ME 412: Experimental Methods in Engineering Mechanics

John M. Kolinski

PATT - EMSI

22.9.2021
[Discussion of the syllabus]
Module 1 in brief – analog electronics

• We need an analog front-end to make sense of any quantity we wish to measure. The analog / digitized signal can then be used for quantifying or control. This is **ubiquitous** in mechanical sciences; thus the motivation to introduce the basics of analog circuitry.

• We will probe the speed of a dynamic crack in a brittle solid:

![Diagram of a crack probing system with applied force, initial crack, and associated electronic components including an amplifier, 20 MHz digitizer, and personal computer.](image)
Modules 2 & 3

• Stretching of gels and bars: Use digital image correlation to measure constitutive law of hydrogel gel sample. UTMs in DLL will be used for the experiment. Process images with open-source NCORR software.

• Brownian motion & micro-rheology. Use Stokes-Einstein relation to measure viscosity from images of particles in solution. Become familiar with open-source particle tracking software package for Python / MATLAB
How will your grade be determined?

• For each of the 3 modules, your group will write a lab report within a group of 3-4 students; this will count for 80% of your grade. The peer assessment in modules 2 and 3 will be factored into this 80% for each module.

• Lab reports will be evaluated by other student groups as a team. The evaluation will count for 20% for modules 2 and 3; for module 1, the grading will be done by me.

• Contributions to the wiki, or experimental “Je ne sais quoi” in a given module can add up to 5% to your grade. These should be included in the appendix to the report.

• Lab groups will rotate groupings in each module.
Deliverables – lab reports for each module

• The lab reports will consist of a 4-5 page two-column document, with a focus on the scientific outcome of the experiment. Format should follow Physical Review Letters
• Figures and captions are an extremely important component to the report, and should be clear, legible, and concise.
• Writing should be clear and concise
• Exercises unrelated to the scientific results should be included in the appendix
• Wiki contributions should be included in the appendix and can receive additional credit toward the report grade
The format is simple: an abstract summarizes the results. The first paragraph is dedicated to relevant background information, and culminates in the unanswered question that the report will address. The next paragraph will then proceed to explain how the work contained in the report makes progress toward addressing the question. The remainder of the report is dedicated to presenting results. A brief discussion establishes context for the work.
A grading rubrick for grading the final reports:

1. Follows the PRL format (ca. 5% of total).
   • Title, abstract, authors list using revtex template for PRL. No more than 5 pages for the main report, including figures, etc.
   • 1-2 paragraph introduction for background, going from general to specific.
   • Conclude introductory paragraph with the open question you will address with the report.
   • Next paragraph introduces your method to address the question - `In this letter, we will measure... We simulate ... This paragraph should consist of a concise summary of your results.
   • Specifics of the measurement / simulation approach should be provided, and supported with figures that validate the chosen approach. This portion should be anywhere from 2-5 paragraphs and 1-3 figures.
   • Results should follow – what did you find, and why are you confident in your conclusions? Support with data in figures!
   • Discussion / conclusion – 1-2 paragraphs. Review what you learned, and broaden the discussion to how your measurement / simulation advances the state of the art.
   • References, formatted appropriately.
A grading rubrick for grading the final reports:

2. Figures clear, legible and appropriately captioned (ca. 30 % of the grade)
   • I should be able to understand what you did by reviewing only the figures and captions. They should be clear, legibly labelled, and completely captioned in concise text.
   • All data should be presented with as quantitative a treatment of experimental / simulation error as you can provide.
A grading rubrick for grading the final reports:

3. Writing is clear and concise (ca. 30% of total grade)
   • 5 pages with figures is SHORT! Don’t waste words …
   • Good writing: the process of eliminating *unnecessary words!*
   • Style and clarity count
A grading rubrick for grading the final reports:

4. ‘Je ne sais quoi’ – the effort of the measurement & attempts to reduce noise, and provide excellent, reliable measurements. (ca. 20+5% of total grade)
   • You’ve learned a lot of methods for reducing noise, and enhancing measurement precision.
   • Here is your opportunity to show off – how precise are your measurements? How do they stand in comparison with the state of the art?
   • How well have you understood and quantified the various sources of error in the measurement?
   • Have you provided sufficient detail in the supplementary document for your report to support your error quantification & technical approach, including details of your simulation?
   • You can also score points for identifying paths toward an improved measurement in the discussion – perhaps changing the actuator in some way, or enhancing the load cell’s precision, or changing the current range ... however you might go about it, if you were given another chance, more time and unlimited equipment, what would you do to improve your measurement?
A grading rubric for grading the final reports:

• 5. Appropriate use of statistics and error quantification (5%)
  • Use of error analysis where possible to generate error bars
  • Any discussion of error should distinguish between sources of error
  • Clear and consistent use of accuracy vs. precision
  • Clear and consistent recognition of experimental error and measured fluctuations of the physical phenomenon
  • Error propagation is carried out when relevant for derived quantities
5. Use of the scientific method (10%)
   • Science is inherently hypothesis driven – the rationale for a given hypothesis is presented in the introduction, leading to the open question the experiment intends to address
   • Confidence is built by hypothesis formulation and testing, which can rule out spurious / errant conclusions. This process should be described in the report or supplementary information.
   • Conclusions are well-supported by data, within measurement error.
   • Future prospects, prospective hypotheses suggested by the work and broad scientific outlook are presented in ‘discussion’ or ‘conclusion’ paragraph(s) in the report.

En.Wikipedia: Scientific_method
A summary of the rubrick for grading the final reports:

<table>
<thead>
<tr>
<th>Grading Category</th>
<th>% of report grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRL format (title, abstract, technical layout, etc.)</td>
<td>5</td>
</tr>
<tr>
<td>Figures clear, well-annotated and captioned</td>
<td>30</td>
</tr>
<tr>
<td>Clear, concise writing</td>
<td>30</td>
</tr>
<tr>
<td>Proper use of any statistical analysis / error quantification</td>
<td>5</td>
</tr>
<tr>
<td>Use of scientific method</td>
<td>10</td>
</tr>
<tr>
<td>Experimental “Je ne sais quoi” / wiki / etc.</td>
<td>20 + 5 (5 bonus)</td>
</tr>
<tr>
<td>Total</td>
<td>105 / 100</td>
</tr>
</tbody>
</table>
Good vs. Bad captions:

**Figure 1.** Radish plants showing the effects of freezing at -15°C for 2h (A1 and A2) compared with control plants (A4 and B3) kept at room temperature. The plants in pots A1 and A4 were cold acclimated for 2 days at 2.5°C prior to freezing or room temperature treatments. The plants in pots A2 and B3 were not cold acclimated and were kept at room temperature (~ 25°C) for 2 days prior to freezing or room temperature treatments. Following the freezing or room temperature treatments, plants were kept in a greenhouse for one week.

**Figure 1.** Radish plants subjected to a freezing treatment.
Contact hours & teaching staff

• The course is a 4 hour class. This means that you should budget 4 hours for lab activities on Wednesday morning, and an additional 4 hours off campus to complete the readings, and complete module reports.

• Ramin Kaviani will help to teach the class & attend ad-hoc office hours on Tuesday afternoons.
<table>
<thead>
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Crack speed – dynamic fracture testing using analog electronics


ME 412
John M. Kolinski
Equipment for analog electronics module

You’ll be using the lab benches in MED 2 2419. The lab bench layout is standard at each desk, and includes a 3-channel power supply, a function generator, two multi-meters and an oscilloscope (see the layout at right).

Also available are a standard breadboard and jumper wire, as well as BNC connectors (not shown) and banana plug connectors.

Circuit components are found near the entrance to the room, and include all components from passives (resistors, capacitors, inductors and diodes) to actives, up to and including the op-amps and precision voltage references.

On day one of your on campus presence, you’ll form your group for the semester by our algorithm. Groups are formed at random.
How do things Break?

The Griffith approach (1922) –

Materials break when there is enough *elastic energy* ($J/m^2$) to drive a crack

Number of bonds per unit area ($\sim 10^{20} m^{-2}$)  
Bond energy ($\sim 10^{-19} J$)  
Fracture energy (1-10 J/m$^2$)

The energy supply comes from the elastic energy stored in the solid – due to the applied stress

$\Rightarrow$ Fracture is a matter of energy balance

Elastic energy stored in the sample $\equiv$ dissipation / surface energy

$G = \Gamma$
How do things Break?

Linear Elastic Fracture Mechanics (LEFM) 1950’s -

**Cracks** create **singular stresses** at their tip

As $r \to 0$, the stress **diverges** (Freund, 1990)

\[
\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta, v) + O(1)
\]

\[
K = \text{stress intensity factor} \ (\sim \sigma_\infty \sqrt{l})
\]

\[
f_{ij}(\theta, v) \ - \text{dimensionless functions of angle and crack speed}
\]
There are 3 fracture modes: characterized by the symmetry of the loading on the crack plane.

\[ \sigma_{ij} = \sum_{\alpha=1}^{\text{III}} \frac{K_\alpha}{\sqrt{2\pi r}} f_{ij}^{\alpha}(v, \theta) \]

Because the stress is singular, there is no a-priori path selected for a crack.

Our applied loading is tensile loading. But remember ....
How do things Break?

**LEFM predictions for dynamic cracks**

- Stresses diverge asymptotically close to the crack tip:
  \[ \sigma \sim \frac{K}{\sqrt{r}} \]  
  (Irwin '56)

- The “stress intensity factor” \( K \) can be computed or measured:
  \[ K \sim k(v)\sigma_{\infty}\sqrt{l} \]

- The elastic energy flux \( G \) is related to \( K \) by:
  \[ G \sim \frac{A(v)K^2}{E} \sim \frac{g(v)\sigma_{\infty}^2l}{E} \]

  - Freund (1990)
  - Eshelby '72
  - Kostrov '75

\( C_R \) - Rayleigh wave velocity – the speed of elastic shear waves at a free surface
The background: LEFM predicts a crack tip equation of motion - $\ell(v)$

The crack tip’s velocity increases as it propagates into the sample.

This is a consequence of more elastic energy flowing to the crack tip as it propagates.

Such behavior is predicted by Linear Elastic Fracture Mechanics Theory.
Testing dynamic fracture mechanics using gels

A simple crack in an infinite medium and constant stress, \( \sigma_\infty \), using \( G = \Gamma \):

\[
\Gamma = G(v, l) \approx \frac{6}{E \pi} \sigma_\infty^2 \left( 1 - \frac{v}{c_R} \right)
\]


Linear Elastic Fracture mechanics works!
The experimental set-up:

Here we see a Wheatstone bridge configuration that is used with a high-speed digitizer to record at 20 MHz the resistance of a thin coating of aluminum applied to brittle plastic.

As the crack propagates, the cross-section of the aluminum coating reduces, increasing the resistance measured across the sample – recall our expression for resistivity, where $R = \rho \frac{\ell}{A}$. Here, $A$ reduces as the crack propagates. However, the coating thickness and gauge of the resistor are held constant; thus, the measured resistance is inversely proportional to the coating’s unruptured length.

FIG. 1. Schematic representation of the experimental apparatus: (a) velocity measurements; the path of the crack (dotted line) and its direction of propagation are indicated. (b) Surface profile measurements.
Profilometry indicates a non-smooth crack surface

Using the mechanical profilometer, a 3D representation of the crack’s surface is obtained. It is clear that whereas the crack initially propagates smoothly, *sharp features emerge* as the crack progresses further.

These features could be important in understanding the observations of crack tip velocity ...
Crack tip speeds: the measurement and the prediction

Here we see a huge departure between theory and experiment: In the inset, the predicted $v(\ell)$ curve is plotted.

The experimental data, however, look nothing like this (!) There are all sorts of spikes, and acceleration that is observed. Nevertheless, we see the crack accelerate on average.
Cross-correlation of velocity with surface profilometry indicates that the surface roughness features occur in tandem with velocity fluctuations.

In (a), we see the correlation of the velocity with the surface profile in the `noisy’ region. Clearly the surface features and velocity are well-correlated at a timescale of a few microseconds.

The frequency spectrum of the surface features at two different velocity windows. At top, the mean velocity is 3/5 what it is at the bottom, indicating that the frequency of the observed surface features is velocity-independent.
The velocity at the onset of oscillations is singly-valued, indicating a critical transition. Here we see that the velocity immediately prior to the onset of $v$ fluctuations is nearly constant, around 320 m/sec, as the applied stress is varied (a longer run-out to reach a given velocity). This suggests that the transitional velocity is a uniquely valued, and indicates a critical transition to unstable crack propagation.
I. Rationale:
Experimental measurement is the cornerstone of all scientific progress. Being able to design, construct apparatus, carry out measurements and interpret data is essential to scientific and engineering endeavors. In the field of mechanics, there are some experimental methods that have proved their utility, including the use of electronics in experiment, and imaging. This class provides a broad introduction to these methods with specific classical examples in three modules. Through these examples, students will develop a foundation for future research and development in applied mechanics.

II. Course Aims and Outcomes:
Aims
At the end of this class, you should be comfortable identifying an experimental method to measure a given mechanical quantity. Furthermore, you will become familiar with the three experiments and the science behind them. You will be required to seek out information in the course of each module, and this should prepare you to face challenging, open-ended experimental problems in engineering or science.

Specific Learning Outcomes:
Students who have taken this class can expect
- to understand enough analog electronics to build simple circuits using passive and active components with minimal assistance from e.g. Google
- to read a manual or data sheet and then successfully use the IC or apparatus
- to understand the basics of imaging and geometric optics
- to understand the construction and essential elements of a microscope
- to use image processing to extract data from digitally recorded images
- to learn about the mechanics of fracture and Brownian motion
- to be familiar with particle tracking and image correlation methods

III. Format and Procedures:
The course structure is unconventional, so please ask questions if you do not understand something. Warning: this is not a lab class where I provide you with a recipe. You will be required to identify problems, and take the appropriate actions to address them, often without substantial input from the teaching staff. Part of the process of science and engineering is collaborative effort to solve open-ended
problems. Not all students appreciate this course structure – if you don’t think you will like this course structure, I encourage you to consider the other great course offerings at EPFL instead of EMEM.

Each module will be introduced with a preliminary lecture, where the scientific motivation for the measurement and experimental approach are introduced. Some guided laboratory exercises will provide a platform for engagement with the necessary experimental apparatus. Following these preliminary lab exercises, each experimental group will proceed to construct some components of the experiment, calibrate the constructed components, record data and compile the laboratory procedure and results into a report.

Experimental groups will comprise three students each, for a total of approximately 10 groups. Groups will prepare a report to be assessed after the module. Contributions to the course wiki should be included in the appendix, and will be assessed with the report. Wiki contributions can only enhance the grade of the report, and will not detract from it.

I will grade the first reports to provide an example of the standards expected in a report, and the two subsequent modules will be graded by other groups, whose assessment I will use in determining the grade. As long as the assessment is not capricious, the peer assessment will be used to determine the grade for the report. The assessment of the other group’s report will comprise a portion of your grade for the report on the respective module.

In each module, groups must rotate so that each individual student is working with students they have not worked with in a previous module.

**Grading:** lab reports form the bulk of the grade, with each module weighted equally. For modules 2 and 3, the grade will consist of 80% of the composite assessment (mine and the evaluation of another group), and 20% of your grade will be based on your group’s *assessment* of the peer report. Contributions to the wiki can add up to 5% toward the report grade.

The format for the lab reports should follow the 2-column APS letter format (see [https://journals.aps.org/prl/authors](https://journals.aps.org/prl/authors) for technical formatting guidelines), with a focus on concise explanations, and an intense focus on the results of the experiment. All technical details concerning the lab exercises and contributions to the wiki can be submitted as an appendix, or as a second document. Close attention must be paid to **figure presentation and captioning**, as this is a critical aspect of clearly communicating your results from the experiment. Bullet points describing expectations are as follows:

- Figures should be clear, with legible axis labels and legends.
- Figure captions should be complete and concise – all aspects of the figure should be explained, but excess verbiage is to be avoided.
- Writing style should be focused in concise and clear exposition.
- Rigor in the scientific approach, including hypothesis formulation, hypothesis testing, development of results with data to support the conclusions drawn from the experiment, and contextualization of the result in a brief discussion convey the rigor of the scientific approach to each experiment.

Expectations of group reports for each module:
- The report on the final measurement will follow the format of a PRL paper. Some rough guidelines on this writing style can be found here: [https://www.asc.ohio-state.edu/wilkins.5/onepage/prl.html](https://www.asc.ohio-state.edu/wilkins.5/onepage/prl.html). I would encourage more a focus on the overall spirit of the guidelines (points 1, 2, 8 and 9) are the most important. Having a precise line count is less important, but any text on page 5 is too much text. A supplementary document or appendix including technical details, materials and methods and calibration experiments is allowed, but should only be used to fill in the gaps in the paper.
- The grading breakdown, briefly, is:
IV. Background for the class:
This course assumes a strong background in fluid and solid mechanics, as well as familiarity with electronics and associated apparatus.

V. Course Resources: I encourage you to take good notes during introductory lecture for each module, and maintain a running lab book with figures and illustrations. These will make the preparation of the lab reports much easier.

Selected readings will be provided each week on the wiki

The wiki forms a core component of the class. Here, explanations of a particularly helpful procedure, or extra information that you found useful should be posted. Contributions to the wiki should be carefully documented and presented to increase their utility. Wiki contributions should be included with lab reports in the appendix, and can be counted as additional credit for the lab report.

- An incomplete list of course readings:
  - The art of electronics by Horowitz and Hill
  - Theory of Elasticity by Landau and Lifshitz
  - Fluid Mechanics by Landau and Lifshitz
  - Dynamic Fracture Mechanics by L.B. Freund
  - …. Selected papers to be posted to the Moodle
  - …. Data sheets for pertinent op-amps, etc.

VI. Academic Integrity
As laboratory reports constitute the bulk of the assessment, a high standard for proper citation practice is essential. Any deliberate plagiarism will result in a failing grade for the report. You will not lose credit for proper attribution; on the contrary, proper attribution will enhance the clarity of the report and likely increase your grade.

VII. Students with disabilities
In compliance with the EPFL LEX 2.6.5, I am available to discuss appropriate academic accommodations that may be required for student with disabilities.

IX. Tentative Course Schedule: (Subject to change)

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<tr>
<td>Week</td>
<td>Block</td>
<td>Activity/Assigned reading</td>
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### X. Covid-specific measures

As a class, we will abide by all Covid-related regulations at EPFL in EMEM. This currently means that you will require a Covid certificate to participate in the class. Given the group-work nature of the class, I will be checking certificates periodically, until I get to know you. The Covid pandemic is not over yet – and there are virus variants that are highly contagious, and furthermore have proven infectious even to the vaccinated. Thus I request that you obtain FFP2-level or higher respiratory protection. Furthermore, we will likely take periodic breaks outside to relieve the stress of wearing a mask at all times.
Analog Electronics I: resistors & V-dividers
pg. 1-44 in Harowitz & Hill, The Art of Electronics.

2 Fundamental laws:
- Ohm's law
- Kirchoff's laws

\[ V = IR \]  

Voltage (potential)  
Flow of e-

Analogy to hydrodynamics:
\[ H = J \cdot R \]  
Hydrostatic pressure  
Fluid flow  
Hydraulic resistance

"A volt pushes an amp through an ohm."

Voltage is measured relative to a reference potential.

"Ohmic" vs. "non-Ohmic" behavior

\[ V \]  
\[ I \]  
Ideal current source  
Diode behavior
Power: \[ P = IV = I^2R \]

- \( I \): current, change/time
- \( V \): work or energy/charge

\[ [IV] \frac{\text{work}}{\text{time}} = \frac{\text{energy}}{\text{time}} = [P] \]

**Kirchhoff's Laws:**

**I:**
- Sum of voltages around a closed loop is 0.

**II:**
- Sum of the currents at a node is 0.

\[ I_1 + I_2 + I_5 + I_4 + I_3 = 0 \]

**Examples:**

- **Series Resistance:**
  \[ V \]
  \[ I = I_1 = I_2 \text{ by KCL} \]
  \[ V_{\text{tot}} = V_1 + V_2 \text{ (KVL)} \]

- **Equivalent Resistance:**
  \[ V_{\text{tot}} = I_t R_{\text{eq}} = I_t R_1 + I_t R_2 \]
\[ V_{tot} = I_{tot} R_{eq} = I_{tot} R_1 + I_{tot} R_2 \]

\[ R_{eq} = R_1 + R_2 \]

**Parallel Resistors:**

\[ I_{tot} = I_1 + I_2 \quad \text{(k2)} \]

\[ V_{tot} = V_1 = V_2 \quad \text{(k1)} \]

**Equivalent R:**

\[ V_{tot} = I_{tot} R_{eq} = V_1 = U_2 \]

\[ I_{tot} R_{eq} = I_1 R_1 = I_2 R_2 \]

\[ I_{tot} = I_1 + I_2 \quad I_1 = \frac{V}{R_1} \quad I_2 = \frac{V}{R_2} \]

\[ I_{tot} = \frac{V}{R_{eq}} \]

\[ \frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2} \]

\[ R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \]

**Consider these cases:**

\[ R || R = \frac{R \cdot R}{R + R} = \frac{R}{2} \]

\[ \frac{1}{\frac{1}{R} + \frac{1}{2R}} = \frac{2R}{3R} = \frac{2}{3}R \]

\[ \frac{1}{\frac{1}{R} + 2R} = \frac{R}{1 + 2R} \]

\[ 10 \cdot \frac{R}{2R} = \frac{10R}{2} = 5R \]
R_{eq} = 2R

\frac{R+10R}{10R} \quad \frac{10R}{R} \quad R_{eq} = 11R \approx 10R.

V - divider (a device)

\begin{align*}
I &= \frac{V_m}{(R_1 + R_2)j} \quad V_{out} = IR_2 \\
V_{out} &= \frac{V_m R_2}{R_1 + R_2} = \frac{V_m}{2}
\end{align*}

Using device:

\begin{align*}
3V & \quad I = \frac{1}{2} \quad 10k \quad 10k \quad 10k \quad V_{out} \quad 10k \quad \text{"Resistor load"} \quad 100k \\
30k & \quad 40k \\
\end{align*}

V_{op} = \frac{V}{3} \\
R_0 = 10k \quad 100k = 5k \quad 10k
Thevenin equivalent circuits

- For any two nodes in a circuit, irrespective of the circuit's complexity (only linear circuit elements @ steady-state):
  An equivalent circuit comprised of a voltage source $V_{Th}$ and a resistance $R_{Th}$ can be found.

$V_{Th}$ & $R_{Th}$ can be found in 2 steps:

1. $V_{Th} = V$ (open circuit) - no load!
2. $R_{Th} = \frac{V_{Th}}{I_{Th}}$ (short circuit)

$R_{Th}$ is the output impedance of such a circuit.
The voltage divider revisited: the Thevenin equivalent circuit and output impedance

Let’s revisit the humble voltage divider circuit from last week:

1. \( V_{\text{open circuit}}: V_o = \frac{V_m R_2}{R_1 + R_2} = V_{\text{Th}}. \)

2. \( R_{\text{Th}} = \frac{V_m}{I_{\text{short}}} \), if \( R_L = 0 \Omega \), a short circuit is achieved.

   If this circuit is shorted, we find:

   \[ I = \frac{V_m}{R_1}. \]

   Thus, \( R_{\text{Th}} = \frac{V_m}{I} = \frac{R_1 R_2}{R_1 + R_2} \).

   Note, this is \( R_1 R_2 \).
Reactive circuit elements & transient circuit response: capacitors

A capacitor with capacitance $C$ stores charge ($Q$) in proportion to the applied voltage $V$. This can be expressed with an equation as:

$$Q = CV$$

$Q$ is hard to quantify or measure in practice; instead, we reformulate this expression for current $I$ to make it more manageable, by taking the time derivative of both sides:

$$\frac{dQ}{dt} = C \frac{dV}{dt} \rightarrow I = C \frac{dV}{dt}$$

Since $I$ now depends on the temporal behavior of $V$, we call capacitance a `reactive' circuit element -> current `reacts' to changes in voltage. A natural question is how this manifests itself with a simple circuit: let us look at the transient response of a simple circuit:

$$I = \frac{V_i - V(t)}{R} = \frac{C}{R} \frac{dV}{dt}.$$
We can find $A$ by considering the initial condition:

$V(0) = 0 = V_i + Ae^0 \Rightarrow A = -V_i.$

Thus, $V(t) = V_i (1-e^{-t/RC}).$
A/C circuit analysis – looking at sinusoidal* signals

Since our circuits at this stage are Linear, the output of a sinusoidally driven circuit is itself a sinusoid with the same frequency and at most a modified amplitude and phase.

Sinusoids are parameterized by an amplitude and a phase – this structure lends itself well to a description with complex variables, which also have two quantities of merit – the real and imaginary part. The phasor description, with e.g. $V = V_0 e^{j\varphi} = V_0 (\cos \varphi + j \sin \varphi)$; for a signal oscillating with frequency $\omega$, the phase of this signal is $\omega t$.

* For the math experts, the Fourier series is a sum of sinusoids, and with appropriate weights can approximate any periodic function over an interval. Thus analyzing the response to an arbitrary sinusoid, and the linear property of RLC circuits means we can describe the circuits response to an arbitrary forcing.
Generalizing resistance: Impedance and the transfer function for ‘forcing’ (V) to ‘response’ (I)

Let’s first consider a sinusoidal voltage applied to a capacitor:

\[ V(t) = V_0 \sin(\omega t) \]

Here, \[ I = C \frac{dV}{dt} = C \omega V_0 \cos(\omega t) \]

Neglecting the phase, we can obtain the current as:

\[ I = \frac{V}{\frac{1}{\sqrt{\omega C}}} \]

Thus, the capacitor behaves like a \( \omega \)-dependent resistor!

To account for phase correctly, we’ll write \[ V = \text{Re} (V_0 e^{j\omega t}) \].

Since \[ I = C \frac{dV}{dt}, \] \[ I(t) = \text{Re} (j\omega C V_0 e^{j\omega t}) = \text{Re} \left( \frac{V_0 e^{j\omega t}}{-\frac{1}{\sqrt{\omega C}}} \right) \].
Thus, we define a reactance $X_c$ for the capacitor:

$$X_c = \frac{V}{I} = -\frac{j}{\omega C}$$

(n.b. this is structurally similar to $R$ from Ohms law! $R = \frac{V}{I}$)

We can now define a generalized impedance, $Z$, which in general is complex:

$$I = \frac{V}{Z} \quad \Rightarrow \quad V = IZ.$$

$Z$'s follow $R$'s rules for addition in $\| \& \|$ series:

$$Z_{eq} = Z_1 + Z_2 + Z_3.$$
In II: \[ Z_\text{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} \]

Generalized impedance for passive components:

\[ Z_R = R \text{ (resistors)} \]

\[ Z_L = j\omega L \text{ (inductors)} \]

\[ Z_C = -\frac{1}{j\omega C} \text{ (capacitors)} \]
Generalizing the V-divider: frequency dependent response and filter devices

- The analysis is essentially the same as before:
  \[ V_{out} = I Z_2 = \frac{V_{in} Z_2}{Z_1 + Z_2} \]

- \( I = \frac{V_{in}}{Z_{tot}} \)

- But now, \( Z \) is in general complex—and thus \( f \)-dependent!

Low-pass filter: \[ \text{R} \text{-} \text{C} \text{-} \text{V} \]

High-pass filter: \[ \text{R} \text{-} \text{C} \text{-} \text{V} \]

Band-pass filter: \[ \text{R} \text{-} \text{C} \text{-} \text{V} \]
Analysis of the high-pass filter

By generalized Ohm's law: \[ I = \frac{V_{\text{in}}}{Z_{\text{tot}}} = \frac{V_{\text{in}}}{R - j\omega C} = \frac{V_{\text{in}} (R + j\omega C)}{R^2 + (j\omega C)^2} \]

\[ \Rightarrow V_{\text{out}} = I Z_R = \frac{V_{\text{in}} (R + j\omega C) R}{R^2 + (j\omega C)^2} \]

Since we'll focus on amplitude, not phase, we'll work w/ \[ |V_{\text{out}}| = (V_{\text{out}}^2 + V_{\text{out}}) \frac{1}{2} = \frac{R V_{\text{in}}}{R^2 + (\frac{1}{\omega C})^2} = \frac{R \omega C V_{\text{in}}}{(1 + (R\omega C)^2)^{1/2}} \]