The Hertz Contact Problem: Johnson's Loadback

Here, \( P \) denotes an applied load.
A few assumptions of Hertz theory:

1. The surfaces are continuous & non-conforming.

2. Strains are small: $a \ll R$

3. Bodies I & II are elastic half-spaces. $a \ll R_I, R_{II}$; also $a \ll l$
4. The surfaces are frictionless:

\[ q_x, q_y = 0 \]

... Traction in shear

i.e. no tangential stress.

Recall: our measurement:

\[ \text{geometry of contact is identical for X-rod vs. Y-rod spheres.} \]
\[ z_1 = A_1 x^2 + B_2 y^2 \text{ (sph. cap)} \]

A complementary spherical cap has a similar equation:

\[ z_2 = -\left( A_2 x^2 + B_2 y^2 \right). \text{ 2nd sphere} \]

For spherical caps, \( A = B = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \)

- \( \delta \) contours of constant separation are circles.
The distance between points on bodies 1 & 2:
\[ h = A x^2 + By^2 = \frac{1}{2} \left( \frac{x^2}{R'} + \frac{y^2}{R''} \right) \]
\( R', R'' \) are relative radii of curvature; for spheroids, \( R' = R'' \)

Upon applying a load \( P \), caps displace by \( \delta_1 \) & \( \delta_2 \):

Bodies deform by \( u_{31} (x, y) \); \( u_{32} (x, y) \) relative to points \( T_1 \) & \( T_2 \) on surfaces prior to contact.

Points in contact: \( u_{31} + u_{32} + h = \delta_1 + \delta_2 \)

Define \( \delta = \delta_1 + \delta_2 \rightarrow u_{31} + u_{32} = \delta - Ax^2 - By^2 \)

Outside of contact area: \( u_{31} + u_{32} > \delta - Ax^2 - By^2 \)
Dimensional Analysis:

Sphere-sphere contact, radius of contact a:

We know $U_1(0,0) = \sigma_1$, $U_{II}(0,0) = \sigma_{II}$.

$\left( \frac{U_1(0) - U_1(x)}{a} \right) + \left( \frac{U_{II}(0) - U_{II}(x)}{a} \right) = \frac{x^2}{2a} \left( \frac{1}{R_1} + \frac{1}{R_{II}} \right)$

Substitute $x = a$ & write $U_1(0) - U_1(a) = d$,

$\frac{d}{a} + \frac{d}{a} = \frac{a}{2} \left( \frac{1}{R_1} + \frac{1}{R_{II}} \right)$

Assume deformation is small: $d \ll a$; $\frac{d}{a}$ defines a strain scale
- strain $\sim$ contact pressure/elastic modulus (lin. Elastic)
Define $p_m$ to be a ch. contact pressure, which is the same in both bodies.

$$\frac{p_m}{E_1} + \frac{p_m}{E_2} \propto a \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \Rightarrow p_m \propto a \frac{\left(\frac{1}{E_1} + \frac{1}{E_2}\right)}{\frac{1}{R_1} + \frac{1}{R_2}}.$$  

Recognizing that for an external load $P = \pi a^2 p_m$, we have

$$a \propto P \left(\frac{\frac{1}{E_1} + \frac{1}{E_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\right)^{\frac{1}{3}}.$$  

$$\Rightarrow p_m = \frac{P}{\pi a^2} \propto \left[\frac{P \left(\frac{1}{E_1} + \frac{1}{E_2}\right)^2}{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2}\right]^{\frac{1}{3}}.$$
For a 3-D solid, compressions $\delta_1$ & $\delta_2$ are proportional to displacements $d_1$ & $d_2$.

$$\Rightarrow \delta \propto d_1 + d_2$$

$$\Rightarrow \frac{\delta}{a} = \frac{a}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \Rightarrow \delta \propto \frac{a^2}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

(from geometry)

$$\Rightarrow \delta \propto \rho^{\frac{3}{2}} \left( \frac{1}{E_2} + \frac{1}{E_1} \right)^{\frac{3}{2}}$$

$$\frac{1}{(\frac{1}{R_1} + \frac{1}{R_2})^{\frac{3}{2}}}$$
“Spreading Resistance”

For small contact diameter $a$, junction conductance $\Lambda = 2a\lambda$

From Maxwell: $R = \frac{1}{2a\lambda}$ (sphere-plane contact)

sph·sph: $R \approx \frac{1}{2\lambda} \left( \frac{1}{a} - \frac{2}{\pi} \right)$ for $r > a$. 

Specific conductivity (inverse resistance per unit volume).

Theory behind measurement:

Current $\lambda$

Spreading"
From Hele-Shaw theory: \( \Lambda = 2 \lambda a = k_1 \rho^{4/3} \)

\[ \Delta V \]

\[ a = \frac{\Lambda}{2 \lambda} = \frac{k_1}{2 \lambda} \rho^{4/3} \]

\[ \Lambda = \left( \frac{I}{\Delta V} \right) \text{ by Ohm's law.} \]