CORRIGE 4

(1) On rappelle $g_1 = \frac{\partial}{\partial x} f, g_2 = \frac{\partial}{\partial y} f$.

(a) We calculate:

(i) $g_1 = 12x^2y + 6xy$, $g_2 = 4x^3 + x^2$

(ii) $g_1 = \frac{3x^2}{y}$, $g_2 = \frac{-x^3}{y^2}$

(iii) $g_1 = 2xe^{x^2-y^2}$, $g_2 = -2ye^{x^2-y^2}$

(iv) $g_1 = 2x\cos(x^2 + y^2)$, $g_2 = 2y\cos(x^2 + y^2)$

(v) $g_1 = \frac{x}{\sqrt{x^2+y^2-1}}$, $g_2 = \frac{y}{\sqrt{x^2+y^2-1}}$

(vi) $g_1 = \frac{\cos(x)\sin(y)}{\sin(x)\sin(y)} = \text{cotan}(x)$, $g_2 = \frac{\sin(x)\cos(y)}{\sin(x)\sin(y)} = \text{cotan}(y)$

(b) On calcule:

(i) $\frac{\partial}{\partial y} g_1 = 12x^2 + 2x$, et $\frac{\partial}{\partial x} g_2 = 12x^2 + 2x$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(ii) $\frac{\partial}{\partial y} g_1 = \frac{-3x^2}{y^2}$, et $\frac{\partial}{\partial x} g_2 = \frac{-3x^2}{y^2}$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(iii) $\frac{\partial}{\partial y} g_1 = -4xye^{x^2-y^2}$, et $\frac{\partial}{\partial x} g_2 = -4xye^{x^2-y^2}$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(iv) $\frac{\partial}{\partial y} g_1 = 4xycos(x^2+y^2)$, et $\frac{\partial}{\partial x} g_2 = 4xycos(x^2+y^2)$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(v) $\frac{\partial}{\partial y} g_1 = x\frac{\partial}{\partial y}(x^2 + y^2 - 1)^{-\frac{1}{2}} = -xy(x^2 + y^2 - 1)^{-\frac{3}{2}}$, et $\frac{\partial}{\partial x} g_2 = y\frac{\partial}{\partial y}(x^2 + y^2 - 1)^{-\frac{1}{2}} = -xy(x^2 + y^2 - 1)^{-\frac{3}{2}}$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(vi) $\frac{\partial}{\partial y} g_1 = \frac{\partial}{\partial y}(cotan(x)) = 0$, et $\frac{\partial}{\partial x} g_2 = \frac{\partial}{\partial x}(cotan(y)) = 0$. Alors, $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$.

(2) Question 2:

(a) Pour $(x, y) \neq (0, 0)$, on a:

$$g_1 = \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{y\sqrt{x^2+y^2} - x^2y\frac{1}{\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{y(x^2+y^2) - x^2y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

puisque $f(x, y)$ est symmetric en x et y, on a:

$$g_2 = \frac{\partial}{\partial y}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}.$$

Maintenant, pour $(x,y)=(0,0)$ il faut s'en remettre à la définition de la dérivée partielle:

$$\frac{\partial}{\partial x}f(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\lim_{h\to 0}\frac{(\frac{0}{\sqrt{h^2+0^2}})}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

et

$$\frac{\partial}{\partial y}f(0,0)=\lim_{h\to 0}\frac{f(0,y)-f(0,0)}{h}\lim_{h\to 0}\frac{(\frac{0}{\sqrt{0^2+h^2}})}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Donc, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ existent pour tout $(x,y)\in\mathbb{R}^2$.

(b) g_1 n'est pas continue au point $(0,0)$ parce que, par example

$$\lim_{y\to 0}g_1(0,y)=\lim_{y\to 0}1=1\neq g_1(0,0)=9$$

(3) Plan tangent:

(a) Le plan tangent au point $(2,1,4)$ est:

$$z=z(2,1)+\frac{\partial f}{\partial x}(2,1)\cdot(x-2)+\frac{\partial f}{\partial y}(2,1)\cdot(y-1).$$

On a $\frac{\partial f}{\partial x}(2,1)=2xy(2,1)=2\cdot 2\cdot 1=4$ et $\frac{\partial f}{\partial y}(2,1)=x^2(2,1)=2^2=4$ alors:

$$z=4+4(x-2)+4(y-1)=4x+4y-8.$$

(b) Soit $f(x,y)=\sqrt{-x^2-y^2+25}$. On a $\frac{\partial f}{\partial x}(x,y)=\frac{-x}{\sqrt{-x^2-y^2+25}},\frac{\partial f}{\partial y}(x,y)=\frac{-y}{\sqrt{-x^2-y^2+25}}$ donc au point $(-3,0,4)$ on a $\frac{\partial f}{\partial x}(-3,0)=\frac{3}{4},\frac{\partial f}{\partial y}(x,y)=0$ donc le plan tangent est

$$z=4+\frac{3}{4}(x+3)$$

(c) On a $\frac{\partial f}{\partial x}(x,y)=-y^2\frac{\partial}{\partial x}(\frac{1}{x})=\frac{y^2}{x^2},\frac{\partial f}{\partial y}(x,y)=\frac{-2y}{x}$ donc au point $(1,1,-1)$, on a $\frac{\partial f}{\partial x}(1,1)=1,\frac{\partial f}{\partial y}(1,1)=-2$ donc le plan tangent est

$$z=-1+1(x-1)-2(y-1)=x-2y.$$

Le plan tangent est horizantal si et seulement s'il est de la forme

$$z=c,$$

et donc si et seulement si

$$\frac{\partial f}{\partial x}(x_0,y_0)=\frac{\partial f}{\partial y}(x_0,y_0)=0$$

Mais il n'y a pas un point (x_0,y_0) qui satisfait cela.