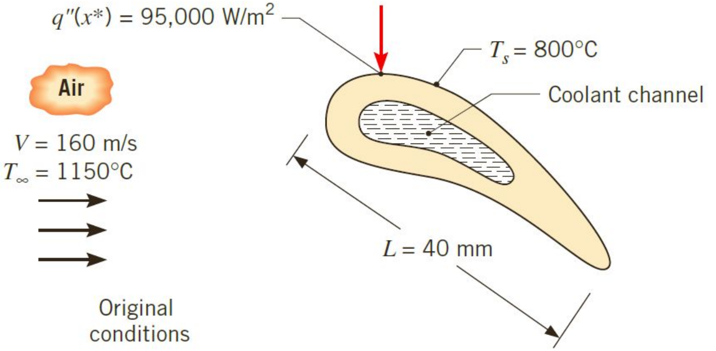
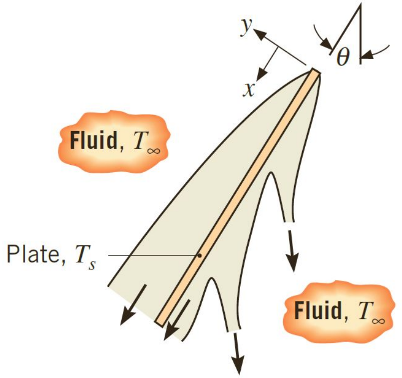
<table>
<tr><th></th><th>Forced Convection</th><th>Free Convection</th></tr>
<tr><td>External Flow</td><td>
</td><td>
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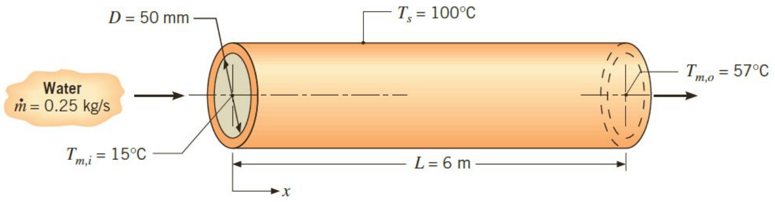
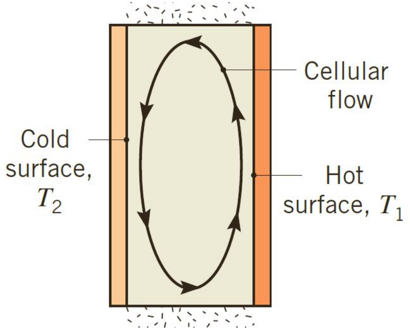
<tr><td>Internal Flow</td><td>
</td><td>
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Forced Convection in External flow

The Problem of Convection

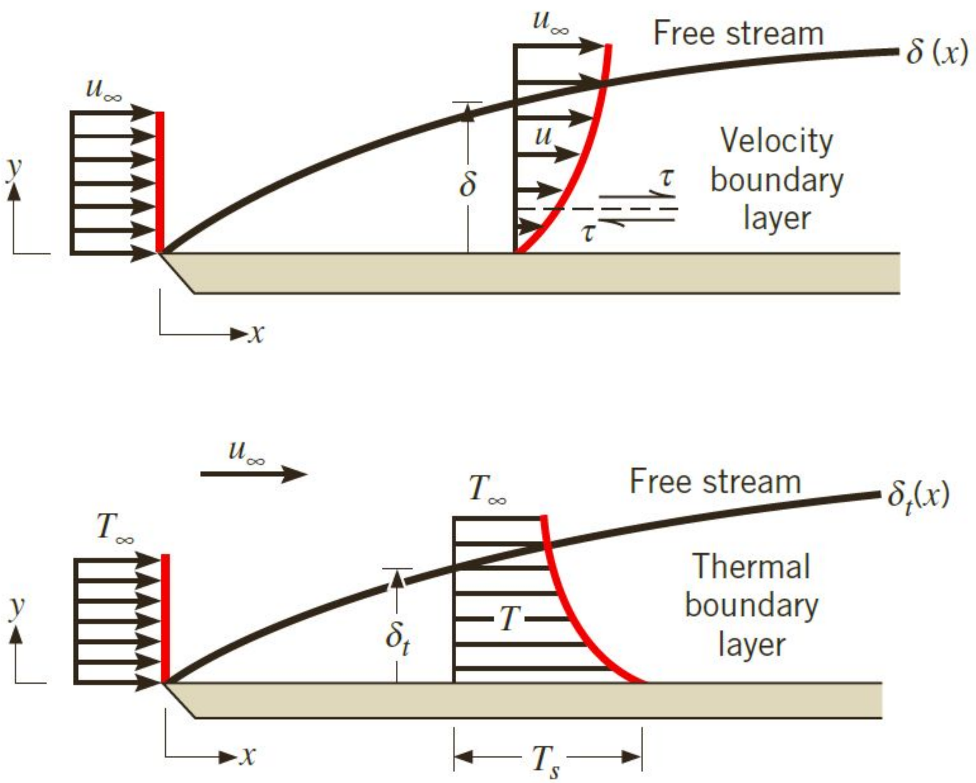
Skin friction coefficient

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \qquad C_f \equiv \frac{\tau_w}{\rho u_\infty^2/2}$$

Nusselt Number

$$\underbrace{-k_f \frac{\partial T}{\partial y}\bigg|_{y=0}}_{\substack{\text{conduction}\\ \text{into the fluid}}} = h(T_w - T_\infty)$$

$$\frac{\partial\left(\dfrac{T_w - T}{T_w - T_\infty}\right)}{\partial(y/L)}\Bigg|_{y/L=0} = \frac{hL}{k_f} = \mathrm{Nu}_L, \text{ the Nusselt number}$$



Flow boundary layer

Fluids flowing past solid bodies adhere to them, so a region of variable velocity must be built up between the body and the free fluid stream, as indicated in Fig. 6.1. This region is called a *boundary layer*, which we abbreviate as b.l. The b.l. has a thickness, δ. The boundary layer thickness is arbitrarily defined as the distance from the wall at which the flow velocity approaches to within 1% of u_∞. The boundary layer is normally very thin in comparison with the dimensions of the body immersed in the flow.[1]

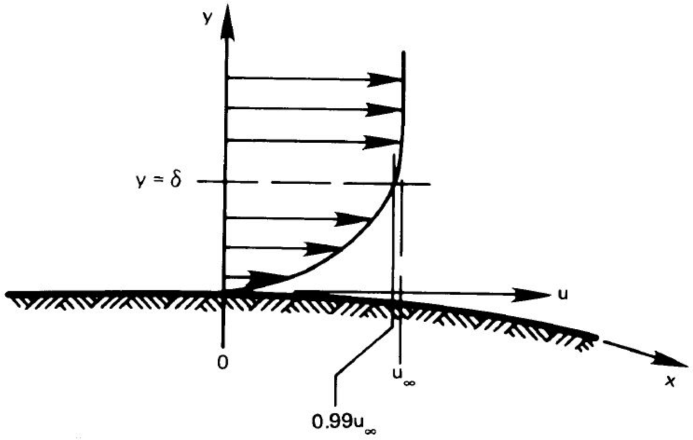


Figure 6.1 A boundary layer of thickness δ.

$$\mathrm{Re}_x \equiv \frac{\rho u_\infty x}{\mu} = \frac{u_\infty x}{\nu} \tag{6.1}$$

where ν is the kinematic viscosity μ/ρ and Re_x is called the *Reynolds number*. It characterizes the relative influences of inertial and viscous forces in a fluid problem. The subscript on Re—x in this case—tells what length it is based upon.

Laminar-Turbulent transition

$$Re_{x,c} \equiv \frac{\rho u_\infty x_c}{\mu} = 5 \times 10^5$$

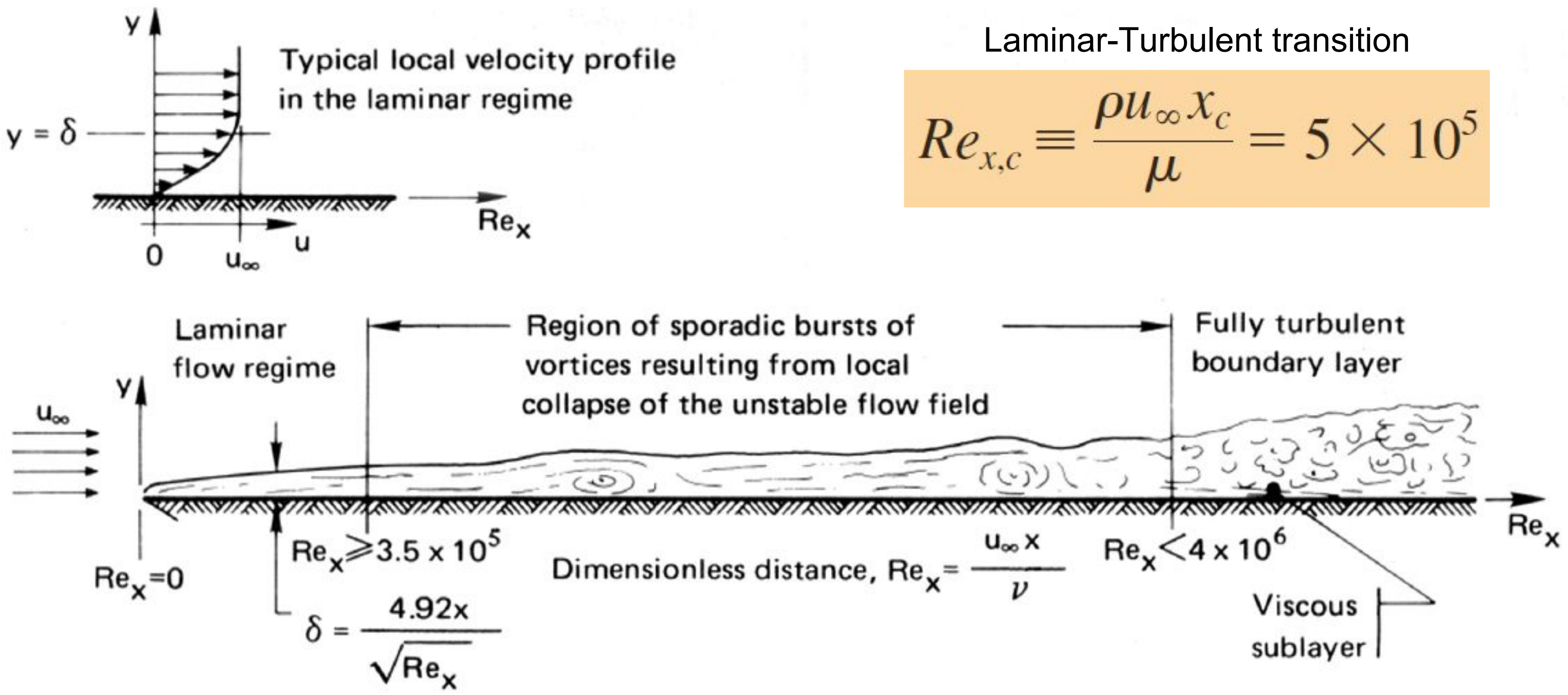
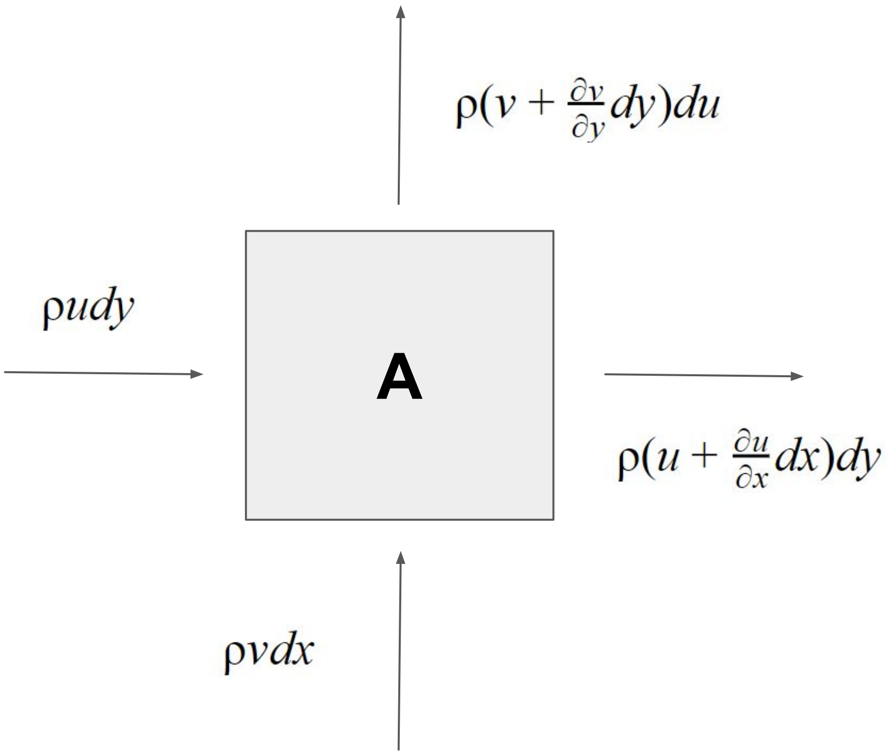


Figure 6.4 Boundary layer on a long, flat surface with a sharp leading edge.

6.2 Laminar incompressible boundary layer on a flat surface



Incompressible
$\rho \approx$ const.

Conservation of mass—The continuity equation

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \tag{6.11}$$

6.2 Laminar incompressible boundary layer on a flat surface

Conservation of momentum

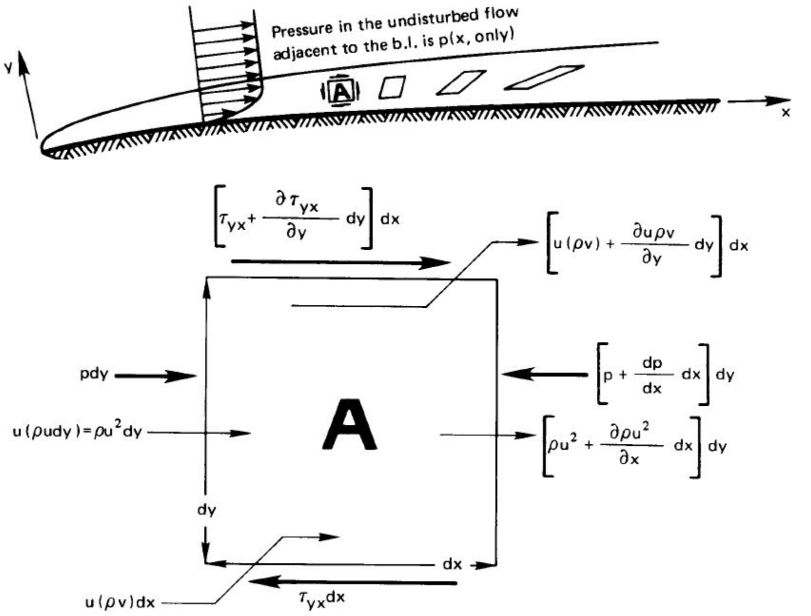


Figure 6.9 Forces acting in a two-dimensional incompressible boundary layer.

Equation (6.13) has a number of so-called *boundary layer approximations* built into it:

- $|\partial u/\partial x|$ is generally $\ll |\partial u/\partial y|$.
- v is generally $\ll u$.
- $p \neq \text{fn}(y)$

The shear stress in this result can be eliminated with the help of Newton's law of viscous shear:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}} \qquad (6.13)$$

6.2 Laminar incompressible boundary layer on a flat surface

Conservation of momentum

Equation (6.13) has a number of so-called *boundary layer approximations* built into it:

- $|\partial u/\partial x|$ is generally $\ll |\partial u/\partial y|$.
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$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}} \qquad (6.13)$$

We obtain the pressure gradient by writing the Bernoulli equation for the free stream flow just above the boundary layer where there is no viscous shear. Thus,

$$\frac{p}{\rho} + \frac{u_\infty^2}{2} = \text{constant}$$

Differentiate this and use it to eliminate the pressure gradient,

$$\frac{1}{\rho}\frac{dp}{dx} = -u_\infty \frac{du_\infty}{dx}$$

so from eqn. (6.12):

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} = u_\infty \frac{du_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (6.14)$$

6.2 Laminar incompressible boundary layer on a flat surface

Conservation of momentum

Predicting the velocity profile in the laminar boundary layer without a pressure gradient

And if there is no pressure gradient in the flow—if p and u_∞ are constant as they would be for flow past a flat plate—then eqns. (6.12), (6.13), and (6.14) become

$$\boxed{\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}} \tag{6.15}$$

6.2 Laminar incompressible boundary layer on a flat surface

Conservation of momentum

Predicting the velocity profile in the laminar boundary layer without a pressure gradient

Boundary Conditions

$$u(y=0)=0$$

$$u(y=\infty)=u_\infty$$

$$v(y=0)=0$$

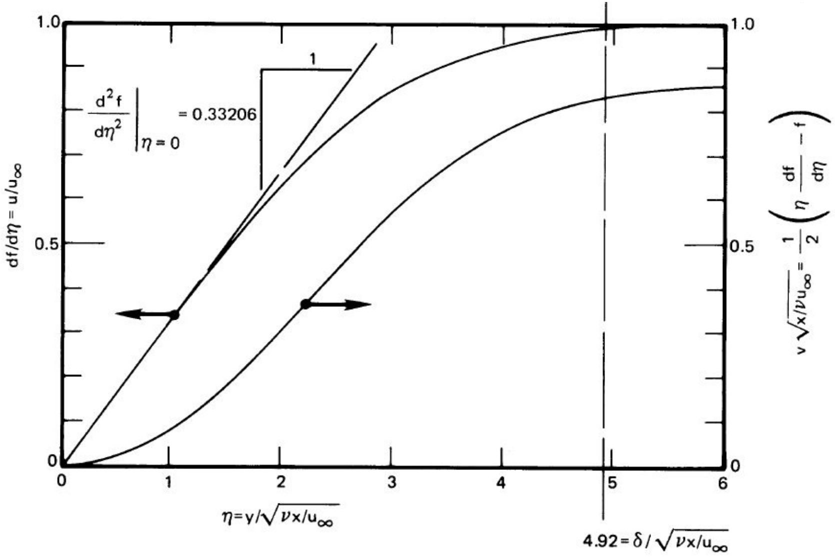


Figure 6.10 The dimensionless velocity components in a laminar boundary layer.

Thus, the b.l. thickness is given by

$$4.92 = \frac{\delta}{\sqrt{\nu x/u_\infty}}$$

or, as we anticipated earlier [eqn. (6.2)],

$$\frac{\delta}{x} = \frac{4.92}{\sqrt{u_\infty x/\nu}} = \frac{4.92}{\sqrt{\mathrm{Re}_x}}$$

$$\boxed{\frac{\delta}{x} = \frac{4.92}{\sqrt{\mathrm{Re}_x}}}$$

6.2 Laminar incompressible boundary layer on a flat surface

The skin friction coefficient

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\tau_w = 0.332 \frac{\mu u_\infty}{x} \sqrt{\mathrm{Re}_x} \qquad (6.32)$$

The *local skin friction coefficient,* or local skin drag coefficient, is defined as

$$\boxed{C_f \equiv \frac{\tau_w}{\rho u_\infty^2/2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}} \qquad (6.33)$$

The *overall skin friction coefficient,* $\overline{C}_f$, is based on the average of the shear stress, τ_w, over the length, L, of the plate

$$\overline{\tau}_w = \frac{1}{L}\int_0^L \tau_w \, dx = \frac{\rho u_\infty^2}{2L}\int_0^L \frac{0.664}{\sqrt{u_\infty x/\nu}} \, dx = 1.328 \frac{\rho u_\infty^2}{2}\sqrt{\frac{\nu}{u_\infty L}}$$

so

$$\boxed{\overline{C}_f = \frac{1.328}{\sqrt{\mathrm{Re}_L}}} \qquad (6.34)$$

Thermal boundary layer

When a wall is at a temperature T_w, different from that of the free stream, T_∞, a *thermal boundary layer* is present, and it has a thickness, δ_t, different from the *flow* b.l. thickness, δ. A thermal b.l. is pictured in Fig. 6.5. Now, with reference to this picture, we equate the heat conducted away from the wall by the fluid to the same heat transfer expressed in terms of a convective heat transfer coefficient:

$$\underbrace{-k_f \left.\frac{\partial T}{\partial y}\right|_{y=0}}_{\substack{\text{conduction}\\ \text{into the fluid}}} = h(T_w - T_\infty) \tag{6.5}$$

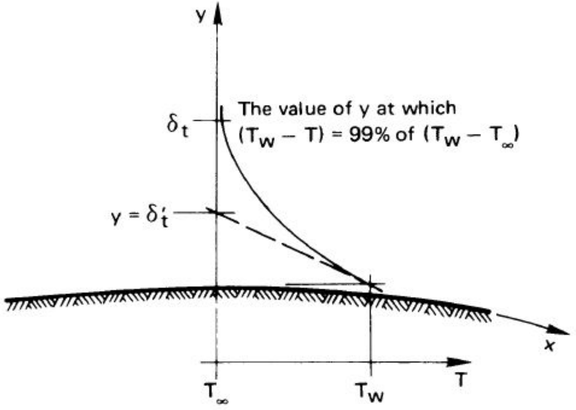


Figure 6.5 The thermal boundary layer during the flow of cool fluid over a warm plate.

where k_f is the conductivity of the fluid. Notice two things about this result. In the first place, it is correct to express heat removal *at the wall* using Fourier's law of conduction, because no fluid moves in the direction of q. Also, while eqn. (6.5) looks like a b.c. of the third kind, it is not. This condition *defines h within the fluid* instead of specifying it as known information on the boundary. Equation (6.5) can be arranged in the form

$$\left.\frac{\partial\left(\dfrac{T_w - T}{T_w - T_\infty}\right)}{\partial(y/L)}\right|_{y/L=0} = \frac{hL}{k_f} = \mathrm{Nu}_L, \text{ the Nusselt number} \tag{6.5a}$$

6.3 The energy equation

$$\underbrace{\frac{d}{dt}\int_R \rho\hat{u}\,dR}_{\substack{\text{rate of internal}\\\text{energy increase}\\\text{in } R}} = -\underbrace{\int_S (\rho\hat{h})\,\vec{u}\cdot\vec{n}\,dS}_{\substack{\text{rate of internal energy and}\\\text{flow work out of } R}} - \underbrace{\int_S (-k\nabla T)\cdot\vec{n}\,dS}_{\substack{\text{net heat conduction}\\\text{rate out of } R}} + \underbrace{\int_R \dot{q}\,dR}_{\substack{\text{rate of heat}\\\text{generation in } R}} \qquad (6.36)$$

$$\boxed{\rho c_p\left(\underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{energy}\\\text{storage}}} + \underbrace{\vec{u}\cdot\nabla T}_{\substack{\text{enthalpy}\\\text{convection}}}\right) = \underbrace{k\nabla^2 T}_{\substack{\text{heat}\\\text{conduction}}} + \underbrace{\dot{q}}_{\substack{\text{heat}\\\text{generation}}}} \qquad (6.37)$$

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}} \qquad (6.40)$$

6.3 The energy equation

Heat and momentum transfer analogy

$$u\frac{\partial}{\partial x}\left(\frac{u}{u_\infty}\right)+v\frac{\partial}{\partial y}\left(\frac{u}{u_\infty}\right)=\nu\frac{\partial^2}{\partial y^2}\left(\frac{u}{u_\infty}\right)\qquad\begin{cases}\left.\dfrac{u}{u_\infty}\right|_{y=0}=0\\ \left.\dfrac{u}{u_\infty}\right|_{y=\infty}=1\\ \dfrac{\partial}{\partial y}\left(\dfrac{u}{u_\infty}\right)_{y=\infty}=0\end{cases}\tag{6.41}$$

And the energy equation (6.40) can be expressed in terms of a dimensionless temperature, $\Theta = (T - T_w)/(T_\infty - T_w)$, as

$$u\frac{\partial\Theta}{\partial x}+v\frac{\partial\Theta}{\partial y}=\alpha\frac{\partial^2\Theta}{\partial y^2}\qquad\begin{cases}\Theta(y=0)=0\\ \Theta(y=\infty)=1\\ \left.\dfrac{\partial\Theta}{\partial y}\right|_{y=\infty}=0\end{cases}\tag{6.42}$$

$$\boxed{\frac{\delta_t}{\delta}=\mathrm{Pr}^{-1/3}}\qquad 0.6\leqslant \mathrm{Pr}\leqslant 50\tag{6.55}$$

$$\frac{\nu}{\alpha}\equiv \mathrm{Pr},\ \text{Prandtl number}$$

- For simple monatomic gases, $\mathrm{Pr} = \frac{2}{3}$.
- For diatomic gases in which vibration is unexcited (such as N_2 and O_2 at room temperature), $\mathrm{Pr} = \frac{5}{7}$.
- For liquids composed of fairly simple molecules, excluding metals, Pr is of the order of magnitude of 1 to 10.
- For liquid metals, Pr is of the order of magnitude of 10^{-2} or less.
- If the molecular structure of a liquid is very complex, Pr might reach values on the order of 10^5. This is true of oils made of long-chain hydrocarbons, for example.

Nusselt number for a flat surface at uniform T wall

$$\boxed{\mathrm{Nu}_x = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}} \qquad 0.6 \leqslant \mathrm{Pr} \leqslant 50 \tag{6.58}$$

This expression gives very accurate results under the assumptions on which it is based, namely a laminar two-dimensional b.l. on a flat surface, with T_w = constant and $0.6 \leqslant \mathrm{Pr} \leqslant 50$.

Normally, in using eqn. (6.43) or any other forced convection equation, properties should be evaluated at the *film temperature*, $T_f = (T_w + T_\infty)/2$.

$$\overline{h} = \frac{1}{L}\int_0^L \underbrace{h(x)\,dx}_{\frac{k}{x}\,\mathrm{Nu}_x} \quad \Rightarrow \quad \boxed{\overline{\mathrm{Nu}}_L = \frac{\overline{h}L}{k} = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}} \tag{6.68}$$

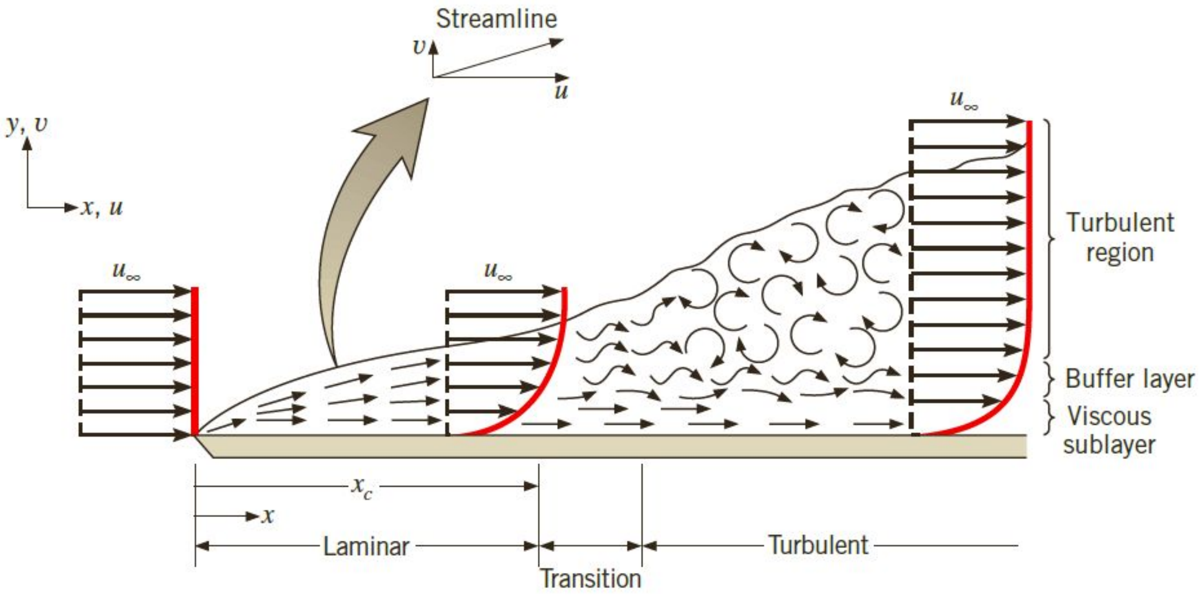
The problem of uniform wall heat flux

When the heat flux at the heater wall, q_w, is specified instead of the temperature, it is T_w that we need to know. We leave the problem of finding Nu_x for q_w = constant as an exercise (Problem 6.11). The exact result is

$$\boxed{\mathrm{Nu}_x = 0.453\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}} \qquad \text{for} \quad \mathrm{Pr} \geqslant 0.6 \tag{6.71}$$

$$\overline{\mathrm{Nu}}_L = \overline{h}L/k = 0.6795\,\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3}$$

Turbulent Flow



Flow in the fully turbulent boundary layer is, in general, highly irregular and is characterized by random, three-dimensional motion of relatively large parcels of fluid. Mixing within the boundary layer carries high-speed fluid toward the solid surface and transfers slower-moving fluid farther into the free stream. Much of the mixing is promoted by streamwise vortices called streaks that are generated intermittently near the flat plate, where they rapidly grow and decay.

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$T = \bar{T} + T'$$
$$\rho = \bar{\rho} + \rho'$$
etc.

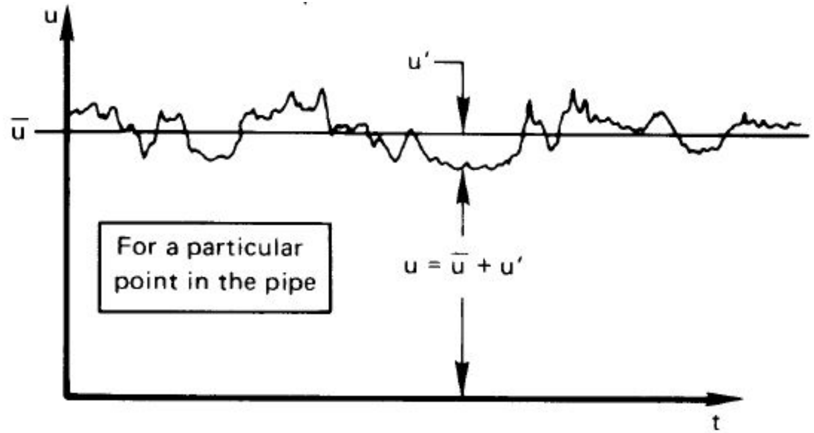


Figure 6.17 Fluctuation of u and other quantities in a turbulent pipe flow.

$$\overline{u} \equiv \frac{1}{\mathbf{T}} \int_0^{\mathrm{T}} u \, dt \qquad (6.81)$$

$$\overline{u} = \underbrace{\frac{1}{\mathbf{T}} \int_0^{\mathrm{T}} \overline{u} \, dt}_{=\overline{u}} + \underbrace{\frac{1}{\mathbf{T}} \int_0^{\mathrm{T}} u' \, dt}_{=\overline{u'}} \qquad (6.82)$$

$$\overline{u'} \text{ or any other average fluctuation} = 0 \qquad (6.83)$$

Local skin friction coefficient

$$C_f \cong \frac{0.0592}{\mathrm{Re}_x^{1/5}}, \qquad 5 \times 10^5 \leqslant \mathrm{Re}_x \leqslant 10^7 \qquad (6.112)$$

Average skin friction coefficient

$$\overline{C}_{f,L} = 0.074\, Re_L^{-1/5} - \frac{2A}{Re_L}$$

$$[Re_{x,c} \lesssim Re_L \lesssim 10^8]$$

$$A = 0.037\, Re_{x,c}^{4/5} - 0.664\, Re_{x,c}^{1/2}$$

Local Nusselt Number

$$\mathrm{Nu}_x = 0.0296\, \mathrm{Re}_x^{0.8}\, \mathrm{Pr}^{0.43} \qquad (6.114)$$

Average Nusselt Number

$$\boxed{\overline{\mathrm{Nu}}_L = 0.037\, \mathrm{Pr}^{0.43} \left\{ \mathrm{Re}_L^{0.8} - \left[\mathrm{Re}_{\mathrm{trans}}^{0.8} - 17.95\, \mathrm{Pr}^{-0.097} \left(\mathrm{Re}_{\mathrm{trans}} \right)^{1/2} \right] \right\}} \qquad (6.118)$$

Forced Convection for an external flat surface	*Skin Friction Coefficient*		*Nusselt Number*	
Laminar	$$\boxed{C_f \equiv \frac{\tau_w}{\rho u_\infty^2/2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}}$$ $$\boxed{\overline{C}_f = \frac{1.328}{\sqrt{\mathrm{Re}_L}}}$$	T_{wall} = const	$$\boxed{\mathrm{Nu}_x = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}}$$ $$\boxed{\overline{\mathrm{Nu}}_L = \frac{\overline{h}L}{k} = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}}$$	$0.6 \leqslant \mathrm{Pr} \leqslant 50$
		q_{wall} = const	$$\boxed{\mathrm{Nu}_x = 0.453\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}}$$ $$\overline{\mathrm{Nu}}_L = \overline{h}L/k = 0.6795\,\mathrm{Re}_L^{1/2}\mathrm{Pr}^{1/3}$$	for $\mathrm{Pr} \geqslant 0.6$
Turbulent	$$C_f \cong \frac{0.0592}{\mathrm{Re}_x^{1/5}}, \qquad 5\times10^5 \leqslant \mathrm{Re}_x \leqslant 10^7$$ $$\overline{C}_{f,L} = 0.074\,Re_L^{-1/5} - \frac{2A}{Re_L}$$ $$[Re_{x,c} \lesssim Re_L \lesssim 10^8]$$ $$A = 0.037\,Re_{x,c}^{4/5} - 0.664\,Re_{x,c}^{1/2}$$		$$\mathrm{Nu}_x = 0.0296\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{0.43}$$ $$\boxed{\overline{\mathrm{Nu}}_L = 0.037\,\mathrm{Pr}^{0.43}\left\{\mathrm{Re}_L^{0.8} - \left[\mathrm{Re}_{\mathrm{trans}}^{0.8} - 17.95\,\mathrm{Pr}^{-0.097}\left(\mathrm{Re}_{\mathrm{trans}}\right)^{1/2}\right]\right\}}$$ $$0.6 \leqslant \mathrm{Pr} \leqslant 50,$$	