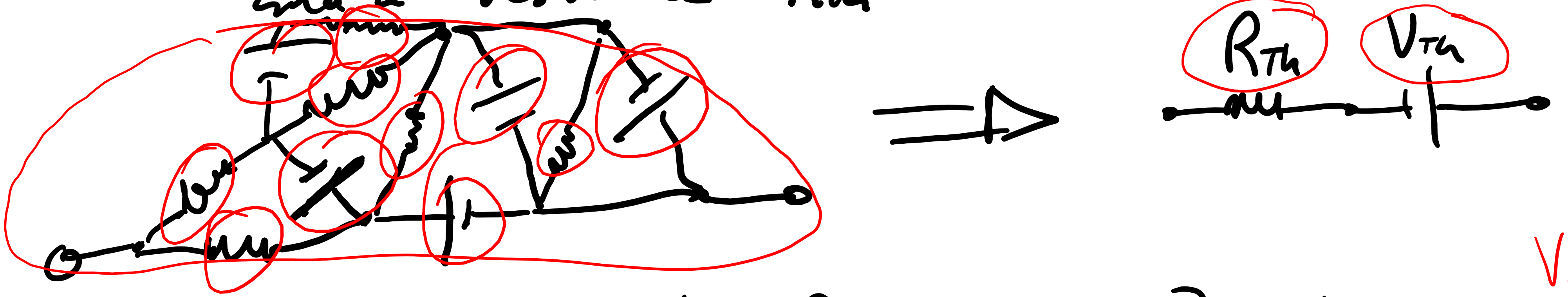
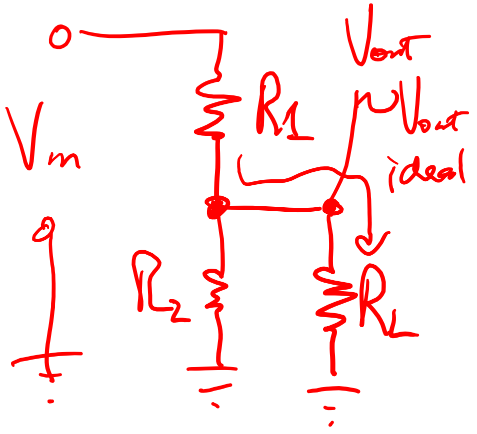
Passive circuit elements continued: Thevenin equivalence, reactive elements, complex impedence, and some useful devices.

Thevenin equivalent circuits

- For any two nodes in a circuit, irrespective of the circuits complexity (only linear circuit elements @ steady-state):

An equivalent circuit composed of a voltage source V_{Th} and a resistance R_{Th} can be found.





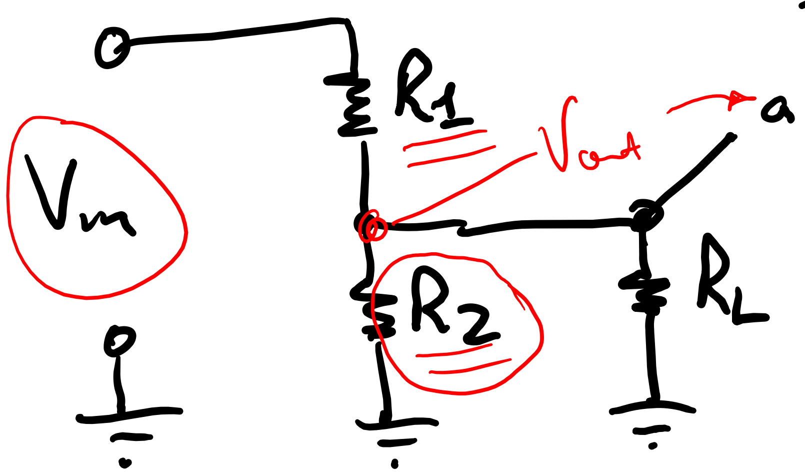
* V_{Th} & R_{Th} can be found in 2 steps:

1. $V_{Th} = V$ (open circuit) – no load!
2. $R_{Th} = \dfrac{V_{Th}}{I \text{ (short circuit)}}$ $\qquad \dfrac{V_{Th}}{I \text{ (short-circuit)}}$

* R_{TH} is the output impedance of such a circuit

The voltage divider revisited: the Thevenin equivalent circuit and output impedance

Let's revisit the humble voltage divider circuit from last week:



1. V open circuit: $V_a = \frac{V_{in} R_2}{R_1 + R_2} = V_{Th}$.

2. $R_{Th} = \frac{V_{Th}}{I\,(\text{short})}$! If $R_L = 0\,\Omega$, a short circuit is achieved.

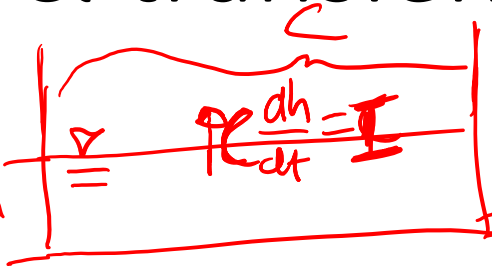
If this circuit is shorted, we find:

$$\boxed{I = \frac{V_{in}}{R_1}.}$$

$$R_{Th} = \frac{V_{Th}}{I_{\text{short}}} = \frac{\cancel{V_{in}} R_2 \; R_1}{(R_1 + R_2)\cancel{V_{in}}} = \frac{R_1 R_2}{R_1 + R_2}.$$

Thus, $R_{Th} = \frac{V_{Th}}{I} = \frac{R_1 R_2}{R_1 + R_2}$. Note, this is $R_1 \| R_2$!

Reactive circuit elements & transient circuit response: capacitors



$V \equiv \rho g h$

$C \equiv$ Vol. of tank

$I \equiv I$

$Q =$ vol. of H_2O in tank.

A ***capacitor with capacitance*** *C* stores charge (*Q*) in proportion to the applied voltage *V*. This can be expressed with an equation as:

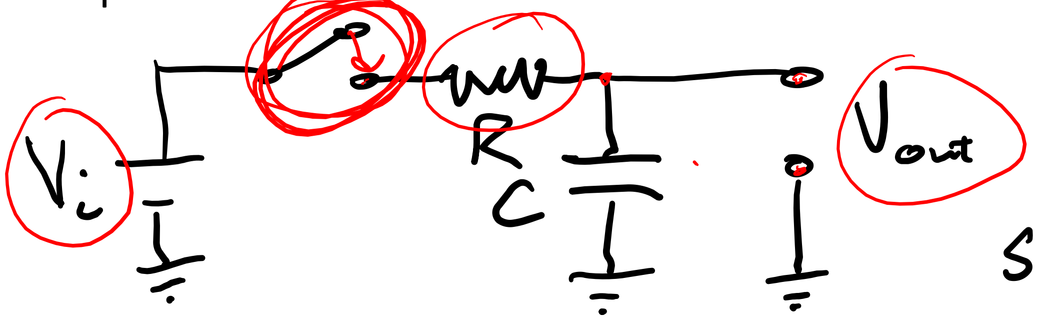
$$\boxed{Q = CV}$$

Q is hard to quantify or measure in practice; instead, we reformulate this expression for current *I* to make it more manageable, by taking the time derivative of both sides:

$$\frac{dQ}{dt} = C\frac{dV}{dt} \rightarrow I = C\frac{dV}{dt}$$

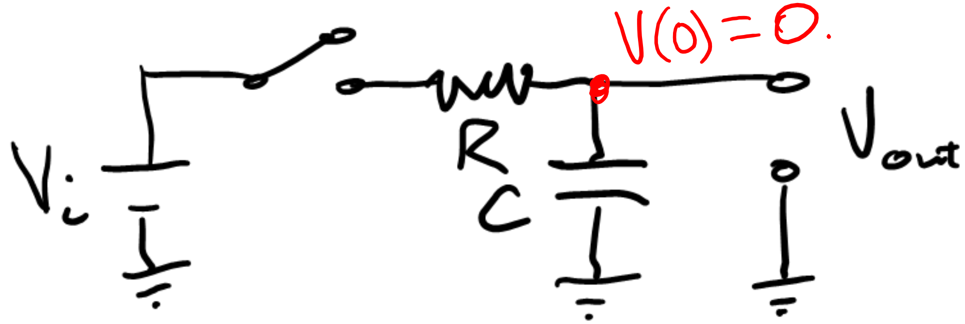
Capacitor

Since *I* now depends on the *temporal behavior* of *V*, we call capacitance a `reactive' circuit element -> current `reacts' to changes in voltage. A natural question is how this manifests itself with a simple circuit: let us look at the transient response of a simple circuit:



here, $I = \boxed{\frac{V_i - V(t)}{R} = C\frac{dV}{dt}}$.

solving this ODE: $\boxed{V(t) = V_i + Ae^{-t/RC}}$.



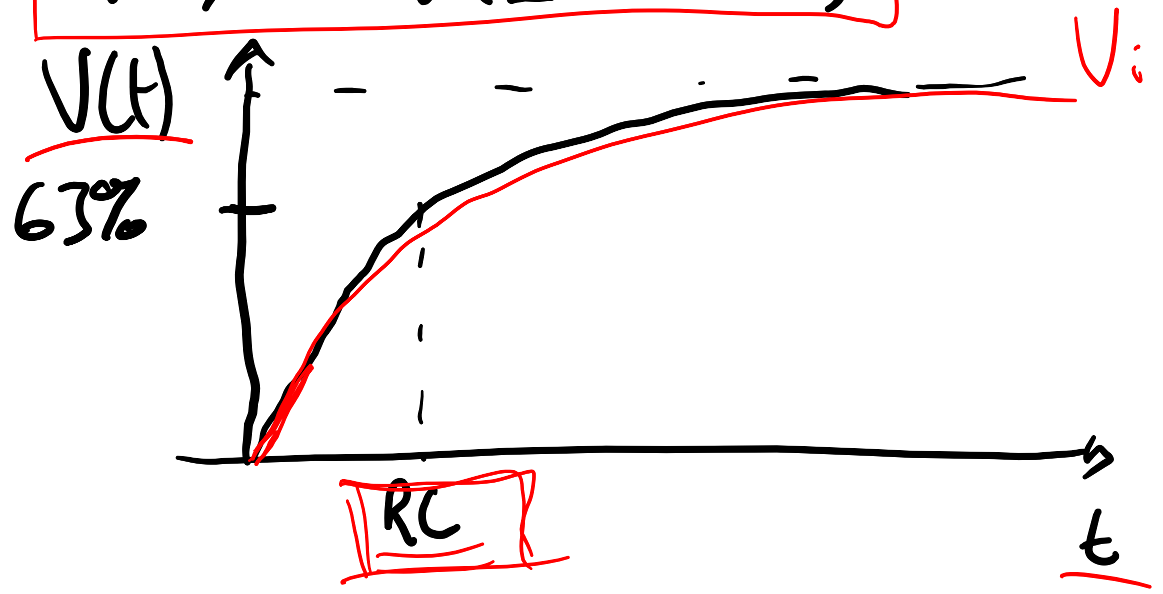
solving this ODE: $\boxed{V(t) = V_i + Ae^{-t/RC}}$.

We can find A by considering the initial condition:

$\boxed{V(0) = 0} = V_i + Ae^{0} \Rightarrow A = -V_i$.

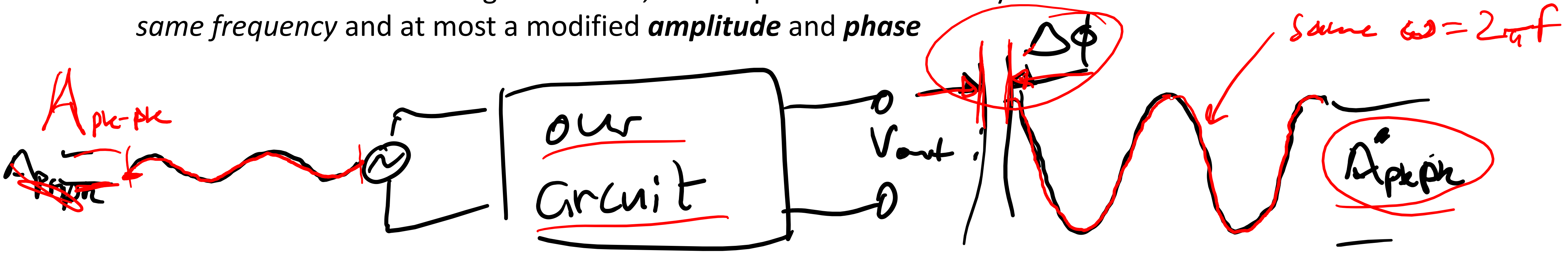
$t=0$

Thus, $\boxed{V(t) = V_i\left(1 - e^{-t/RC}\right)}$.

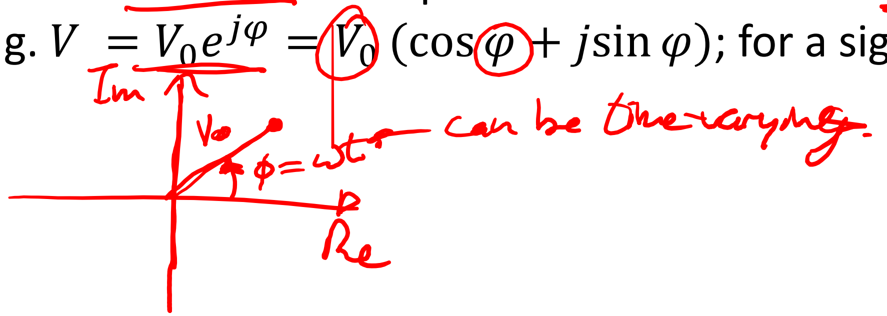


A/C circuit analysis – looking at sinusoidal* signals

Since our circuits at this stage are *Linear,* the output of a sinusoidally driven circuit is itself a sinusoid with the *same frequency* and at most a modified ***amplitude*** and ***phase***



Sinusoids are parameterized by an ***amplitude*** and a ***phase –*** this structure lends itself well to a description with complex variables, which also have two quantities of merit – the ***real*** and ***imaginary*** part. The phasor description, with e.g. $V = V_0 e^{j\varphi} = V_0\,(\cos\varphi + j\sin\varphi)$; for a signal oscillating with frequency ω, the phase of this signal is ωt

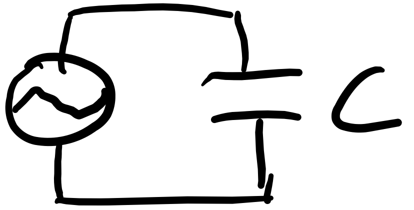


* For the math experts, the Fourier series is a sum of sinusoids, and with appropriate weights can approximate any periodic function over an interval. Thus analyzing the response to an arbitrary sinusoid, and the linear property of RLC circuits means we can describe the circuits response to an *arbitrary* forcing.

Generalizing resistance: Impedance and the transfer function for `forcing' (V) to `response' (I)

Let's first consider a sinusoidal voltage applied to a capacitor:

$V(t) = V_0 \sin(\omega t)$



Here, $I = C\frac{dV}{dt} = C\omega V_0 \cos(\omega t)$

Neglecting the phase, we can obtain the current as:

$I = \frac{V_0}{1/\omega C}$; thus the capacitor behaves like an ω-dependent resistor!

$V = IR \Rightarrow R = \frac{V}{I} \Rightarrow \frac{1}{\omega C} = \frac{|V_0|}{|I|}$

To account for phase correctly, we'll write $V = Re\left(V_0 e^{j\omega t}\right)$.

Since $I = C\frac{dV}{dt}$, $\Rightarrow I(t) = Re\left(j\omega C V_0 e^{j\omega t}\right) = Re\left(\frac{V_0 e^{j\omega t}}{-j/\omega C}\right) = Re\left(\frac{V}{X_c}\right)$.

Thus, we define a reactance X_c for the capaci:

$$\boxed{X_c = \frac{V}{I} = \frac{-j}{\omega C}}.$$

(n.b. this is structurally similar to R from Ohm's law! $R = \frac{V}{I}$.)

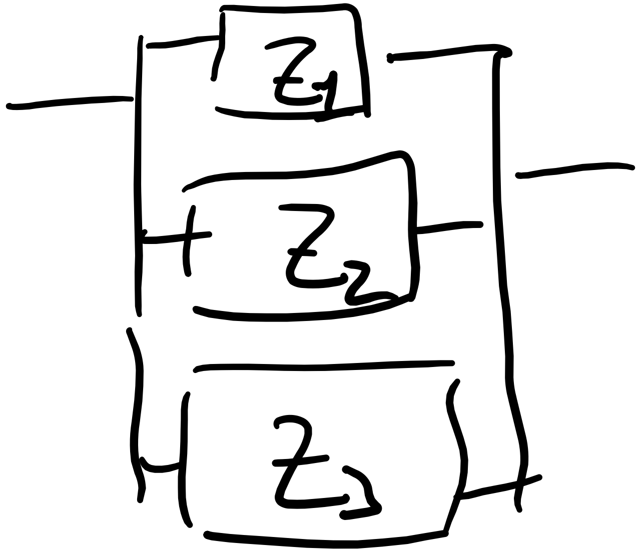
We can now define a generalized Impedance, Z, which in general is complex:

$$I = \frac{V}{Z} \Longleftrightarrow V = IZ.$$

Z's follow R's rules for addition in $\parallel$ & series:

: $Z_{eq} = Z_1 + Z_2 + Z_3$.

In ||:



: $Z_{eq} = \dfrac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}$.

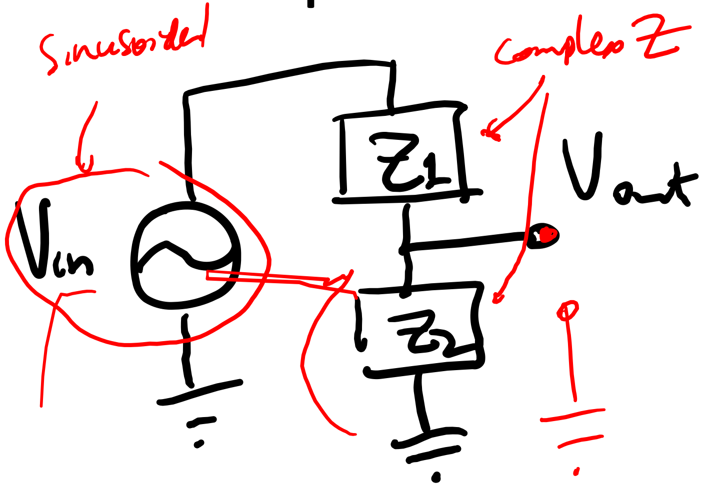
Generalized impedance for passive circuit elements:

$Z_R = R$ (resistors)

$Z_I = j\omega L$ (inductors)

$Z_C = -j/_{\omega C}$ (capacitors).

Generalizing the V-divider: frequency dependent response and filter devices

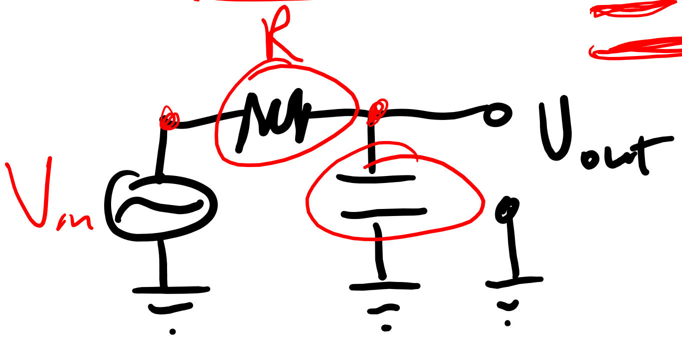


- The analysis is essentially the same as before:
  - $I = \frac{V_{in}}{Z_{tot}}$, $Z_{tot} = Z_1 + Z_2$.
  - $V_{out} = I Z_2 = \boxed{\frac{V_{in} Z_2}{Z_1 + Z_2}}$.

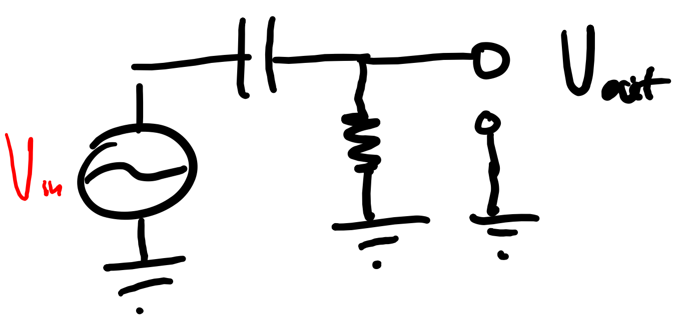
But now, Z is in general complex – and thus ω-dependent!

Low-pass filter:

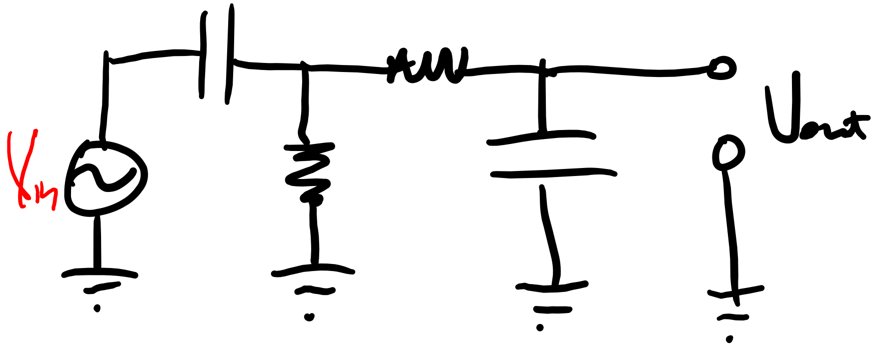
$X_c = \frac{-j}{\omega C}$



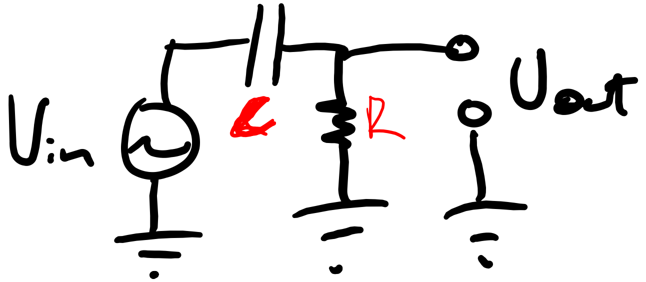
high-pass filter:



band-pass filter:



Analysis of the high-pass filter



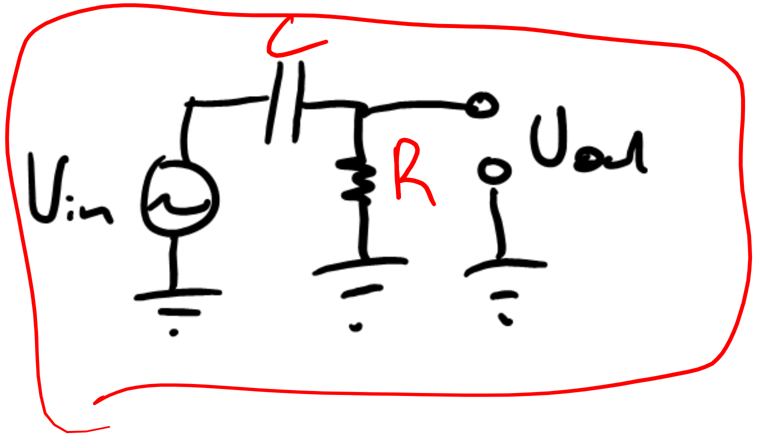
By generalized Ohm's law: $I = \frac{V_{in}}{Z_{tot}} = \frac{V_{in}}{R - j/\omega C}$

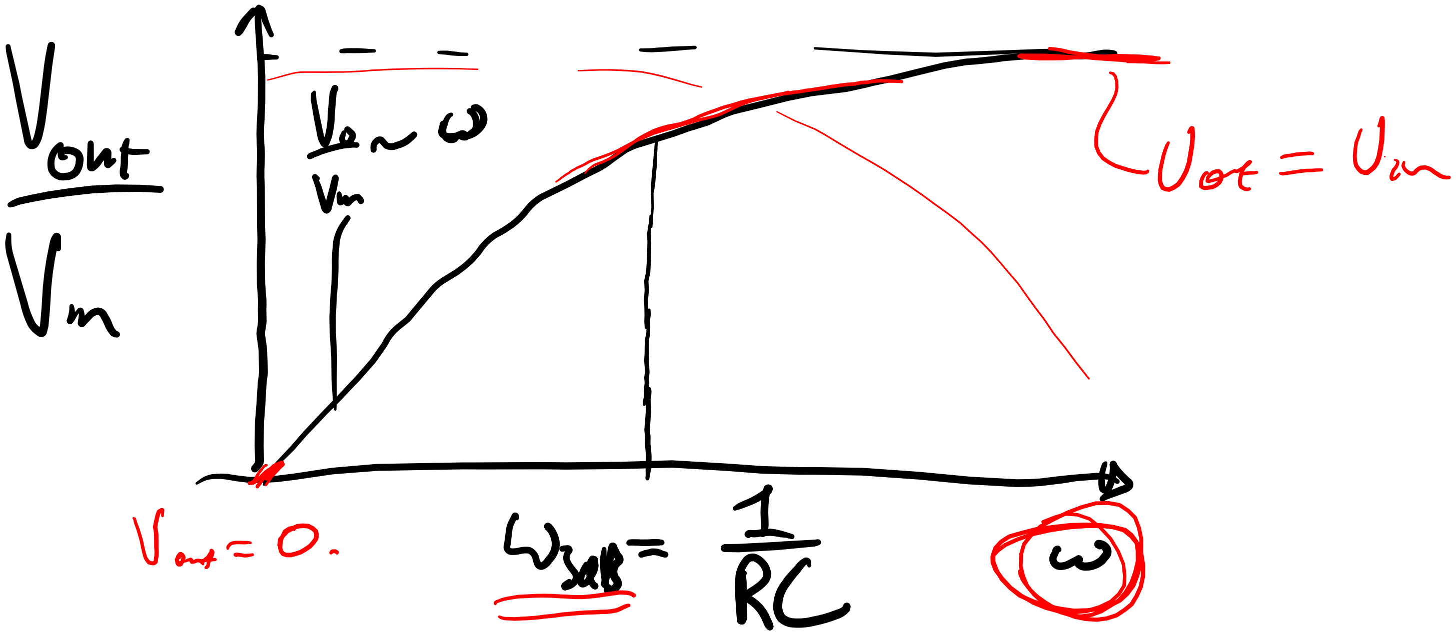
$$= \frac{V_{in}(R + j/\omega C)}{R^2 + (1/\omega C)^2}$$

$$\Rightarrow V_{out} = I Z_R = \frac{V_{in}(R + j/\omega C)R}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

Since we'll focus on amplitude, not phase, we'll work w/

$$|V_{out}| = (V_{out}^* V_{out})^{1/2} = \frac{R|V_{in}|}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^{1/2}} = \frac{R\omega C}{(1 + (R\omega C)^2)^{1/2}} |V_{in}|$$





* A note on dB: $1\,dB$ (decibel) $= 20 \log_{10} \frac{A_2}{A_1}$.

Thus, at ω_{3dB}: $\frac{V_o}{V_{in}} = \frac{1}{\sqrt{2}} \simeq 0.7$

$\omega_{3dB} \Rightarrow V_{out} \sim 0.7 \cdot V_{in}.$