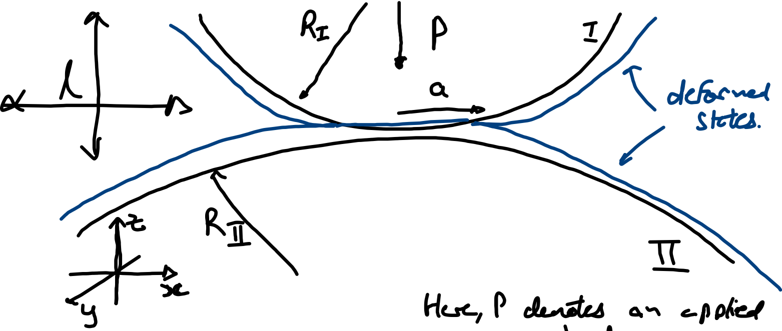
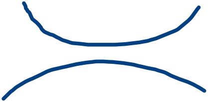
## The Hertz Contact Problem: Johnson's textbook




Here, $P$ denotes an applied load.

# A few assumptions of Hertz theory:

1. The surfaces are continuous & non-conforming.



and not

2. Strains are small: $a \ll R$

3. Bodies I & II are elastic half-spaces.

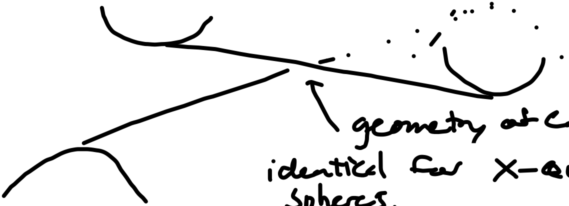
   $a \ll R_{I}, R_{II}$ ; also $a \ll l$

4. The surfaces are <u>frictionless</u>,

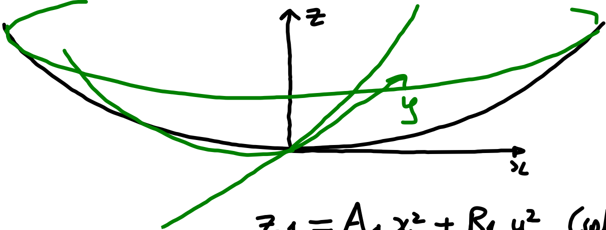
$$\underbrace{q_x, q_y}_{\text{Tractions in shear}} = 0$$

i.e. **no** tangential stress.

Recall: our measurement:



geometry of contact is identical for X-ed cyl. & spheres.




$$z_1 = A_1 x^2 + B_1 y^2 \quad \text{(sph. cap)}$$

A complementary spherical cap has a similar equation:

$$z_2 = -(A_2 x^2 + B_2 y^2). \quad \text{2nd sphere}$$

For spherical caps, $A = B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

$\Rightarrow$ Contours of constant separation are <u>circles</u>.

The distance between points on bodies 1 & 2:

$$h = Ax^2 + By^2 = \frac{1}{2}\left(\frac{x^2}{R'} + \frac{y^2}{R''}\right), \quad R', R'' \text{ are}$$

relative radii of curvature; for sph caps, $R' = R''$

Upon applying a load $P$, caps displace by $\delta_1$ & $\delta_2$:

- Bodies deform by $u_{z1}(x,y)$; $u_{z2}(x,y)$ relative to points $T_1$ & $T_2$ on surfaces prior to contact.

Points in contact: $u_{z1} + u_{z2} + h = \delta_1 + \delta_2$

Define $\delta = \delta_1 + \delta_2 \Rightarrow u_{z1} + u_{z2} = \delta - Ax^2 - By^2$

Outside of contact area: $u_{z1} + u_{z2} > \delta - Ax^2 - By^2$

Dimensional Analysis:

Sphere-sphere contact, radius of contact $a$:

We know $u_I(0,0) = \delta_I$, $u_{II}(0,0) = \delta_{II}$.

$$\left(\frac{u_{zI}(0)}{a} - \frac{u_{zI}(x)}{a}\right) + \left(\frac{u_{zII}(0)}{a} - \frac{u_{zII}(x)}{a}\right) = \frac{x^2}{2a}\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right)$$

Substitute $x = a$ & write $u_z(0) - u_z(a) = d$,

$$\frac{d_1}{a} + \frac{d_2}{a} = \frac{a}{2}\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right)$$

Assume deformation is small: $d \ll a$; $\frac{d}{a}$ defines a strain scale

- strain $\propto$ contact pressure / elastic modulus (lin. Elast.)

Define $p_m$ to be a ch. contact pressure, which is the same in both bodies

$$\frac{p_m}{E_1} + \frac{p_m}{E_2} \propto a\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right) \Rightarrow \boxed{p_m \propto \frac{a\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right)}{1/E_I + 1/E_{II}}}.$$

Recognizing that for an external load $P = \pi a^2 p_m$,

$$\boxed{a \propto \left[\frac{P\left(1/E_I + 1/E_{II}\right)}{\left(1/R_I + 1/R_{II}\right)}\right]^{1/3}}.$$

$$\Rightarrow \boxed{p_m = \frac{P}{\pi a^2} \propto \left[\frac{P\left(1/R_I + 1/R_{II}\right)^2}{\left(1/E_I + 1/E_{II}\right)^2}\right]^{1/3}}.$$

For a 3-D solid, compressions $\delta_I$ & $\delta_{II}$ are proportional to displacements $d_1$ & $d_2$.

$$\Rightarrow \delta \propto d_I + d_{II}$$

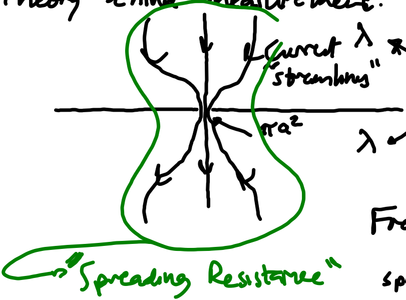
$$\Rightarrow \frac{\delta}{a} = \frac{a}{2}\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right) \Rightarrow \delta \propto \frac{a^2}{2}\left(\frac{1}{R_I} + \frac{1}{R_{II}}\right)$$

(from geometry)

$$\Rightarrow \delta \propto P^{2/3} \frac{\left(1/E_I + 1/E_{II}\right)^{2/3}}{\left(1/R_I + 1/R_{II}\right)^{1/3}}$$

Tabor – available on Wiki

Theory behind measurement:




Specific conductivity (an inverse resistance per unit volume).

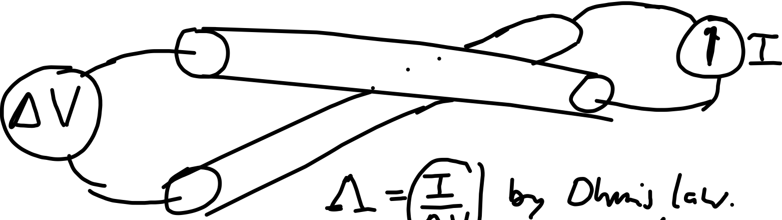
From Maxwell: $R = \frac{1}{2a\lambda}$

(sphere-plane contact)

sph·sph: $R \simeq \frac{1}{2\lambda}\left(\frac{1}{a} - \frac{2}{\pi r}\right)$ $r > a$.

For small contact dimension $a$, junction conductance $\Lambda = 2a\lambda$

From Hertz theory: $\Lambda = 2\lambda a = k_1 P^{1/3}$




$\Lambda = \frac{I}{\Delta V}$ by Ohm's law.

$= k_1 P^{1/3}$

$$a = \frac{\Lambda}{2\lambda} = \frac{k_1 P^{1/3}}{2\lambda}$$