques

- Closest N Activation Model (CNAM)
- Fixed Radius Activation Model (FRAM)
Post-deployment and a priori error performance

- Given sensor network
- Random network (Poisson points)
  - CNAM
    \[ f((x_1, y_1), \ldots, (x_N, y_N)) = (\lambda)^N e^{-\lambda \pi \left((x_s-x_N)^2+(y_s-y_N)^2\right)} \]
  - FRAM
    \[ f((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N) | N) = \frac{1}{(\pi R^2)^N} \]
Cramer-Rao Lower Bound

\[ J_f = E \left[ \frac{\partial \log f}{\partial \theta} \right] \]
Cramer-Rao Lower Bound

\[ \begin{vmatrix} \frac{\partial \log \theta}{\partial \theta} \\ \frac{\partial \log \theta}{\partial \theta} \end{vmatrix} = J_2 = \begin{vmatrix} \frac{\partial \log \theta}{\partial \theta} \\ \frac{\partial \log \theta}{\partial \phi} \end{vmatrix} = J_2 \]
Range Measurements – Post-deployment CRLB

\[ J_1 = \sum_{i=1}^{N} \frac{(I+2x_i)(xx_i)}{\partial^2 (K+i)^2} \]

\[ J_2 = \sum_{i=1}^{N} \frac{x_i}{\partial^2 (K+i)^2} \]

\[ J_3 = \sum_{i=1}^{N} \frac{x_i}{\partial^2 (K+i)^2} \]
Re-deployment strategies

- Constrained optimization
Range Measurements – A priori CRLB

\[ J_{i,j} = -E_{M,L} \left( \frac{\partial^2 \ln(f(M, L))}{\partial X_i \partial X_j} \right) \]

\[ J_{i,j} = JL_{i,j} + JM_{i,j} = -E_L \left( \frac{\partial^2 \ln(f(L))}{\partial X_i \partial X_j} \right) - E_L \left( E_{M|L} \left( \frac{\partial^2 \ln(f(M | L))}{\partial X_i \partial X_j} \right) \right) \]
\[
J_{L_{i,j}} = \begin{cases} 
2\lambda\pi & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

\[
J_{M_{11}} = J_{M_{22}} = E_{\{r_k\}} \left[ \sum_{k=1}^{N} \left( \frac{(2\pi)^N(1 + 2\alpha^2)}{2\alpha^2(K + r_k)^2} \right) \right]
\]

\[
J_{ij} = \begin{cases} 
2\lambda\pi + E_{\{r_k\}} \left( \sum_{k=1}^{N} \left( \frac{(2\pi)^N(1 + 2\alpha^2)}{2\alpha^2(K + r_k)^2} \right) \right) & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]
\[ \overline{J}_{11} = \overline{J}_{22} = 2\lambda \pi + \frac{\lambda \pi (2\alpha^2 + 1)(\sum_{i=2}^{N} \frac{1}{i-1})}{2\alpha^2} \]
Plot of CRLB vs $\lambda$ with $\alpha$ as the parameter

- $\alpha = 0.00625$ (triangle down)
- $\alpha = 0.025$ (asterisk)
- $\alpha = 0.1$ (circle open)
Plot of CRLB vs $\alpha$ with $\lambda$ as the parameter.
Plot of CRLB vs number of sensors used in localization (N)
CRLB 3D plot for the nearest 6 nodes as a function of $\alpha$ and $\lambda$
\[ J_{i,j} = -E_N \left( E_{M,L} \left[ \frac{\partial^2 \ln(f(M, L, N))}{\partial X_i \partial X_j} \right] \right) \]
\[ = -E_N \left( E_{M,L} \left( \frac{\partial^2 \ln(f(N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(L | N))}{\partial X_i \partial X_j} + \frac{\partial^2 \ln(f(M | L, N))}{\partial X_i \partial X_j} \right) \right) \]
\[ = E_N \left( JN_{ij} + JL_{ij} + JM_{ij} \right) \]

\[
J_{ij} = \begin{cases} 
\frac{\lambda \pi (1 + 2\alpha^2)}{\alpha^2} \left[ \log \left( \frac{K + R}{R} \right) - \frac{R}{K + R} \right] & \text{for } i = j \\
0 & \text{for } i \neq j 
\end{cases}
\]

\[
CRLB_x = CRLB_y = \frac{\alpha^2}{\lambda \pi (1 + 2\alpha^2) \left[ \log \left( \frac{K + R}{R} \right) - \frac{R}{K + R} \right]}^{-1}
\]
Error Comparisons

\[(x_s - x_i)^2 + (y_s - y_i)^2 = m_i^2\]

\[2x_s(x_1 - x_i) + 2y_s(y_1 - y_i) = (m_i^2 - m_1^2) - ((x_i^2 + y_i^2) - (x_1^2 + y_1^2))\]

\[AX = B + \epsilon\]

\[\hat{X} = (A^T A)^{-1} A^T B \quad \text{LSE}\]

\[\hat{X} = (A^T W^{-1} A)^{-1} A^T W^{-1} B \quad \text{WLSE}\]

\[W = \text{Cov}(\epsilon \epsilon^T)\]
Error Comparisons

For $\lambda = 16 \ a = 0.1$

CNAM: Comparison of LS and WLS estimators with CRLB
Error Comparisons

CNAM: Comparison of LS and WLS estimators with CRLB for varying \( \lambda \)
Error Comparisons

$\lambda = 16 \; \alpha = 0.1$

FRAM: Comparison of LS and WLS estimators with CRLB as localizing radius $R$ varies
Error Comparisons

FRAM: Comparison of LS and WLS estimators with CRLB for varying $\lambda$
Range Difference Measurements

- Joint Gaussian model

\[
f(m_{21}, m_{31}, \ldots, m_{N1}|(x_1, y_1), \ldots, (x_N, y_N); (x_s, y_s)) = \frac{1}{(\sqrt{2\pi})^N \det(C)} e^{-\frac{1}{2} [m-r]^T C^{-1} [m-r]}
\]

\[
C = \frac{\sigma^2}{2} \begin{pmatrix}
2 & 1 & 1 & \ldots \\
1 & 2 & 1 & \ldots \\
1 & 1 & 2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}_{(N-1) \times (N-1)}
\]

\[
P_{ij} = \begin{cases} 
\frac{2(N-1)}{N\sigma^2} & \text{if } i = j \\
\frac{2(-1)}{N\sigma^2} & \text{if } i \neq j 
\end{cases}
\]
Joint Gaussian model

- CNAM

\[
CRLB_x = CRLB_y = \frac{1}{2\lambda \pi + \frac{2(\lambda A - 1)}{\sigma^2}}
\]

- FRAM

\[
CRLB_x = CRLB_y = \frac{\sigma^2}{2(\lambda A - 1)}
\]
Derived Gaussian model

- CNAM

\[
CRLB_x = CRLB_y = \frac{1}{2\lambda \pi + \frac{(\lambda A - 1)}{2\sigma^2}}
\]

- FRAM

\[
CRLB_x = CRLB_y = \frac{2\sigma^2}{(\lambda A - 1)}
\]
Error Comparisons

\[ AX = B + \epsilon \]

\[
A(i, 1) = x_1 m_{2i} + x_2 m_{i1} + x_i m_{12}
\]

\[
A(i, 2) = y_1 m_{2i} + y_2 m_{i1} + y_i m_{12}
\]

\[
B(i, 1) = \frac{1}{2} \left( m_{12} m_{2i} m_{i1} + m_{2i} (x_1^2 + y_1^2) + m_{i1} (x_2^2 + y_2^2) + m_{12} (x_i^2 + y_i^2) \right)
\]

\[
\hat{X} = (A^T A)^{-1} A^T B \quad \text{LSE}
\]
Error Comparisons – Joint Gaussian Model

CNAM: Comparison of PI estimator with CRLB as localizing nodes $N$ varies for Joint Gaussian Error Model

FRAM: Comparison of PI estimator with CRLB as activation radius $R$ varies for Joint Gaussian Error Model
Error Comparisons – Derived Gaussian Model

CNAM: Comparison of PI estimator with CRLB as localizing nodes $N$ varies for Derived Gaussian Error Model

FRAM: Comparison of PI estimator with CRLB as activation radius $R$ varies for Derived Gaussian Error Model
Thank You