6. Laminar and turbulent boundary layers

John Richard Thome

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6.1 Some introductory ideas

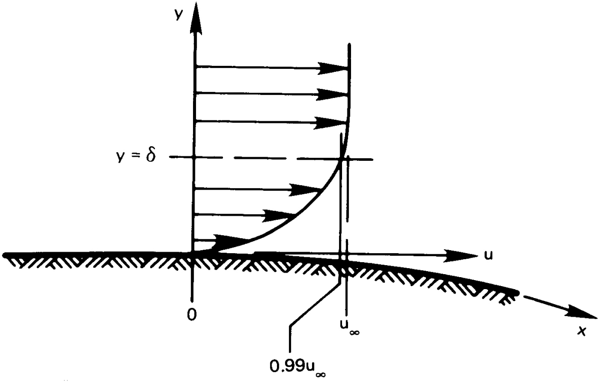

Figure 6.1 A boundary layer of thickness δ
Boundary layer thickness on a flat surface $\delta = fn(u_\infty, \rho, \mu, x)$

6.1 Some introductory ideas

The dimensional functional equation for the boundary layer thickness on a flat surface.

$$\frac{\delta}{x} = fn(Re_x) \qquad Re_x \equiv \frac{\rho u_\infty x}{\mu} = \frac{u_\infty x}{\nu} \quad (6.1)$$

- $\nu = \frac{\mu}{\rho}$:kinematic viscosity.
- Re_x : Reynolds number.

For a flat surface, where u_∞ remains constant.

$$\frac{\delta}{x} = \frac{4.92}{\sqrt{Re_x}} \quad (6.2)$$

6.1 Some introductory ideas

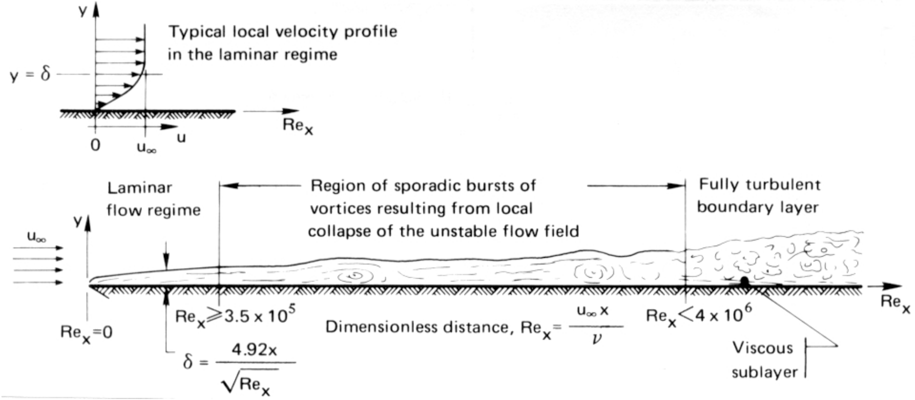

Figure 6.4 Boundary layer on a long, flat surface with a sharp leading edge.

6.1 Some introductory ideas

$(u_{av})_{crit}$: Transitional value of the average velocity.

$$Re_{critical} \equiv \frac{\rho D (u_{av})_{crit}}{\mu} \quad (6.3)$$

$$Re_{x\,critical} = \frac{u_\infty x_{crit}}{\nu} \quad (6.4)$$

Transition from laminar to turbulent flow.

$$Re_{x,c} = 5 \cdot 10^5$$

Thermal boundary layer

The wall is at temperature T_w

$$-k_f(\frac{\partial T}{\partial y}|_{y=0}) = (T_w - T_\infty)h \quad (6.5)$$

Where k_f is the conductivity of the fluid. The following condition defined h within the fluid instead of specifying it as known information on the boundary.

$$\frac{\partial \left(\frac{T_w - T}{T_w - T_\infty}\right)}{\partial \left(\frac{y}{L}\right)}|_{\frac{y}{L}=0} = \frac{hL}{k_f} = Nu_L \quad (6.5a)$$

The physical significance of Nu is given by

$$Nu_L =\equiv \frac{hx}{k_f} = \frac{L}{\delta_t'} \quad (6.6)$$

The nusselt number is inversely proportional to the thickness of the thermal boundary layer δ_t'.

Thermal boundary layer

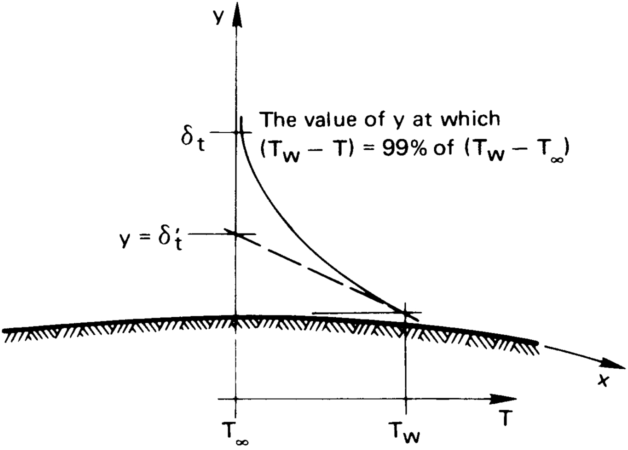

Figure 6.5 The thermal boundary layer during the flow of cool fluid over a warm plate.

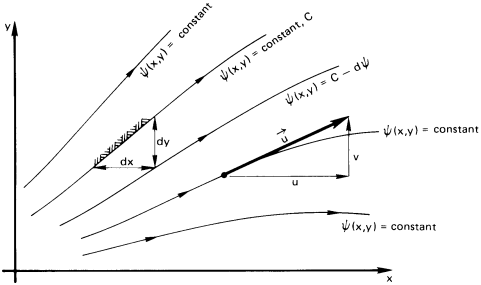

Figure 6.7 A steady, incompressible, two-dimensional flow field represented by streamlines, or lines of constant ψ.

For an incompressible flow, continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.11)$$

$$\bigtriangledown \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Conservation of momentum

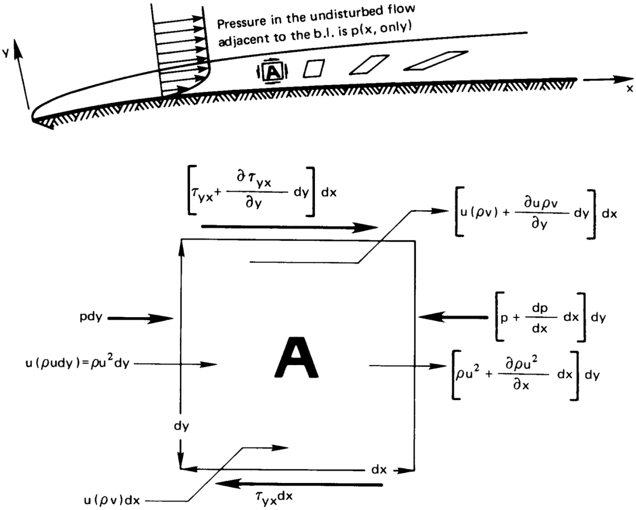

Figure 6.9 Forces acting in a two-dimensional incompressible boundary layer.

Conservation of momentum

The external forces are :

$$\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx - \tau_{yx} dx + p dy - \left(p - \frac{\partial p}{\partial x} dx\right) dy = \left(\frac{\partial \tau_{yx}}{\partial y} - \frac{\partial p}{\partial x}\right) dxdy$$

The rate at which A loses x-directed momentum to its surroundings is :

$$\left(\rho u^2 + \frac{\partial \rho u^2}{\partial x} dx\right) dy - \rho u^2 dy + \left[u(\rho v) + \frac{\partial \rho u v}{\partial y} dy\right] dx - \rho u v dx$$

$$= \left(\frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y}\right) dxdy$$

Conservation of momentum

We obtain one form of the steady, two-dimensional, incompressible boundary layer momentum equation.

$$\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2} \quad (6.12)$$

A second form of the momentum equation.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2} \quad (6.13)$$

If there is no pressure gradient in the flow, if p and u_∞ are constant as they would be for flow past a flat plate, so we obtain,

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} \quad (6.15)$$

The skin friction coefficient

The shear stress can be obtained by using Newton's law of viscous shear.

$$\tau_w = \mu \frac{\partial u}{\partial y}|_{y=0} = 0.332 \frac{\mu u_\infty}{x} \sqrt{Re_x}$$

The skin local friction coefficient is defined as :

$$C_f \equiv \frac{\tau_w}{\rho u_\infty^2/2} = \frac{0.664}{\sqrt{Re_x}} \quad (6.33)$$

The overall skin local friction coefficient is based on the average of the shear stress :

$$\bar{C}_f = \frac{1.328}{\sqrt{Re_L}} \quad (6.34)$$

6.3 The energy equation

Using Fourier's law

$$h = \frac{q}{T_w - T_\infty} = -\frac{k}{T_w - T_\infty}\frac{\partial T}{\partial y}|_{y=0} \quad (6.35)$$

- Pressure variations in the flow are not large enough to affect thermodynamic properties.
- Density changes result only from temperature changes and will also be small (incompressible behaviour).
- Temperature varaitions in the flow are not large enough to change k significantly.
- Viscous stresses do not dissipate enough energy to warm the fluid significantly.

6.3 The energy equation

We write conservation of energy in the form.

$$\underbrace{\frac{d}{dt}\int_R \rho\hat{u}\,dR}_{\text{rate of internal energy increase in } R} = -\underbrace{\int_S \left(\rho\hat{h}\right)\vec{u}\cdot\vec{n}\,dS}_{\text{rate of internal energy and work out of } R}$$

$$-\underbrace{\int_S (-k\bigtriangledown T)\cdot\vec{n}\,dS}_{\text{net heat conduction rate out of } R} + \underbrace{\int_R \dot{q}\,dR}_{\text{rate of heat generation in } R} \qquad (6.36)$$

For a constant pressure flow field.

$$\rho c_p\left(\underbrace{\frac{\partial T}{\partial t}}_{\text{energy storage}} + \underbrace{\vec{u}\cdot\bigtriangledown T}_{\text{enthalpy convection}}\right) = \underbrace{k\bigtriangledown^2 T}_{\text{heat conduction}} + \underbrace{\dot{q}}_{\text{heat generation}} \qquad (6.37)$$

In a steady two-dimensional flow field without heat sources, equation 6.37 takes the following forme.
With this assumption, $\partial^2 T/\partial x^2 \ll \partial^2 T/\partial y^2$, so the boundary layer form is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

6.4 The Prandtl number and the boundary layer thickness

To look more closely at the implications of the similarity between the velocity and the thermal boundary layers.

$$h = fn(k, x, \rho, c_p, \mu, u_\infty)$$

We can find the following number by dimension analysis.
Prandtl number

$$\mathrm{Pr} \equiv \frac{\nu}{\alpha}$$

Relative effectiveness of momentum and energy transport by diffusion in the velocity and thermal boundary layers where for laminar flow.
Relationship with other dimensionless number.

$$Nu_x = fn(Re_x, Pr)$$

6.4 The Prandtl number and the boundary layer thickness

- For simple monatomic gases, $Pr = \frac{2}{3}$.
- For diatomic gases in which vibration is unexcited, $Pr = \frac{5}{7}$.
- As the complexity of gas molecules increase, Pr approaches an upper value of unity.
- For liquids composed of fairly simple molecules, excluding metals, Pr is of the order of magnitude of 1 of 10.
- For liquid metals, Pr is of the order of magnitude of 10^{-2} or less.

Boundary layer thickness, δ and δ_t, and the Prandtl number

When $Pr > 1, \delta > \delta_t$, and when $Pr < 1, \delta < \delta_t$. This is because high viscosity leads to a thick velocity boundary layer, and a high thermal diffusivity should give a thick thermal boundary layer.

$$\frac{\delta}{\delta_t} = fn\left(\frac{\nu}{\alpha} only\right).$$

6.5 Heat transfer coefficient for laminar, incompressible flow over a flat surface

The following equation expresses the conservation of thermal energy in integrated form.

$$\frac{d}{dx}\int_0^{\delta_t} u(T - T_\infty)dy = \frac{q_w}{\rho c_p} \quad (6.47)$$

Predicting the temperature distribution in the laminar thermal boundary layer

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \quad (6.50)$$

Predicting the heat flux in the laminar boundary layer

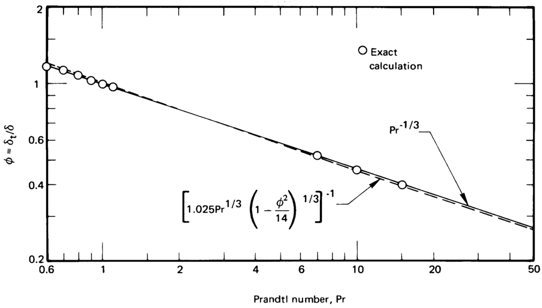

Figure 6.14 The exact and approximate Prandtl number influence on the ratio of boundary layer thicknesses.

$$\frac{\delta_t}{\delta} = \frac{1}{1.025Pr^{1/3}\left[1-\left(\delta_t^2/14\delta^2\right)\right]^{1/3}} \simeq \frac{1}{1.025Pr^{1/3}} \quad (6.54)$$

So we can write for $0.6 \leq Pr \leq 50$.

$$\frac{\delta_t}{\delta} = Pr^{1/3}$$

The following expression gives very accurate results under the assumptions on which it is based : a laminar two-dimensional boundary layer on a flat surface, with T_w constant and $0.6 \leq Pr \leq 50$

$$Nu_x = 0.332 Re^{1/2} Pr^{1/3} \quad (6.58)$$

High Pr At high Pr, equation 6.58 is still close correct. The exact solution is

$$Nu_x \rightarrow 0.339 Re_x^{1/2} Pr^{1/3}, \ Pr \rightarrow \infty$$

Some other laminar boundary layer heat transfer equations

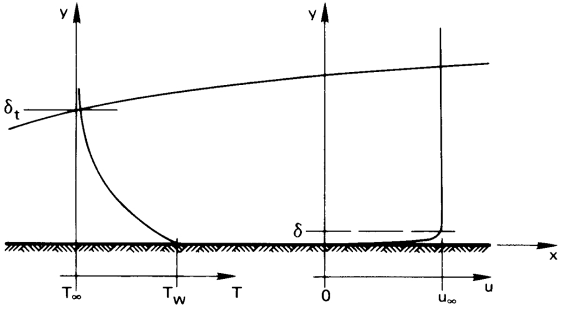

Figure 6.15 A laminar boundary layer in a low-Pr liquid. The velocity boundary layer is so thin that $u \simeq u_\infty$ in the thermal boundary layer.

Low Pr In this case, $\delta_t \gg \delta$, the influence of the viscosity were removed from the problem and for all pratical purposes $u = u_\infty$ everywhere. With a dimension analysis, we can find :

$$Nu_x = \frac{hx}{k}$$

Some other laminar boundary layer heat transfer equations

We can define a new dimensionless number.

Peclet number

$$Pe_x \equiv Re_x Pr = \frac{u_\infty x}{\alpha} \quad (6.61)$$

Peclet number can be interpreted as the ratio of **heat capacity rate of fluid in the b.l.** to **axial heat conductance of b.l.**

The exact solution of the boundary layer equations gives, in this case : For $Pe_x \geq 100$, $Pr \leq \frac{1}{100}$, $Re_x \geq 10^4$.

$$Nu_x = 0.565 Pe_x^{1/2} \quad (6.62)$$

$$\overline{Nu}_L = 1.13 Pe_L^{1/2}$$

Some other laminar boundary layer heat transfer equations

Churchill-Ozoe correlation : For laminar flow over a flat isothermal plate for all Prandtl numbers is the following for $Pe_x > 100$

$$Nu_x = \frac{0.3387 Re_x^{1/2} Pr^{1/3}}{\left(1 + (0.0468/Pr)^{2/3}\right)^{1/4}} \quad (6.63)$$

And $\overline{Nu}_x = 2Nu_x$

Some other laminar boundary layer heat transfer equations

Boundary layer with an unheated starting length

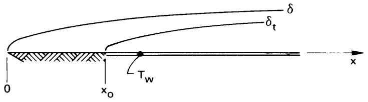

Figure 6.16 A b.l. with an unheated region at the leading edge.

For laminar flow, with $x > x_0$

$$Nu_x = \frac{0.332 Re_x^{1/2} Pr^{1/3}}{\left(1 - (x_0/x)^{3/4}\right)^{1/3}} \quad (6.64)$$

- Uniform wall temp. : $\bar{h} \equiv \frac{\bar{q}}{\triangle T} = \frac{1}{L}\int_0^L h(x)dx$
- Uniform heat flux : $\bar{h} \equiv \frac{q}{\overline{\triangle T}} = \frac{q}{\frac{1}{L}\int_0^L \triangle T(x)dx}$

The problem of uniform wall heat flux

The exact result for $Pr \geq 0.6$ is

$$Nu_x = 0.453 Re_x^{1/2} Pr^{1/3} \quad (6.71)$$

$$\overline{Nu}_L = 0.6795 Re_L^{1/2} Pr^{1/3}$$

Churchill and Ozoe equations for the problem of uniform wall heat flux. For $Pe_x > 100$.

$$Nu_x = \frac{0.4637 Re_x^{1/2} Pr^{1/3}}{\left(1 + (0.02052/Pr)^{2/3}\right)^{1/4}} \quad (6.73)$$

6.6 The Reynolds analogy

The analogy between heat and momentum transfer can now be generalized. For a flat surface with no pressure gradient : C_f is the skin friction coefficient.

$$\frac{d}{dx}\left[\delta \int_0^1 \frac{u}{u_\infty}\left(\frac{u}{u_\infty}-1\right) d\left(\frac{y}{\delta}\right)\right] = -\frac{1}{2}C_f(x) \quad (6.25)$$

For constant wall temperature case :

$$\frac{d}{dx}\left[\delta \int_0^1 \frac{u}{u_\infty}\left(\frac{T-T_\infty}{T_w-T_\infty}\right) d\left(\frac{y}{\delta_t}\right)\right] = \frac{q_w}{\rho c_p u_\infty (T_w - T_\infty)} \quad (6.74)$$

But the similarity of temperature and flow boundary layers to one another suggests the following approximation, which becomes exact only when $Pr = 1$.

$$\frac{T-T_\infty}{T_w-T_\infty}\delta = \left(1-\frac{u}{u_\infty}\right)\delta_t \;\Rightarrow\; -\frac{1}{2}C_f(x) = -\frac{q_w}{\rho c_p u_\infty (T_w - T_\infty)\,\phi^2} \quad (6.75)$$

The result is one instance of the **Reynolds-Colburn analogy**.

$$\frac{h}{\rho c_p u_\infty} Pr^{2/3} = \frac{C_f}{2} \quad (6.76)$$

6.6 The Reynolds analogy

For use in Reynold's analogy, C_f must be a pure skin friction coefficient.

Stanton number :

$$St \equiv \frac{\textit{actual heat flux to the fluid}}{\textit{heat flux capacity of the fluid flow}}$$

$$St \equiv \frac{h}{\rho c_p u_\infty} = \frac{Nu_x}{Re_x Pr} \quad (6.77)$$

Stanton mass transfer number :

$$St_m \equiv \frac{Sh}{ReSc}$$

We obtain

$$\frac{C_f}{2} = St = St_m$$

This equation is known as the **Reynolds analogy**.

6.6 The Reynolds analogy

For application over a wider range, some corrections are necessary. In particular, the Chilton-Colburn analogies are : For $0.6 < Pr < 60$

$$\frac{C_f}{2} = StPr^{2/3} \equiv j_H$$

For $0.6 < Sc < 3000$

$$\frac{C_f}{2} = St_m Sc^{2/3} \equiv j_m$$

Where j_H and j_m are the **Colburn j factors** for heat and mass transfer.

6.7 Turbulent boundary layers

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$
$$T = \bar{T} + T'$$
$$\rho = \bar{\rho} + \rho'$$

<u>etc.</u>

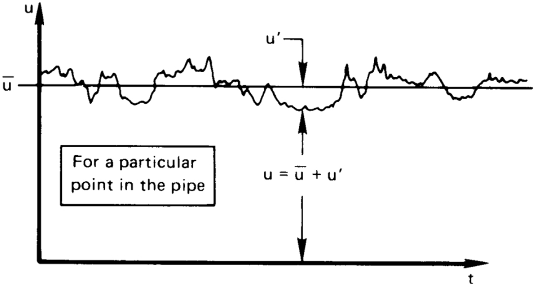

Figure 6.17 Fluctuation of u and other quantities in a turbulent pipe flow.

6.7 Turbulent boundary layers

We define the actual local velocity : $u = \overline{u} + u'$. $\overline{u}$ is the average term and u' is instantaneous magnitude of the fluctuation.

$$\bar{u} = \frac{1}{T}\int_0^T \bar{u}dt + \frac{1}{T}\int_0^T u'dt = \bar{u} + \bar{u'} \quad (6.82)$$

Similary, we have the total shear stress and total fluxes as :

$$\tau_{tot} = \left(\mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'}\right) \quad q''_{tot} = -\left(k\frac{\partial\bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right)$$

And the following eddy diffusivities for these processes :

$$\rho\varepsilon_M\frac{\partial\bar{u}}{\partial y} = -\rho\overline{u'v'} \qquad \tau_{tot} = \rho(\nu + \varepsilon_M)\frac{\partial\bar{u}}{\partial y}$$

$$\varepsilon_H\frac{\partial\bar{T}}{\partial y} = -\overline{v'T'} \qquad q''_{tot} = -\rho c_p(\alpha + \varepsilon_H)\frac{\partial\bar{T}}{\partial y}$$

$$\varepsilon_m\frac{\partial\overline{C_A}}{\partial y} = -\overline{u'C'_A} \quad N''_{A,tot} = -(D_{AB} + \varepsilon_m)\frac{\partial\overline{C_A}}{\partial y}$$

Turbulence near the wall

We define the actual local velocity : $u = \overline{u} + u'$. $\overline{u}$ is the average term and u' is instantaneous magnitude of the fluctuation. For steady, incompressible, constant property flow with time-averaged variables, the x-momentum, energy and species conservation equations are :

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right)$$

$$\left(\bar{u}\frac{\partial \overline{C_A}}{\partial x} + \bar{v}\frac{\partial \overline{C_A}}{\partial y}\right) = \frac{\partial}{\partial y}\left(D_{AB}\frac{\partial \overline{C_A}}{\partial y} - \overline{v'C_A'}\right)$$

6.8 Heat transfer in turbulent boundary layers

For turbulent flow with Re_x up to about 10^7 and above $5 \cdot 10^5$, the local friction factor is correlated by :

$$C_{f,x} = 0.0592 Re_x^{-1/5}$$

For turbulent flow, boundary layer development is dependent on random fluctuation of the fluid, not molecular diffusion, and thus the thermal and species boundary layers do not depend on Pr and Sc. Thus :

$$\delta \approx \delta_t \approx \delta_c$$

With the Chilton-Colburn analogy, the local Nusselt number in turbulent flow is : for $0.6 < Pr < 60$

$$Nu_x = St Re_x Pr = 0.0296 Re_x^{4/5} Pr^{1/3}$$

The increase in mixing of the fluid causes the turbulent boundary layer to grow more rapidly than the laminar boundary layer, and have larger friction and convection coefficients.

Mixed boundary layer conditions

For a laminar layer flowed by a turbulent layer, integrating the global convection coefficient over the laminar zone ($0 < x \leq x_c$) and then over the turbulent zone ($x_c < x \leq L$) :

$$\overline{h_L} = \frac{1}{L}\left\{ \int_0^{x_c} h_{lam} dx + \int_{x_c}^{L} h_{turb} dx \right\}$$

$Re_{x,c}$ is the critical Reynolds number for transition. Setting $Re_{x,c} = 5 \cdot 10^5$, for $0.6 < Pr < 60$ and $5 \cdot 10^5 < Re_L \leq 10^8$

$$\overline{Nu}_L = (0.037 Re_L^{4/5} - 871) Pr^{1/3}$$

$$\overline{C}_{f,L} = \frac{0.074}{Re_L^{1/5}} - \frac{1742}{Re_L}$$

For situations where $L \gg x_c$ and $Re_L \gg Re_{x,c}$, the average Nusselt number reduce to :

$$\overline{Nu}_L = 0.037 Re_L^{4/5} Pr^{1/3}$$

$$\overline{C}_{f,L} = 0.074 Re_L^{-1/5}$$

In the foregoing equations, the fluid physical properties are evaluated at the film temperature.

Guidelines of application of convection methods

- Identify the flow geometry (flat plate, cylinder, etc.) ;
- Decide whether the local or surface average heat transfer coefficient is required for the problem at hand ;
- Choose correct reference temperature and evaluated fluid properties at that temperature ;
- Calculate the Reynolds number to determine if the flow is laminar or turbulent ;
- Calculate the Prandle number ;
- Select the appropriate correlation that respects the restrictions on its use ;
- Double-check your design with a second correlation if application is critical to operation when possible.