Numerical Techniques for Conduction

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Two-dimensional conduction

The general equation for $T\left(\overrightarrow{x}\right)$ with

- steady conduction.
- constant thermal conductivity.
- without internal heat generation.

is called *Laplace's equation* :

$$\nabla^2 T = 0$$

The Laplacian is a sum of several second partial derivatives. Faced with a steady multidimensional problem, four routes are open to us

- Find out whether or not the analytical solution is already available.
- Solve the problem analytically.
- Obtain the solution graphically.
- Solve the problem numerically.

Graphical method : flux plot

The method of flux plotting will solve all steady planar problems in which all boundaries are held at either of two temperatures or are insulated. We identify a series of channels, each which carries the same heat flow, ∂Q W/m. We also include a set of equally spaced isotherms, ∂T apart, between the walls. Since the heat fluxes in all channels are the same,

$$\left|\partial Q\right| = k\frac{\partial T}{\partial n}\partial s$$

Notice that $\partial s/\partial n$ must be the same for each rectangle. We therefore arbitrarily set the ratio equal to unity, so all the elements appear as **distorted squares**. The objective then is to sketch the isothermal lines and the adiabatic, or heat flow, lines which run perpendicular to them. This sketch is subject to two constraints :

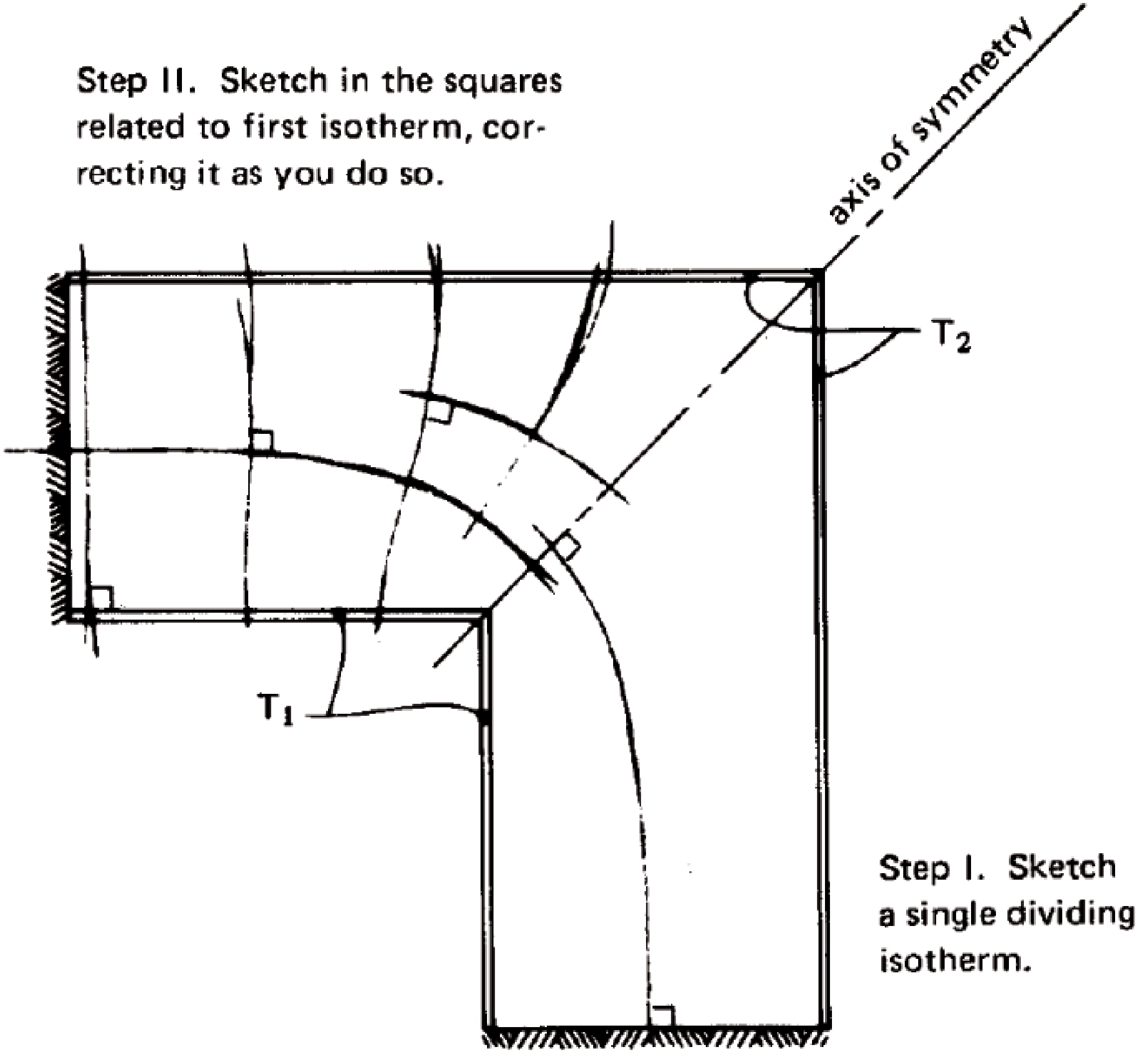
- Isothermal and adiabatic lines must intersect at right angles.
- They must subdivide the flow field into elements that are nearly square.

Graphical method : flux plot

Steps in constructing a flux plot

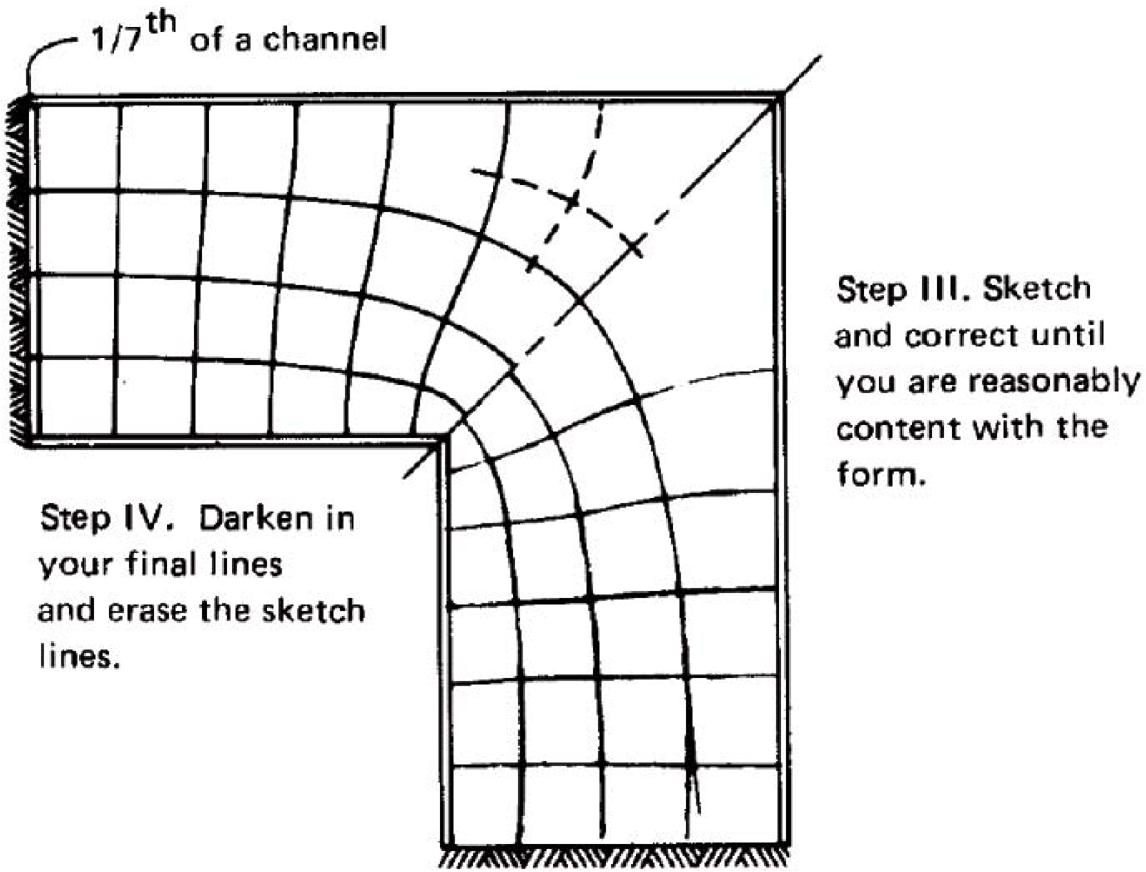
- Identify all lines of symmetry (from thermal and geometrical considerations)
- Note that lines of symmetry are adiabatic and no flux crosses them
- Identify all isotherms at boundaries and then try to sketch in the isotherm lines within the system (with all isotherms normal to the adiabatic lines)
- Heat flow lines are then drawn trying to create curvilinear squares (with heat flow lines and isotherms intersecting at right angles and trying to keep all sides of the square about the same length)
- If you fail, erase and go back to start !

A first rough sketching :



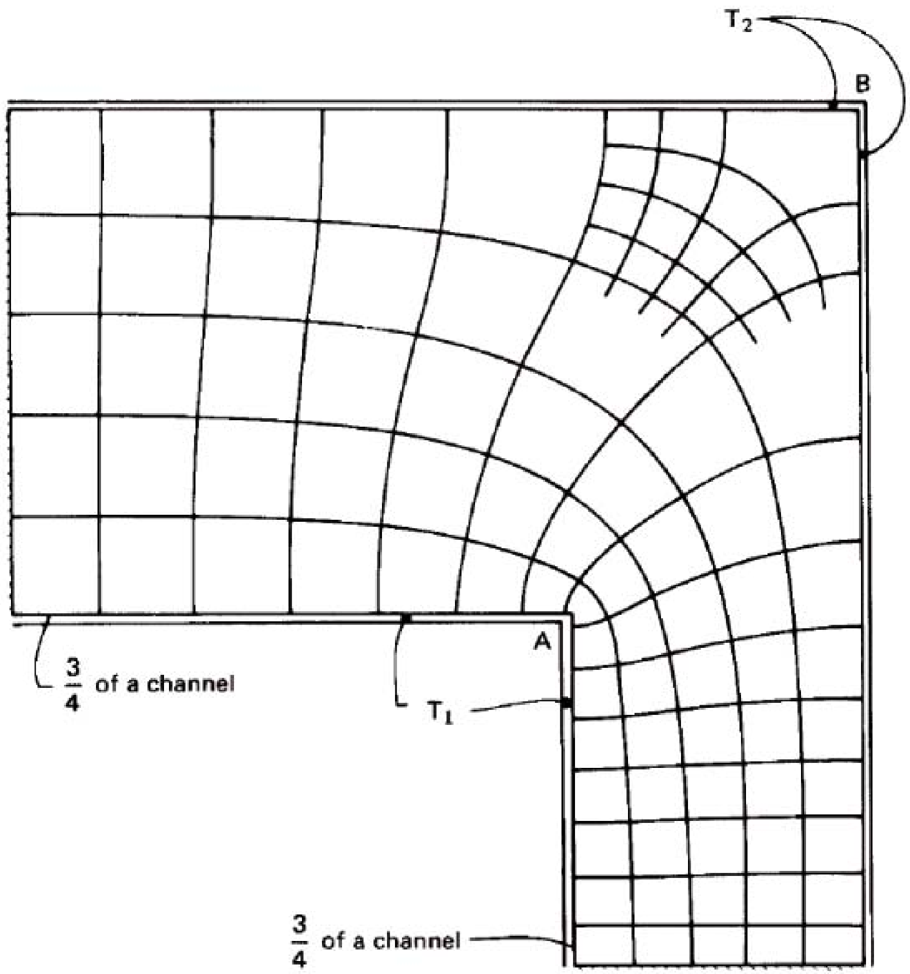
Graphical method : flux plot

The evolution of the flux plot :



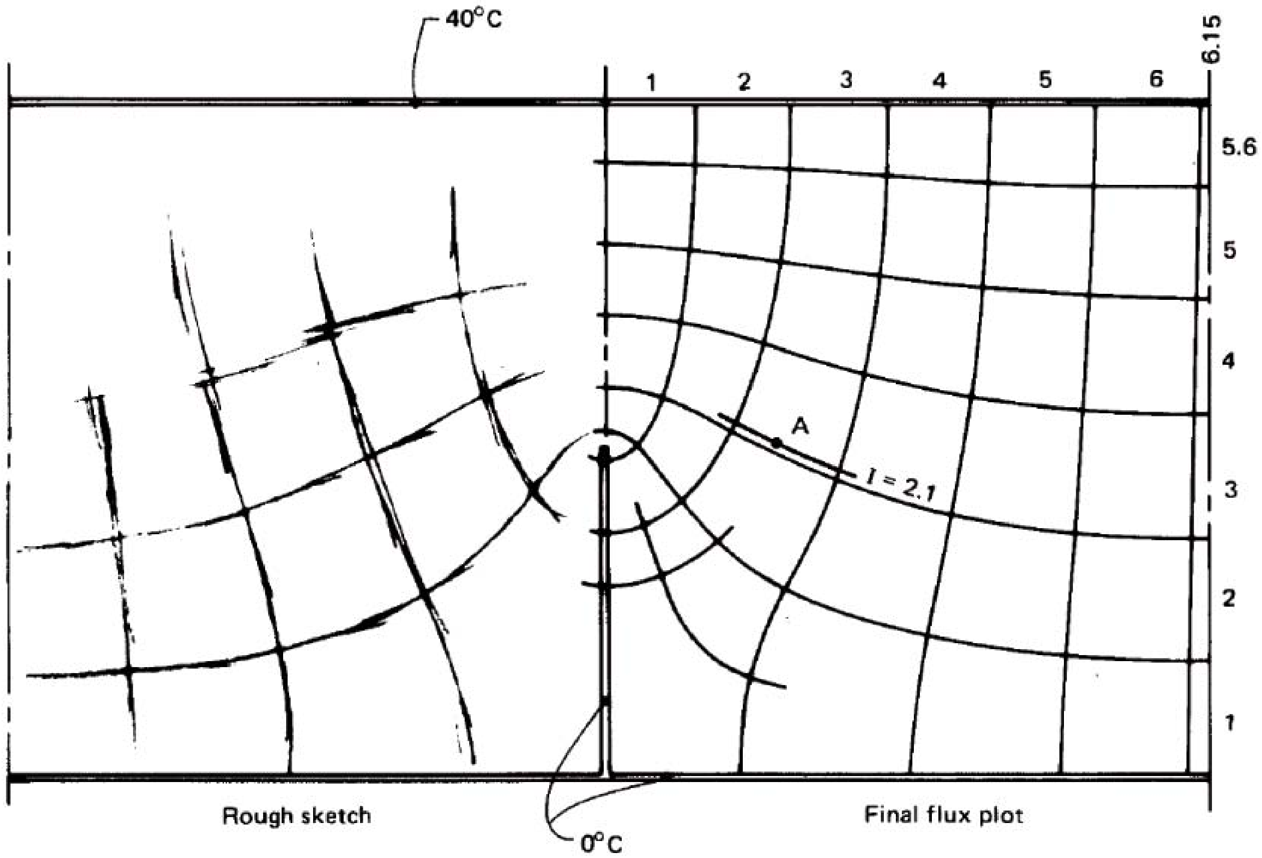
Graphical method : flux plot

A flux plot with no axis of symmetry to guide construction :



Graphical method : flux plot

Heat transfer through a wall with isothermal ribs :



Graphical method : flux plot

Once the grid has been sketched, the temperature anywhere in the field can be read directly from the sketch. And the heat flow per unit depth into the paper is

$$Q = Nk\partial T \frac{\partial s}{\partial n} = \frac{N}{I} k \Delta T$$

where N is the number of heat flow channels and I is the number of temperature increments, $\Delta T / \partial T$.

Graphical method : The shape factor

A heat conduction **shape factor S** may be defined for steady problems involving two isothermal surfaces as follows

$$Q \equiv Sk\Delta T$$

where for a flux plot

$$S = \frac{N}{l}$$

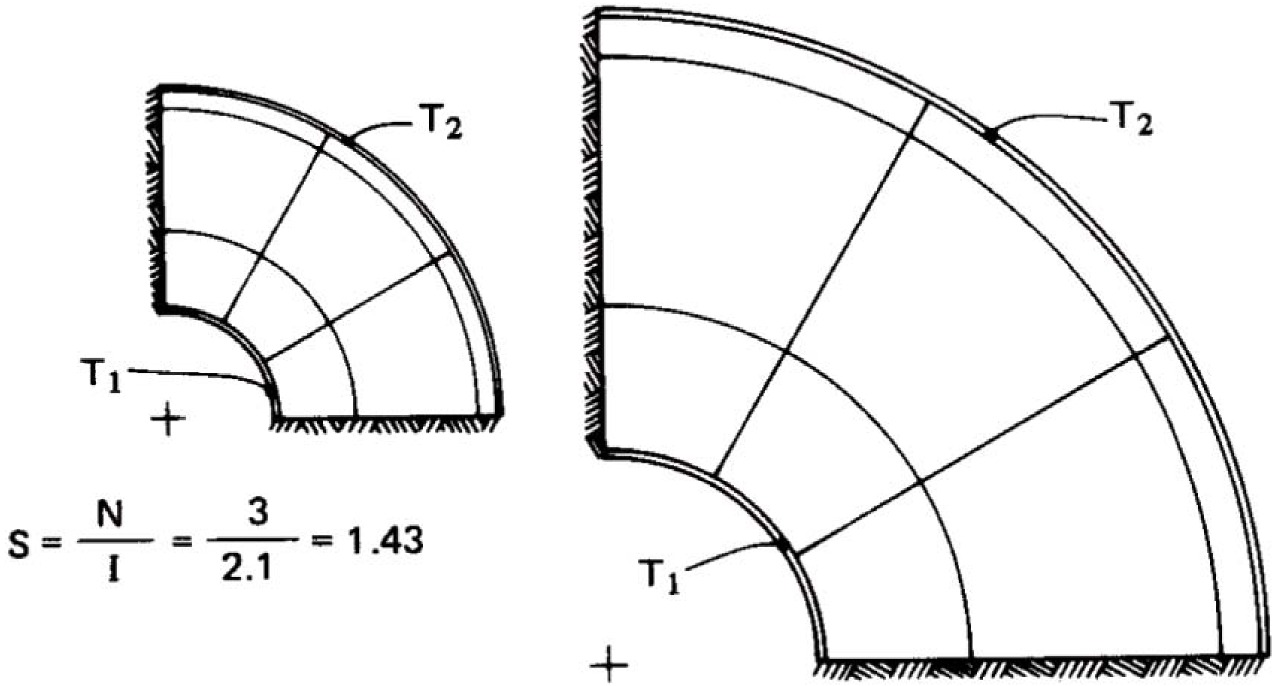
It follows that the thermal resistance of a two-dimensional body is

$$R_t = \frac{1}{kS} \quad \textit{where} \quad Q = \frac{\Delta T}{R_t}$$

The virtue of the shape factor is that it summarizes a heat conduction solution in a given configuration. Once S is known, it can be used again and again.

Graphical method : The shape factor

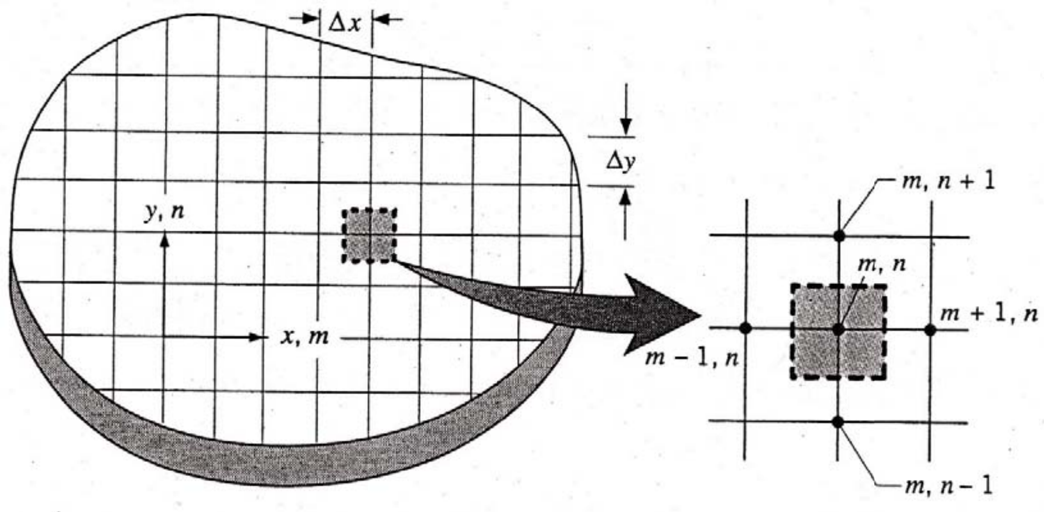
The shape factor for two similar bodies of different size :



Some tables in the course include a number of analytically derived shape factors for use in calculating the heat flux in different configurations.

Numerical method

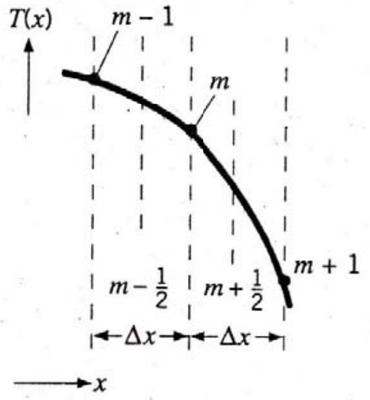
Nodal Network :



(a)

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,\, n} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,\, n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



(b)

Numerical method

Another approximate solution mehtod is the use of numerical techniques : finite-difference, finite-element, and boundary-element methods. We will present the finite-difference method.

Each node represents a small zone with an average temperature of that zone assigned as the node's temperature. The nodes and mesh are set to the user's convenience, the finer the mesh the more accurate the calculation (at increased computational time).

Finite-difference form of heat conduction equation

The value of the second derivative, $\partial^2 T/\partial x^2$ is

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

The temperature gradients in terms of nodal temperatures are

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Substituting equations

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Finite-difference form of heat conduction equation

Similarly, in the y-direction, the analogous expression is

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} = \frac{\left.\partial T/\partial y\right|_{m,n+1/2} - \left.\partial T/\partial y\right|_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

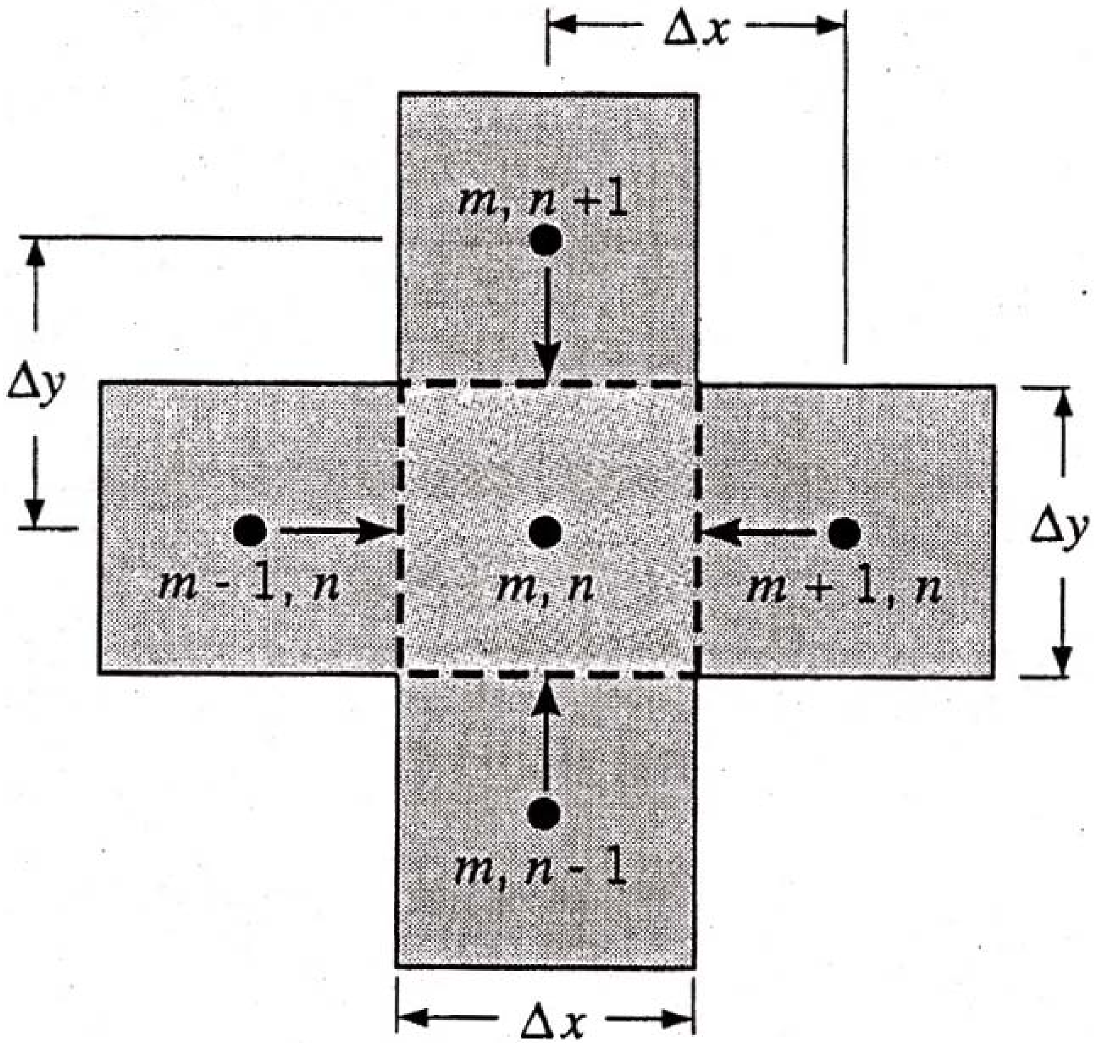
For a mesh in which $\Delta x = \Delta y$, then the heat conduction equation **??** is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\Rightarrow T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

This is an approximate algebraic equation for the heat conduction in this node.

Energy balance approach - **First approach**



Finite-difference form of heat conduction equation

The nodal equation froms an energy balance : here assuming all heat flows into the node, steady-state conditions and internal heat generation

$$Q_{(m-1,n)\longmapsto(m,n)} = k\left(\Delta y \cdot 1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$Q_{(m+1,n)\longmapsto(m,n)} = k\left(\Delta y \cdot 1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$Q_{(m,n+1)\longmapsto(m,n)} = k\left(\Delta x \cdot 1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$Q_{(m,n-1)\longmapsto(m,n)} = k\left(\Delta x \cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

With equilibrium equation

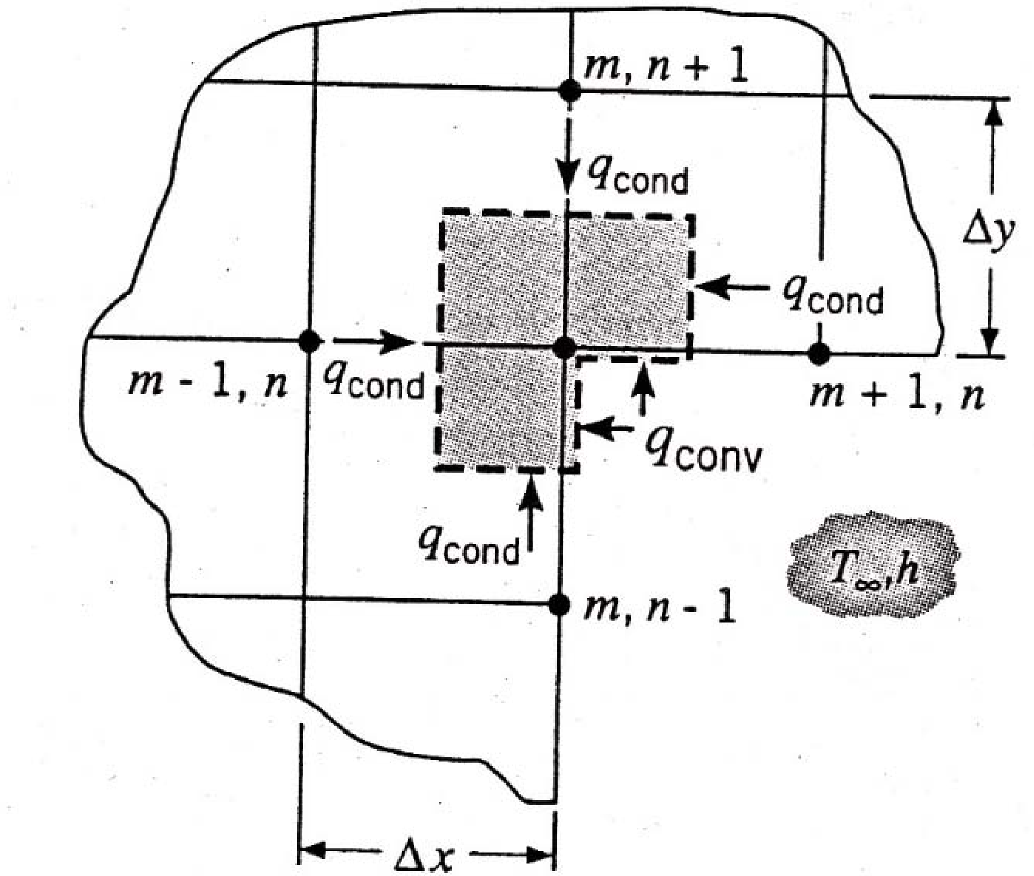
$$\dot{E}_{in} + \dot{E}_g = 0 \Rightarrow \sum_{i=1}^{4} Q_{(i)\longmapsto(m,n)} + \dot{Q}\left(\Delta x \Delta y 1\right) = 0$$

we obtain $(\Delta x = \Delta y)$

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{Q}\left(\Delta x\right)^2}{k} - 4T_{m,n} = 0$$

Finite-difference form of heat conduction equation

Energy balance approach - **Second approach**, with convection



Finite-difference form of heat conduction equation

$$Q_{(m-1,n)\longmapsto(m,n)} = k\left(\Delta y \cdot 1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$Q_{(m+1,n)\longmapsto(m,n)} = k\left(\frac{\Delta y}{2} \cdot 1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$Q_{(m,n+1)\longmapsto(m,n)} = k\left(\Delta x \cdot 1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$Q_{(m,n-1)\longmapsto(m,n)} = k\left(\frac{\Delta x}{2} \cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

And for convection

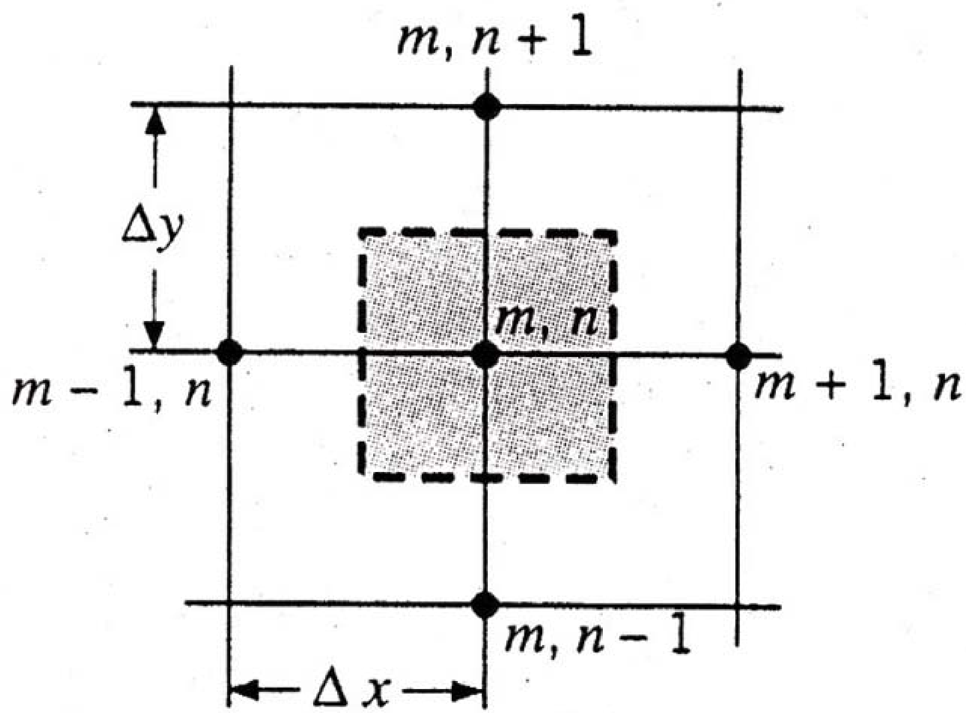
$$Q_{(\infty)\longmapsto(m,n)} = h\left(\frac{\Delta x}{2} \cdot 1\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta y}{2} \cdot 1\right)\left(T_\infty - T_{m,n}\right)$$

Then, for $\Delta x = \Delta y$, we obtain

$$T_{m-1,n} + T_{m,n+1} + \frac{1}{2}\left(T_{m+1,n} + T_{m,n-1}\right) + \frac{h\Delta x}{k}T_\infty - \left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

Summary of nodal finite-difference equations $\Delta x = \Delta y$

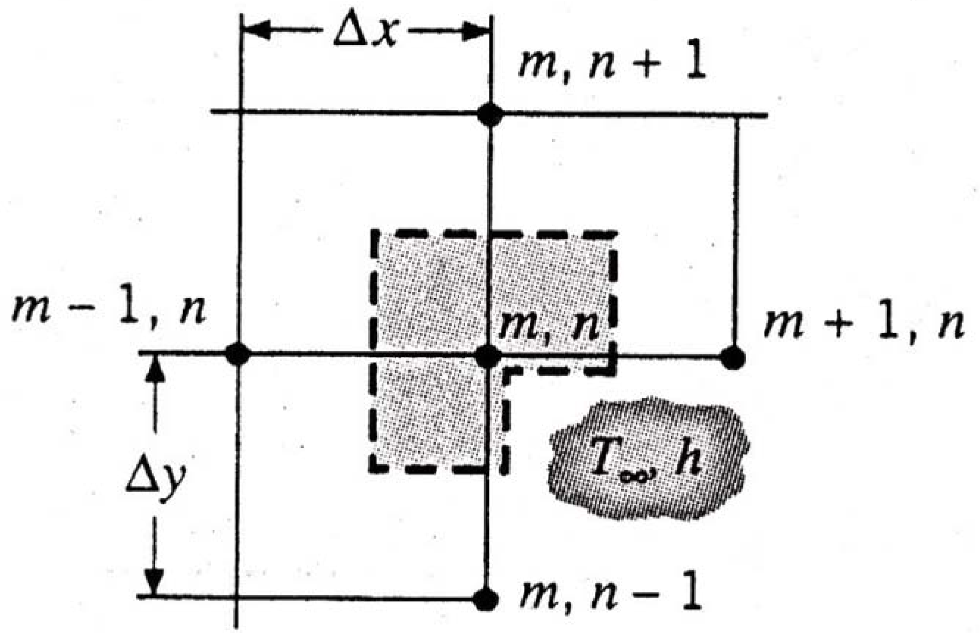
Case 1 : Interior node



$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

Summary of nodal finite-difference equations $\Delta x = \Delta y$

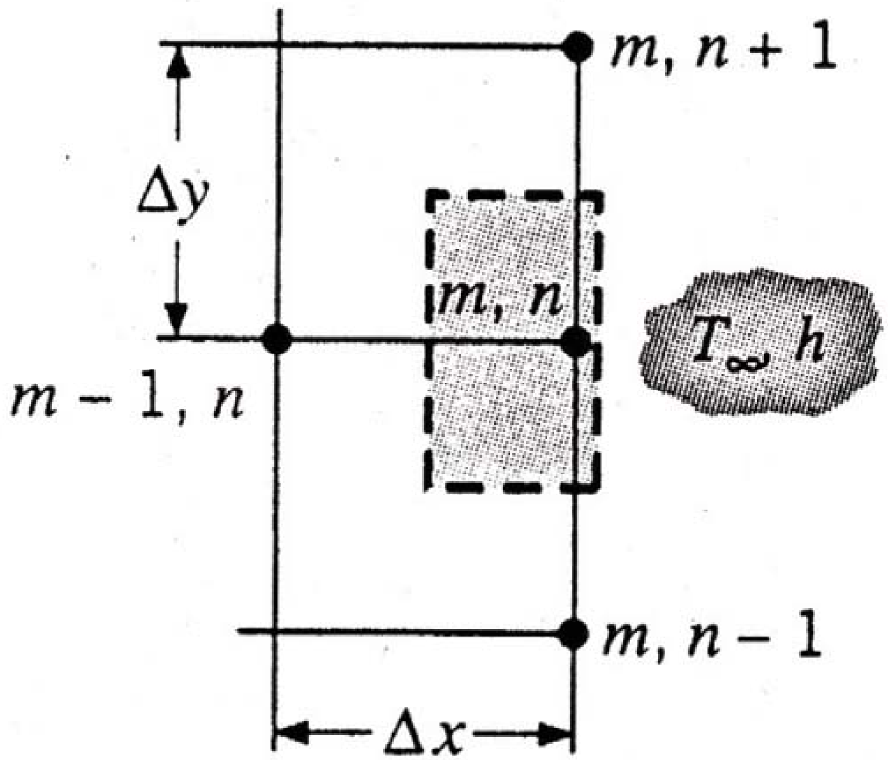
Case 2 : Node at an internal corner with convection



$$2\left(T_{m-1,n} + T_{m,n+1}\right) + \left(T_{m+1,n} + T_{m,n-1}\right) + 2\frac{h\Delta x}{k} T_\infty - 2\left(3 + \frac{h\Delta x}{k}\right) T_{m,n} = 0$$

Summary of nodal finite-difference equations $\Delta x = \Delta y$

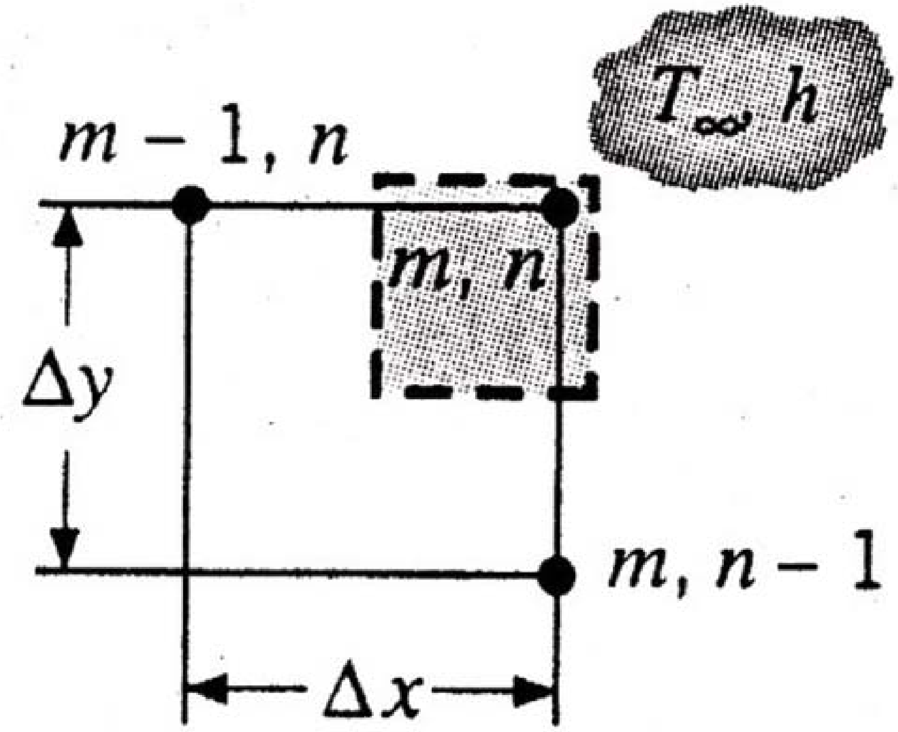
Case 3 : Node at a plane surface with convection



$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right) T_{m,n} = 0$$

Summary of nodal finite-difference equations $\Delta x = \Delta y$

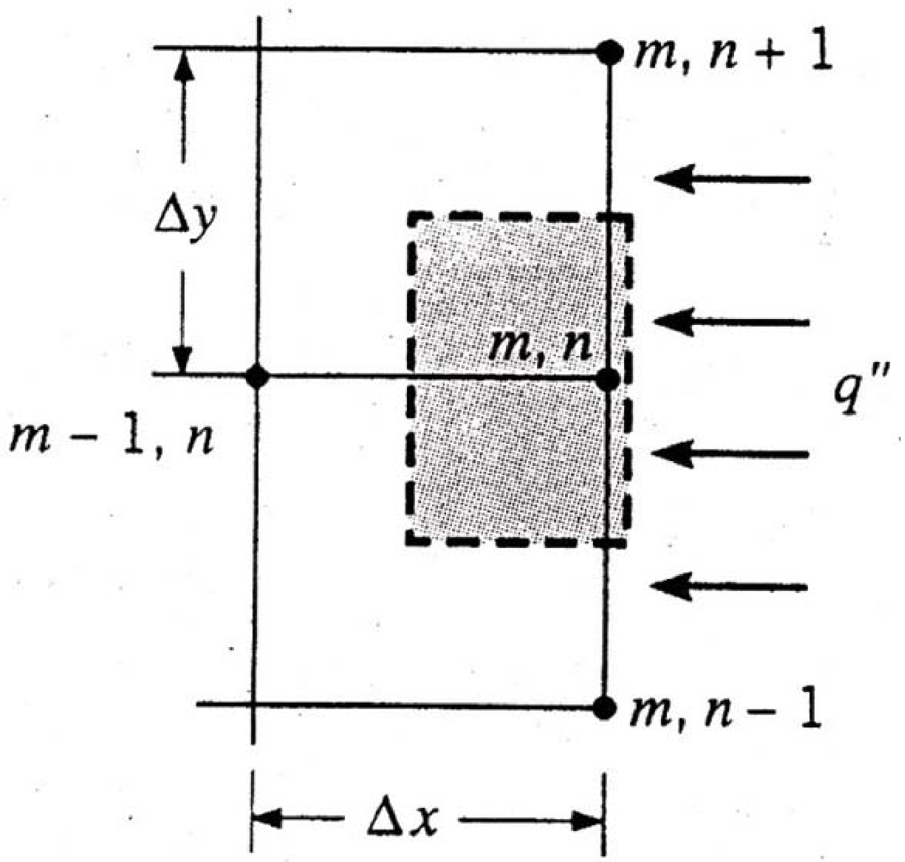
Case 4 : Node at an external corner with convection



$$\left(T_{m,n-1} + T_{m-1,n}\right) + 2\frac{h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right) T_{m,n} = 0$$

Summary of nodal finite-difference equations $\Delta x = \Delta y$

Case 5 : Node at a plane surface with uniform heat flux



$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2Q''\Delta x}{k} - 4T_{m,n} = 0$$

Finite-difference solutions

Depending upon your mathematical background and the specific problem, the numerical solution can be found with

- Matrix inversion.
- Gauss-Seidel iteration.
- ...

The reader who wishes to study such analyses in depth should refer to specific publications.

Transient conduction - Finite-difference methods for transient heat conduction

For transient conditions with two-dimensional effects, constant properties and no internal heat generation, the general expression

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

To obtain the finite-difference form, we can use the central-difference form of

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Transient conduction - Finite-difference methods for transient heat conduction

We discretise in time using the integer p as : $t = p\Delta t$ and obtain

$$\left.\frac{\partial T}{\partial y}\right|_{m,n} = \frac{T_{m,n}^{p+1} + T_{m,n}^{p}}{\Delta t}$$

Hence, the time derivative is in terms of the difference in temperatures at time (p+1) new and (p) previous, separated by the time interval Δt.

- *Explicit method* : the temperatures are evaluated at (**p**)
- *Implicit method* : the temperatures are evaluated at (**p+1**)

Transient conduction - Explicit method

It's a forward-difference method. Evaluate the terms on the right-hand side of equations at **p**.

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta t}=\frac{T_{m+1,n}^{p}+T_{m-1,n}^{p}-2T_{m,n}^{p}}{(\Delta x)^2}+\frac{T_{m,n+1}^{p}+T_{m,n-1}^{p}-2T_{m,n}^{p}}{(\Delta y)^2}$$

Solving for new nodal temperature at p+1 for $\Delta x = \Delta y$:

$$T_{m,n}^{p+1}=Fo\left(T_{m+1,n}^{p}+T_{m-1,n}^{p}+T_{m,n+1}^{p}+T_{m,n-1}^{p}\right)+(1-4\ Fo)\,T_{m,n}^{p}$$

with

$$Fo=\frac{\alpha\Delta t}{(\Delta x)^2}$$

For a one-dimensional transient heat conduction, the expression becomes

$$T_{m}^{p+1}=Fo\left(T_{m+1}^{p}+T_{m-1}^{p}\right)+(1-2\ Fo)\,T_{m}^{p}$$

Transient conduction - Explicit method

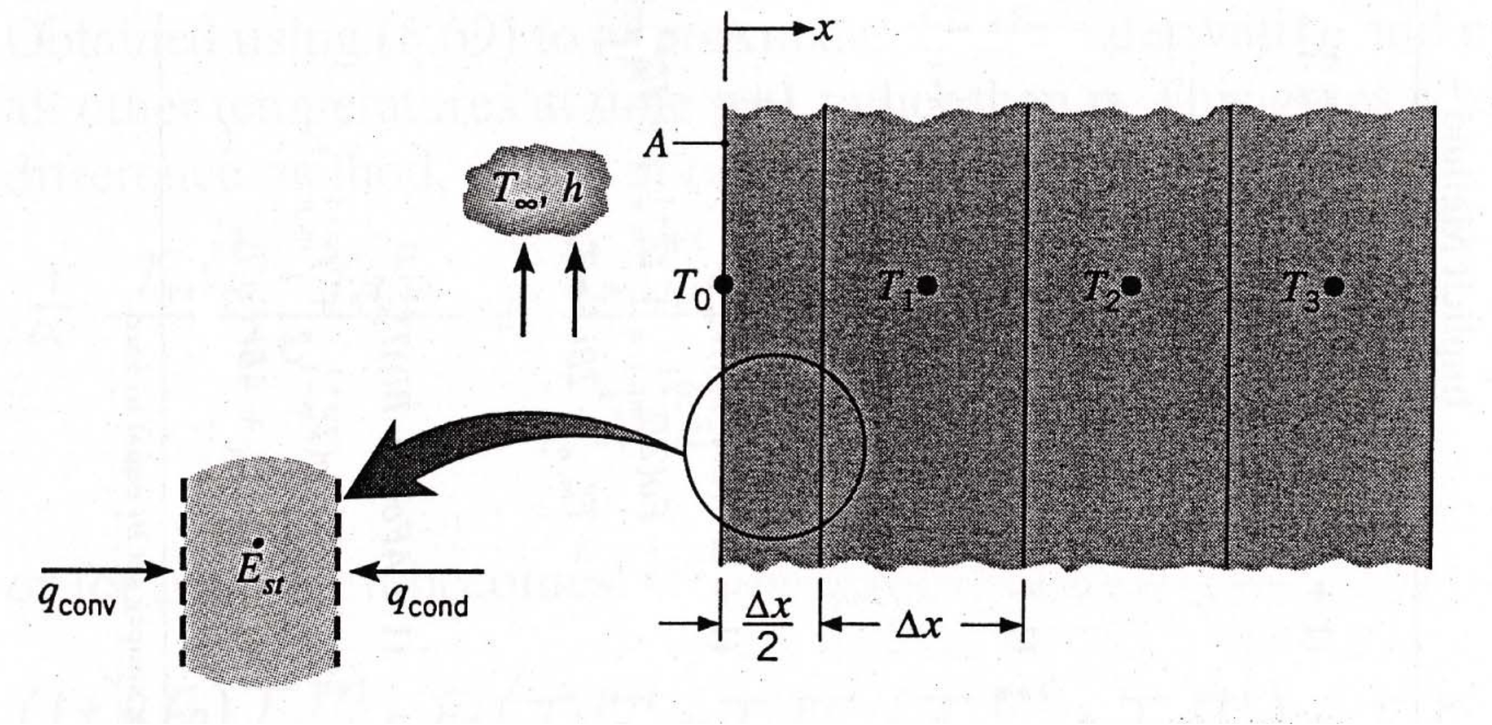
Equations are explicit since the unknown nodal temperatures at time **p+1** are determined with known temperatures at time **p** in each time step.

Initial condition must be known so that the temperature of each node is known at time $t = 0$ when $p = 0$. Then, the temperatures at $t = \Delta t$ for $p = 1$ are calculable and the calculations proceed for $t = 2\Delta t$ for $p = 2$ and so forth.

Accuracy is increased by decreasing the size of the time step Δt and the size of Δx, at the expense of increasing calculation time.

Transient conduction - Stability of calculation

- Stability criterion for 1-D interior node : $Fo \leq \frac{1}{2}$
- Stability criterion for 2-D interior node : $Fo \leq \frac{1}{4}$



Transient conduction - Stability of calculation

Equations may also be derived from an energy balance. For example, for a surface node with a convection boundary 1-D. Starting with

$$\dot{E}_{in} + \dot{E}_g = \dot{E}_{st}$$

We obtain

$$hA\left(T_\infty - T_0^p\right) + \frac{kA}{\Delta x}\left(T_1^0 - T_0^p\right) = \rho c A \frac{\Delta x}{2} \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

And solving for the surface temperature at $t + \Delta t$

$$T_0^{p+1} = \frac{2h\Delta t}{\rho c \Delta x}\left(T_\infty - T_0^p\right) + \frac{2\alpha \Delta t}{(\Delta x)^2}\left(T_1^p - T_0^p\right) + T_0^p$$

since

$$\frac{2h\Delta t}{\rho c \Delta x} = 2\frac{h\Delta x}{k}\frac{\alpha \Delta t}{(\Delta x)^2} = 2\ Bi\ Fo$$

Then

$$T_0^{p+1} = 2Fo\,(T_1^p + Bi\ T_\infty) + (1 - 2Fo - 2Bi\ Fo)\ T_0^p$$

The finite-difference form of the Biot number is

$$Bi = \frac{h\Delta x}{k}$$

For stability, we require that the coefficient for $T_0^p \geq 0$, so

$$1 - 2\ Fo - 2\ Fo\ Bi \geq 0$$

or

$$Fo\,(1 + Bi) \leq \frac{1}{2}$$

Note : The stability limit for the most restrictive requirement must be used !

Transient conduction - Implicit method

Implicit method, obtained using

$$\left.\frac{\partial T}{\partial y}\right|_{m,n} = \frac{T_{m,n}^{p+1} + T_{m,n}^{p}}{\Delta t}$$

to approximate the time derivative and evaluating all other temperatures at time **p+1** rather than **p**. This gives a backward-difference method, which in two-dimensional form is

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2}$$

or for $\Delta x = \Delta y$ it becomes

$$T_{m,n}^{p} = (1 - 4\ Fo)\, T_{m,n}^{p+1} - Fo\left(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}\right)$$

Transient conduction - Implicit method

Notice :

Thus, the new temperature at node m,n depends on the new unknown temperatures at the other adjacent nodes. Consequently, a simultaneous solution is required using Gauss-Seidel iteration or matrix inversion.

The implicit solution scheme is implicitly **unconditionally stable**. Hence, we can choose time steps Δt and node spacings Δx and Δy to our own advantage.

Energy balance method :

Surface node :

$$(1 + 2Fo + 2Bi\ Fo)\, T_0^{p+1} - 2Fo\ T_1^{p+1} = 2Fo\ Bi\ T_\infty + T_0^p$$

Interior node :

$$(1 + 2Fo)\, T_m^{p+1} - Fo\left(T_{m-1}^{p+1} + T_{m+1}^{p+1}\right) = T_m^p$$