2. Heat conduction concepts

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24 février 2008

2.1 Fourier's Law

Consider the general temperature distribution in a three-dimensional body. For some reason, there is a space- and time-dependent temperature field in the body.

$$T = T\left(x, y, z, t\right)$$

The vector that has both the magnitude and direction of the maximum increase of temperature at each point is called the temperature gradient, ∇T :

$$\nabla T \equiv \overrightarrow{i}\ \frac{\partial T}{\partial x} + \overrightarrow{j}\ \frac{\partial T}{\partial y} + \overrightarrow{k}\ \frac{\partial T}{\partial z} \qquad (2.1)$$

Fourier's law

$$\overrightarrow{q} = -k \nabla T \qquad (2.2)$$

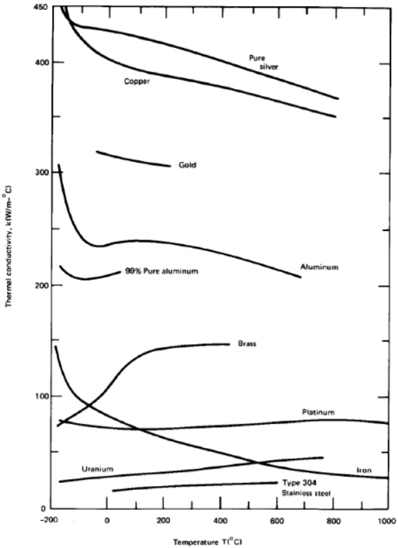
$$q_x = -k \frac{\partial T}{\partial x}$$

Notes

- The medium in wich conduction occurs is isotropic, such k is independent of the direction of the coordinates
- The ratio of thermal conductivity to the heat capacity of a material gives a measure of the ability of the material to conduct thermal energy relative to its ability to store thermal energy. Defined as α in $\frac{m^2}{s}$. A large value of α mean rapid diffusion of heat.

$$\alpha = \frac{k}{\rho\ c}$$

2.1 Fourier's Law

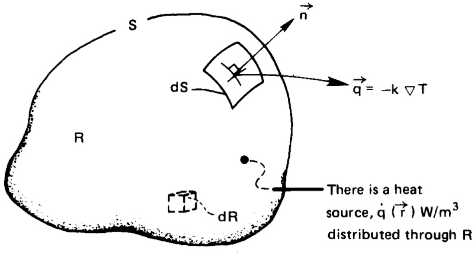


2.1 Heat diffusion equation

Typically, our objectives are to know

- The temperature distribution
- The local heat flux at boundaries or locations of interest

Control volume in a heat-flow field :



2.1 Heat diffusion equation

We apply *the conservation of energy* to a three-dimensional control volume to obtain this

$$\dot{Q}_{in} + \dot{Q}_{generated} - \dot{Q}_{out} = \dot{Q}_{stored}$$

We can write the heat conducted out of dS, in watt, as

$$(-k\nabla T) \cdot \left(\overrightarrow{n}\, dS\right) \qquad (2.4)$$

The heat generated (or consumed) within the region R must be added to the total heat flow into S to get the overall rate of heat addition to R

$$Q = -\int_S (-k\nabla T) \cdot \left(\overrightarrow{n}\, dS\right) + \int_R \dot{q}\, dR \qquad (2.5)$$

The rate of energy increase of the region R is

$$Q = \int_R \left(\rho c \frac{\partial T}{\partial t}\right) dR \qquad (2.6)$$

2.1 Heat diffusion equation

Where the derivate of T is in partial form because T is a function of both $\overrightarrow{r}$ and t. Finally, after rearranging the terms, we obtain

$$\int_S k\nabla T \cdot \overrightarrow{n}\, dS = \int_R \left[\rho c \frac{\partial T}{\partial t} - \dot{q}\right] dR \qquad (2.7)$$

To get the left-hand side into a convenient form, we introduce Gauss's theorem, wich converts a surface integral into a volume integral. We therefore get the heat diffusion equation in three dimensions :

$$\nabla \cdot k\nabla T + \dot{q} = \rho c \frac{\partial T}{\partial t} \qquad (2.10)$$

Heat diffusion equation : At any point in the material the rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must equal the rate of change of thermal energy stored within the volume.

2.1 Heat diffusion equation

The limitation on this equation are :

- Incompressible medium.
- No convection.

Complements :

- If the variation of k with T is small, k can be factored out to get

$$\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.11)$$

- For steady-state conditions

$$\nabla \cdot k \nabla T + \dot{q} = 0$$

- For one-dimension conduction without energy generation

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

The term $\nabla^2 T$ is called the Laplacian. It arises thus in a Cartesian coordinated system

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.12}$$

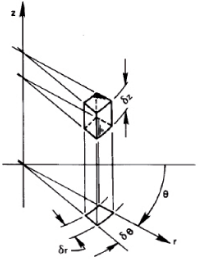
The Laplacian can be expressed in cylindrical coordinates

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.13}$$

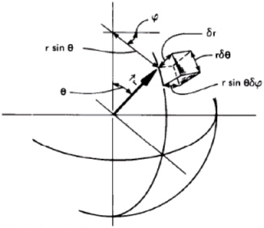
The Laplacian can also be expressed in spherical coordinates

$$\nabla^2 T = \frac{1}{r}\frac{\partial^2 (rT)}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{2.14}$$

2.1 Heat diffusion equation



Polar coordinates



Spherical coordinates

2.2 Solutions of the heat diffusion equation

Step 1 : Pick the coordinate scheme that best fits the problem and identify the independent variables that determine T.

Step 2 : Write the appropriate d.e., starting with one of the forms of the heat equation.

Step 3 : Obtain the general solution of the d.e.

Step 4 : Write the "side conditions" on the d.e. (initial and boundary conditions).

2.2 Solutions of the heat diffusion equation

Step 5 : Substitute the general solution in the boundary and initial conditions and solve for the constants.

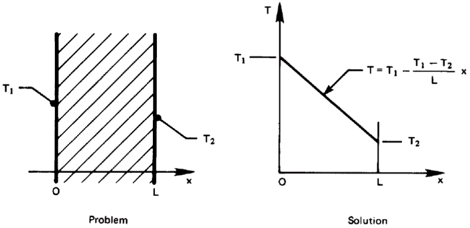
Step 6 : Put the calculated constants back in the general solution to get the particular solution to the problem.

Step 7 : Play with the solution, look it over, see what it has to tell you. Make any checks you can think of to be sure it is correct.

Step 8 : If the temperature field is now correctly established, you can calculate the heat flux at any point in the body by substituting T back into Fourier's law.

2.2 The simple slab

Heat conduction in a slab :



For steady-state, one dimensional heat conduction without internal heat generation, the heat equation reduces to :

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

2.2 The simple slab

Thus, for one-dimensional, steady-state conduction, the heat flux is a constant and independant of x. If the thermal conductivity k is assumed to be constant - often a good assumption -, then integrating twice gives the general solution

$$T(x) = C_1 x + C_2$$

The applicable boundary conditions are

$$T(0) = T_1$$

$$T(L) = T_2$$

So, the solution is

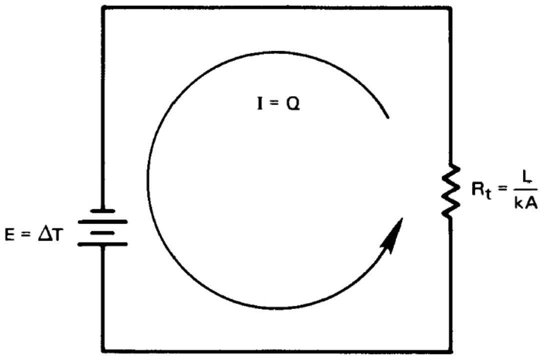
$$T(x) = \frac{T_2 - T_1}{L} x + T_1$$

The heat conduction rate is

$$q = k\frac{\Delta T}{L}$$

2.3 Thermal resistance and the electrical analogy

We can also represent heat flow through the slab with a diagram that is perfectly analogous to an electric circuit.



2.3 Thermal resistance and the electrical analogy

We can write the last equation as a thermal resistance expression for conduction

$$Q = \frac{\Delta T}{L/kA}$$

where L/kA assumes the role of a thermal resistance, to which give the symbol $R_{t,cond}$. The equivalent electrical expression (Ohm's law) is

$$R_e = \frac{L}{\sigma A}$$

Expression for convection is

$$R_{t,conv} = \frac{1}{hA}$$

Expression for radiation is

$$R_{t,rad} = \frac{1}{h_{rad} A}$$

2.3 Thermal resistance and electrical analogy

The total resistance for series is

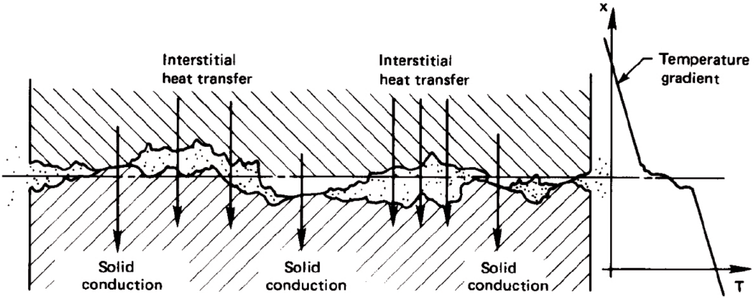
$$R_{tot} = \sum_{i=1}^{n} R_{t_i}$$

Thus heat transfer analysis can be done using circuit representations as

$$Q = \frac{T_1 - T_n}{R_{tot}} = \frac{T_1 - T_2}{R_{t,12}} = \frac{T_2 - T_3}{R_{t,23}} = \ldots = \frac{T_{n-1} - T_n}{R_{t,(n-1)n}}$$

2.3 Contact resistance

No two solid surfaces will ever form perfect thermal contact when they are pressed together. Heat transfer through the contact plane between two solid surfaces :



2.3 Contact resistance

We treat the contact surface by placing an interfacial conductance, h_c, in series with the conducting materials on either side. Then

$$Q = Ah_c\Delta T$$

The interfacial conductance, h_c, depends on the following factors

- The surface finish and cleanliness of the contacting solids.
- The materials that are in contact.
- The pressure with which the surfaces are forced together.
- The substance in the interstitial spaces.
- The temperature at the contact plane.

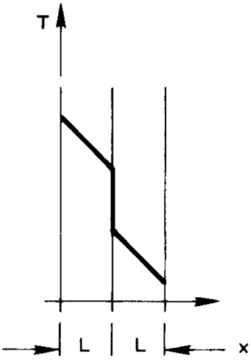
2.3 Contact resistance

Table with some typical interfacial conductances for normal surface finishes and moderate contact pressures (about 1 and 10 atm). Air gaps not evacuated unless so indicated.

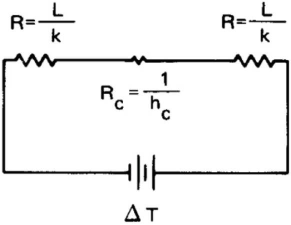
Situation	h_c (W/m^{2}K)
Iron/aluminum (70 atm pressure)	45,000
Copper/copper	10,000 – 25,000
Aluminum/aluminum	2,200 – 12,000
Graphite/metals	3,000 – 6,000
Ceramic/metals	1,500 – 8,500
Stainless steel/stainless steel	2,000 – 3,700
Ceramic/ceramic	500 – 3,000
Stainless steel/stainless steel (evacuated interstices)	200 – 1,100
Aluminum/aluminum (low pressure and evacuated interstices)	100 – 400

2.3 Contact resistance

Conduction through two unit-area slabs with a contact resistance :



Configuration



Thermal circuit

2.3 Radial systems

For steady-state, one-dimensional heat flow, equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left(k\ r\frac{dT}{dr}\right) = 0$$

Applying the same boundary conditions as earlier, we obtain

$$T\left(r\right) = T_i - \frac{\Delta T}{\ln\left(r_o/r_i\right)}\ln\left(\frac{r}{r_i}\right)$$

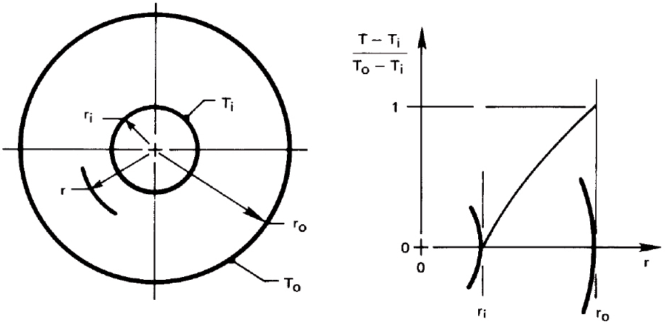
At any station, r :

$$q_{radial} = -k\frac{\partial T}{\partial r}$$

So the heat flux falls off inversely with radius. Let us see if this is the case for a cylinder of length l :

$$Q\left(W\right) = \left(2\pi rl\right)q = \frac{2\pi kl\Delta T}{\ln\left(r_o/r_i\right)}$$

Heat transfer through a cylinder with a fixed wall temperature :



Configuration　　　Temperature profile

Finally, we again recognize Ohm's law in this result and write the thermal resistance for a cylinder

$$R_{t,cyl} = \frac{\ln\left(r_o/r_i\right)}{2\pi lk} \qquad (2.22)$$

This can be compared with the resistance of a plane wall

$$R_{t,wall} = \frac{L}{kA}$$

Both resistances are inversely proportional to k, but each reflects a different geometry. Expression for convection is

$$R_{t,conv} = \frac{1}{2\pi rlh}$$

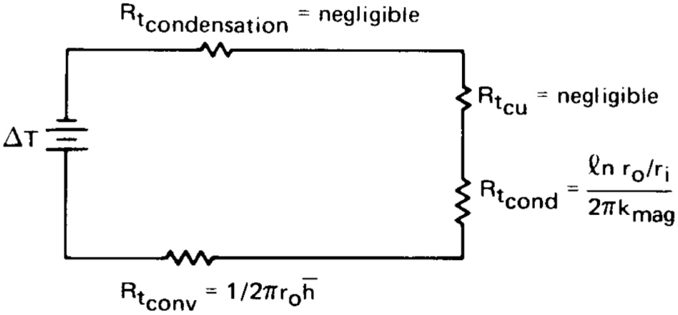
Expression for radiation is

$$R_{t,rad} = \frac{1}{2\pi rlh_{rad}}$$

Example 2.7 : An interesting consequence of the preceding result can be brought out with a specific example. Suppose that we insulate a 0.5 cm O.D. copper steam line with 85 percents magnesia to prevent the steam from condensing too rapidly. The steam is under pressure and stays at 150°C. The copper is thin and highly conductive - obviously a tiny resistance in series with the convective and insulation resistances, as we see in figure (next slide). The condensation of steam inside the tube also offers very little resistance. But on the outside, a heat transfer coefficient of $\overline{h}$=20 W/m^2K offers fairly high resistance. It turns out that insulation can actually improve heat transfer in this case.

2.3 Application

Thermal circuit for an insulated tube :



The two significant resistances, for a cylinder of unit lenght (l=1m) are

$$R_{t,cond} = \frac{\ln\left(r_o/r_i\right)}{2\pi lk} = \frac{\ln\left(r_o/r_i\right)}{2\pi\ 0.0074} K/W$$

$$R_{t,conv} = \frac{1}{2\pi r_o \overline{h}} = \frac{1}{2\pi\ 20\ r_o} K/W$$

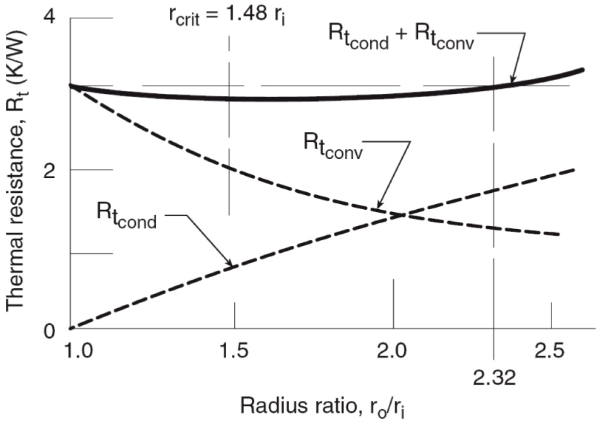
$R_{t,conv}$ falls off rapidly when r_o is increased, because the outside area is increasing. Accordingly, the total resistance passes through a minimum in this case. Such a value could be obtained from the requirement that

$$\frac{dR_{tot}}{dr} = 0$$

Hence

$$r = \frac{k}{\overline{h}}$$

The critical radius of insulation written for a cylinder of unit length (l=1m) :



2.3 Application

To determined whether the foregoing result maximizes or minimizes the total resistance, the second derivative must be evaluated. From the above result

$$r \equiv r_{crit}$$

In the present example, adding insulation will increase heat loss instead of reducing it, until $r_{crit} = k/\overline{h} = 0.0037$m or $r_{crit}/r_i = 1.48$. Indeed, insulation will not even start to do any good until $r_o/r_i = 2.32$. We call r_{crit} the **critical radius of insulation**.

2.4 Overall heat transfer coefficient, U

We often want to transfer heat through composite resistances. It is very convenient to have a number, U, that works like this :

$$Q = UA\Delta T \qquad (2.32)$$

In general, the relationship is

$$R_{tot} = \sum_{i=1}^{n} R_{t_i} = \frac{1}{UA}$$

Overall heat transfer coefficient : This number, U, is defined largely by the system, and in many cases it proves to be insensitive to the operating conditions of the system. It is therefore important to remember which area an overall heat transfer coefficient is based on. It is particularly important that A and U be consistent.

2.4 Overall heat transfer coefficient, U

Typical values of U :

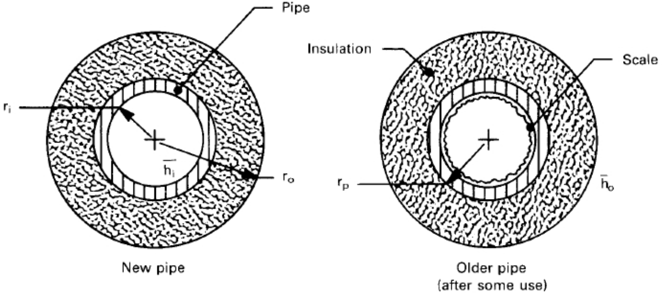
- The fluids with low thermal conductivities (tars, oils, ...) usually yield low values of h. When such fluid flows on one side of an exchanger, U will generally be pulled down.
- Condensing and boiling are very effective heat transfer processes. They greatly improve U but they cannot override one very small value of h on the other side of the exchange.

In fact :

- For a high U, **all** resistances in the exchanger must be low.
- The highly conducting liquids give high values of h and U.

2.4 Fouling resistance

The inside of one of the simplest heat exchanger is new and clean on the left, but on the right it has built up a layer of scale.



$$R_f = \frac{1}{U_{old}} - \frac{1}{U_{new}} \qquad (2.36)$$

2.4 Fouling resistance

$$\frac{1}{UA} = \frac{1}{U_0 A_0} = \frac{1}{U_i A_i}$$

For a old pipe based on A_i

$$U_{old} = \frac{1}{\frac{1}{h_i} + \frac{r_i\, ln(r_0/r_p)}{k_{insul}} + \frac{r_i\, ln(r_p/r_i)}{k_{pipe}} + \frac{r_i}{r_0\, h_0} + R_f}$$

where

U is the overall heat transfer coefficient ($Wm^{-2}K^{-1}$)
h is the heat transfer coefficient ($Wm^{-2}K^{-1}$)
R_f is the fouling factor ($W^{-1}m^2K$)
r is the rayon (m)
k is the thermal conductivity($Wm^{-1}K^{-1}$)