ON SAINT-VENANT'S PRINCIPLE

Author(s): E. STERNBERG

Source: *Quarterly of Applied Mathematics*, JANUARY, 1954, Vol. 11, No. 4 (JANUARY, 1954), pp. 393-402

Published by: Brown University

Stable URL: https://www.jstor.org/stable/43634082

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms

is collaborating with JSTOR to digitize, preserve and extend access to *Quarterly of Applied Mathematics*

JSTOR

This content downloaded from 2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

ON SAINT-VENANT'S PRINCIPLE*

BY
E. STERNBERG
Illinois Institute of Technology

Introduction. The principle bearing his name was introduced by Saint-Venant [1][1] in connection with, and with limitation to, the problem of extension, torsion, and flexure of prismatic and cylindrical bodies. The first universal statement of the principle is apparently due to Boussinesq [2], and reads:[2] "An equilibrated system of external forces applied to an elastic body, all of the points of application lying within a given sphere, produces deformations of negligible magnitude at distances from the sphere which are sufficiently large compared to its radius." Love [3] writes:[3] "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

As pointed out by v. Mises [4], the foregoing statements are in need of clarification[4] since the forces applied to a body at rest must be in equilibrium in any event. Only when the body extends to infinity, and provided we require the tractions at infinity to vanish suitably, is it meaningful to speak of the strains "produced" by a non-equilibrated system of forces applied to a bounded part of its surface. Moreover, in this instance, the strains produced by a given loading are arbitrarily small at points sufficiently far removed from the region of load application, regardless of whether or not the loading is self-equilibrated.[5] On the other hand, the stresses and strains at a fixed point of an elastic body, in the absence of body forces, may be made arbitrarily large or small by choosing the magnitude of the loads sufficiently large or small. These observations further confirm the need for clarification.

What is meant by the statements cited may roughly be expressed as follows:[6] if the forces acting on an elastic body are confined to several distinct portions of its surface, each lying within a sphere of radius ϵ, then the stresses and strains at a fixed interior point of the body are of a smaller order of magnitude in ϵ as $\epsilon \to 0$ when the forces on each of the portions are in equilibrium than when they are not. In this comparison we must evidently assume that the forces remain bounded as $\epsilon \to 0$. The analogous interpretation for distributed surface tractions is immediate.

It should be noted that such an interpretation is implied in the usual applications of Saint-Venant's principle. Moreover, that this is what Boussinesq had in mind is apparent from his efforts to justify the principle. With this objective, Boussinesq [2] considered

*Received January 5, 1953.

[1]Numbers in brackets refer to the bibliography at the end of this paper.

[2]See [2], p. 298. ("Des forces extérieures, qui se font équilibre sur un solide élastique et dont les points d'application se trouvent tous à l'intérieur d'une sphère donnée, ne produisent pas de déformations sensibles à des distances de cette sphère qui sont d'une certaine grandeur par rapport à son rayon.")

[3]See [3], p. 132.

[4]See also, for example, Biezeno and Grammel [5], where the traditional statement of the principle is discussed in detail.

[5]See the general solution to the problem of a semi-infinite medium bounded by a plane, [3], art. 166.

[6]This interpretation follows v. Mises [4].

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

a semi-infinite body under concentrated loads acting perpendicular to its plane boundary. He showed that if the points of application of the loads lie within a sphere of radius ϵ, the stresses at a fixed interior point of the body are of the order of magnitude ϵ provided the resultant force is zero, and of the order ϵ^2 in case the resultant moment also vanishes. Various energy arguments have since been advanced in support of the principle.[7]

In 1945 v. Mises [4], in his illuminating paper on this subject, showed with the aid of two specific examples that the usual statements of the principle, when properly clarified, cannot be valid without qualifications. The two examples chosen by v. Mises are the three-dimensional problem of the half-space and the plane problem of the circular disk, each under concentrated surface loads.[8] On the basis of these examples v. Mises proposed an amended principle.

It is the purpose of this paper to supply a general proof of Saint-Venant's principle as modified by v. Mises. The argument is carried on for the case of piecewise continuous tractions and is later extended to concentrated forces; it applies to finite and infinite domains of arbitrary connectivity.[9]

The dilatation formula of Betti. As a preliminary to the proof, we recall here a formula due to Betti,[10] which is a consequence of Betti's reciprocal theorem. Let D be a regular[11] (not necessarily simply connected) region occupied by an elastic medium, and let B be the boundary of D (Figure 1). Furthermore,[12] let $\mathbf{u}$, e_{ij}, and τ_{ij} be a displacement field, a field of strain, and a field of stress which within D satisfy the fundamental field equations of the linear theory of elasticity in the absence of body forces. If τ_{ij} gives rise to piecewise continuous surface tractions $\mathbf{T} = [X_1, X_2, X_3]$ on B, then the dilatation $\Delta^Q = e_{ii}^Q$ at a fixed interior point $Q(\xi_1, \xi_2, \xi_3)$ of D is given by,[13]

$$c\Delta^Q = \int_B \mathbf{T} \cdot \mathbf{g}\, dB, \qquad c = \frac{8\pi(1-\nu)\mu}{1-2\nu}, \tag{1}$$

where μ and ν are the shear modulus and Poisson's ratio, respectively. In (1), $\mathbf{g}$ is a displacement field which is defined as follows:

$$\mathbf{g} = \mathbf{g}' + \mathbf{g}'', \qquad \mathbf{g}' = -\operatorname{grad} R^{-1}, \qquad R = |\mathbf{R}|, \tag{2}$$

where $\mathbf{R}$ is the position vector with respect to Q of a point $P(x_1, x_2, x_3)$ of D (Figure 1). Moreover, $\mathbf{g}''$ is that displacement field which satisfies the equilibrium equations within D and gives rise to surface tractions on B which are equal and opposite to those associated with $\mathbf{g}'$. Thus $\mathbf{g}$ is characterized by the requirements that (a) it satisfy the equilibrium conditions inside D with the exception of the point Q where it must have the singularity appropriate to a center of dilatation, and (b) its associated surface tractions vanish on B. We emphasize, for future reference, that $\mathbf{g}$ is an analytic function of position on any analytic portion of B. Betti's formula remains valid if D is not bounded, provided that $R^2\tau_{ij} \to 0$ as $R \to \infty$.

---

[7]See References [6] to [11] and [18].

[8]More recently, Erim [12], applied v. Mises' analysis to the half-plane under concentrated loads.

[9]The restriction that the region be simply connected turns out to be unessential.

[10]See [3], p. 235.

[11]The term "regular region" is used in the sense of Kellogg [13], pp. 113, 217.

[12]Throughout this paper letters in boldface designate vectors; the subscripts i, j assume the values 1, 2, 3, and the usual summation convention is employed.

[13]The symbols $\cdot$ and $\times$ designate scalar and vector multiplication of two vectors, respectively.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

The vector field **g** plays a role analogous to that of Green's function in potential theory. Equation (1) reduces the determination of the dilatation in the second boundary-value problem (surface tractions prescribed) to the determination of **g**″. We note that the function **g** is completely characterized by the shape of D and the location of Q, and is independent of the surface tractions **T**. This observation provides the key to the subsequent proof of the modified Saint-Venant principle in which we are confronted with the task of comparing the effects of certain changes in the loading upon the de-

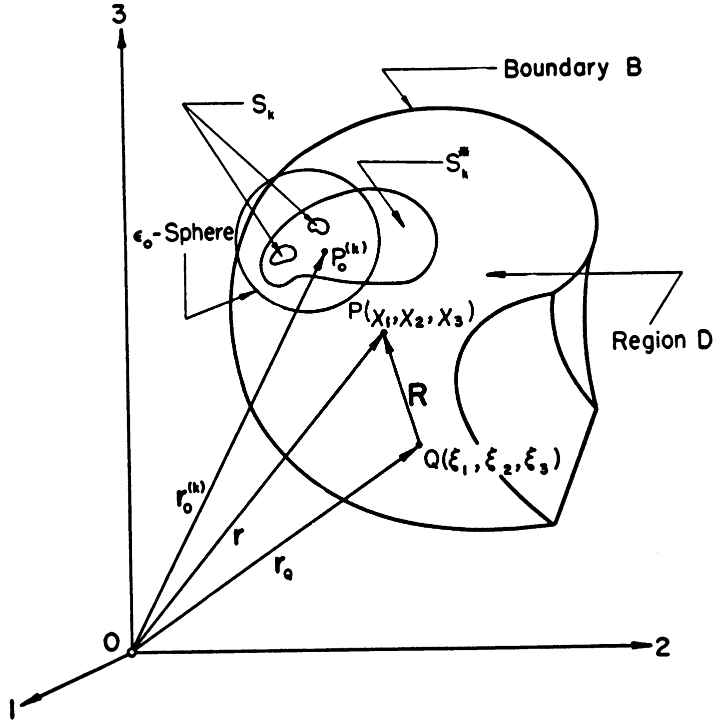

Fig. 1.

formations and stresses at an interior point of the body. Before turning to our main objective, we consider two specific applications of Betti's formula.

The sphere and the half-space as examples. Let D be a sphere with radius a and let Q be its center. Here, trivially,

$$\mathbf{g}'' = \frac{2(1-2\nu)\mathbf{R}}{(1+\nu)a^3}, \tag{3}$$

and by (1), (2), (3),

$$\Delta^Q = \frac{3(1-2\nu)}{8\pi(1+\nu)\mu a^3} \int_B \mathbf{T} \cdot \mathbf{R} \, dB. \tag{4}$$

If the loading, in particular, consists of two equal and opposite concentrated forces, each of magnitude L, applied at the endpoints of a diameter and directed toward Q, a trivial limit process applied to (4) at once yields,

$$\Delta^Q = -\frac{3(1-2\nu)L}{4\pi(1+\nu)\mu a^2}. \tag{5}$$

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

This formula was derived by Synge [14], by Weber [15], as well as by F. Rosenthal and the present author [16], in each case by entirely different means.

Next, let D be the half-space $x_3 \geq 0$ and B the plane $x_3 = 0$. In this special instance a closed representation of $\mathbf{g}$ is available[14] corresponding to any interior point $Q(\xi_1, \xi_2, \xi_3)$. Indeed, let

$$\phi = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 + \xi_3)^2]^{-1/2}, \tag{6}$$

so that ϕ is the reciprocal of the distance between $P(x_1, x_2, x_3)$ and the mirror image of Q in the plane $x_3 = 0$. Then, as is readily verified,

$$\mathbf{g}'' = -(3 - 4\nu)\left[\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_3}\right] - 2x_3 \operatorname{grad} \frac{\partial\phi}{\partial x_3} \tag{7}$$

and, according to (2), (6), (7), on the plane $x_3 = 0$,

$$\mathbf{g} = -4(1 - \nu)[\operatorname{grad} R^{-1}]_{(x_3=0)} . \tag{8}$$

Equations (1), (8) imply,

$$\Delta^Q = -\frac{(1 - 2\nu)}{2\pi\mu} \int_B \mathbf{T} \cdot \operatorname{grad} R^{-1} \, dB. \tag{9}$$

An elementary limit process applied to (9) yields the formula cited by v. Mises[15] for the dilatation at Q induced by a concentrated load acting at the origin $x_i = 0$.

Proof of the modified Saint-Venant principle. Let D, with the boundary B, again be a regular region[16] of space of arbitrary connectivity. Let $S_k(k = 1, 2, \cdots m)$ be m non-intersecting closed subregions of B which lie within neighborhoods of m distinct points $P_0^{(k)}$ (position vectors $\mathbf{r}_0^{(k)}$) of B, each S_k, $P_0^{(k)}$ being wholly contained within a sphere of radius ϵ_0 (Figure 1). We note that S_k need not be simply connected or even connected.

Let $\mathbf{u}$, e_{ij}, τ_{ij} be a solution in D of the field equations of elasticity theory, the body forces being absent, which corresponds to piecewise continuous surface tractions $\mathbf{T} = [X_1, X_2, X_3]$ on B.[17] Moreover, let $\mathbf{T}$ vanish on B with the exception of the subregions S_k.

According to Betti's formula (1),

$$\Delta^Q = \sum_{k=1}^{m} \Delta^Q[S_k], \qquad c\Delta^Q[S_k] = \int_{S_k} \mathbf{g} \cdot \mathbf{T} \, dB \tag{10}$$

where $\Delta^Q[S_k]$ represents the "contribution" from the tractions on S_k to the dilatation Δ^Q at a fixed interior point Q of D. We observe that a $\Delta^Q[S_k]$ possesses individual physical significance only if the tractions on S_k are self-equilibrated, unless D extends to infinity. Furthermore, if D is bounded, either the tractions on each S_k are self-equilibrated, or there are at least two S_k for which this is not true.

We now examine $\Delta^Q[S_k]$ and, for the sake of convenience, henceforth write S, P_0, $\mathbf{r}_0$

[14]This result is apparently due to Cerruti. See [3], p. 239.

[15]See Equation (4) of [4].

[16]See Footnote 11. Again D need not be bounded.

[17]If D is not bounded, we require $r^2\tau_{ij} \to 0$ as $r \to \infty$, where $r = |\mathbf{r}|$, and $\mathbf{r}$ is the position vector of a point P of D.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

in place of S_k, $P_0^{(k)}$, $\mathbf{r}_0^{(k)}$. Let S and P_0 be contained in an open simply connected subregion S^* of B (Figure 1) which admits a parametrization of the form,[18]

$$\mathbf{r} = \mathbf{r}(\alpha, \beta), \qquad \mathbf{r}_\alpha \times \mathbf{r}_\beta \neq 0, \qquad (\alpha, \beta) \quad \text{in} \quad \Sigma^* \tag{11}$$

where $\mathbf{r} = [x_1, x_2, x_3]$ here is the position vector of a point P of S^*, Σ^* is an open simply connected region of the (α,β)-plane, and $\mathbf{r}(\alpha, \beta)$ is assumed to be at least twice continuously differentiable in Σ^*. Thus B is assumed to have finite and continuous curvatures in S^*. The mapping (11) defines a regular curvilinear coordinate net on S^*. It is convenient to require that

$$\mathbf{r}_0 = \mathbf{r}(0, 0). \tag{12}$$

Finally, suppose that the closed subregion Σ of Σ^* is the antecedent in the (α,β)-plane of the subregion S of S^*.

From (10), by virtue of the regularity of $\mathbf{g}$ on B, we have, on expanding $\mathbf{g}(\alpha, \beta)$ in a Taylor series (possibly with a remainder term) at (0, 0),[19]

$$c\Delta^Q[S] = \mathbf{g}^0 \cdot \int_S \mathbf{T}\, d\sigma + \mathbf{g}_\alpha^0 \cdot \int_S \mathbf{T}\alpha\, d\sigma + \mathbf{g}_\beta^0 \cdot \int_S \mathbf{T}\beta\, d\sigma + \cdots \tag{13}$$

where,

$$\int_S d\sigma = \int_\Sigma |\, \mathbf{r}_\alpha \times \mathbf{r}_\beta \,|\, d\alpha\, d\beta. \tag{14}$$

Before drawing any general conclusions, let us apply (13) to the example of the half-space. Here D is the region $x_3 \geq 0$ and B the plane $x_3 = 0$. We may choose the origin $x_i = 0$ at P_0, employ B as S^*, and adopt the parametrization,

$$\alpha = x_1, \qquad \beta = x_2. \tag{15}$$

By (8), in this instance,

$$\left.\begin{aligned} \mathbf{g}(\alpha, \beta) &= -\frac{4(1-\nu)}{R^3}[\xi_1 - \alpha, \xi_2 - \beta, \xi_3] \\ R &= [(\xi_1 - \alpha)^2 + (\xi_2 - \beta)^2 + \xi_3^2]^{1/2} \end{aligned}\right\} \tag{16}$$

Substitution of (16) into (13) yields, except for differences in notation and a constant factor, the expansion derived by v. Mises by other means, provided the integrals in (13) are replaced with the corresponding finite sums.[20]

We now return to (13) and to our main objective, which is to examine the order of magnitude of the dilatation at the fixed interior point Q of D in relation to the size of the region S, under various assumptions regarding the tractions on S (e.g., if the tractions on S are or are not self-equilibrated). To this end we first recall the mathematical

---

[18]The subscripts α, β denote partial differentiation with respect to the argument indicated. The existence of such a regular parametrization is assured in the small, provided S^* is sufficiently smooth. Note that the embedding regions S_k^*, belonging to the various S_k, $P_0^{(k)}$, in general, require different parametrizations.

[19]The superscript zero attached to any function of (α, β) refers to its value at (0, 0).

[20]See Equation (7) of [4]; this equation gives the mean normal stress rather than the dilatation at Q, and applies to concentrated forces. The transition from distributed tractions to concentrated forces will be discussed later. See Equation (29) of this paper.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

meaning of the concept of "order of magnitude". If[21]

$$| f(x)/x^p | < M \quad (M \text{ independent of } x) \quad \text{as} \quad x \to 0, \qquad \text{then} \qquad f(x) = O(x^p), \tag{17}$$

that is, $f(x)$ is said to be of the same order of magnitude as x^p. It is clear from (17) that the question as to the order of magnitude of $\Delta^Q[S]$ in (13), with respect to the radius ϵ_0 of the sphere enclosing S and P_0, has no meaning since ϵ_0 is a number and not a variable.

The foregoing question becomes meaningful, however, if we ask what happens in the limit as the region S is contracted to the fixed point P_0 of B. To make this idea precise, consider a one-parameter family of closed subregions $S(\epsilon)$ of S^* such that for every ϵ in $0 < \epsilon \leq \epsilon_0$, $S(\epsilon)$ together with P_0 lies within a sphere of radius ϵ, $S(\epsilon_0) = S$, and the maximum diameter $d(\epsilon)$ of $S(\epsilon)$ is a monotone increasing function of ϵ. Next, let $\mathbf{u}(\epsilon)$, $e_{ij}(\epsilon)$, $\tau_{ij}(\epsilon)$ $(0 < \epsilon \leq \epsilon_0)$ be a one-parameter family of solutions in D of the field equations of elasticity theory, the body forces being absent, which satisfies the following conditions: $\mathbf{u}(\epsilon_0) = \mathbf{u}$; $\tau_{ij}(\epsilon)$ gives rise to piecewise continuous surface tractions $\mathbf{T}(\epsilon) = [X_1(\epsilon), X_2(\epsilon), X_3(\epsilon)]$ which vanish on B with the exception[22] of the subregions $S(\epsilon)$; $\mathbf{T}(\epsilon)$ remains bounded as $\epsilon \to 0$.

Writing $\Delta^Q(\epsilon)$ for $\Delta^Q[S(\epsilon)]$, we have from (13), (14),

$$c\Delta^Q(\epsilon) = \mathbf{g}^0 \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\, d\sigma + \mathbf{g}^0_\alpha \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma + \mathbf{g}^0_\beta \cdot \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma + \cdots, \tag{18}$$

$$\int_{S(\epsilon)} d\sigma = \int_{\Sigma(\epsilon)} | \mathbf{r}_\alpha \times \mathbf{r}_\beta |\, d\alpha\, d\beta, \tag{19}$$

where $\Sigma(\epsilon)$ is the antecedent of $S(\epsilon)$ in the (α,β)-plane. Furthermore, in view of the regularity of the mapping (11), and from (12),

$$\delta(\epsilon) = \max_{\Sigma(\epsilon)} (\alpha^2 + \beta^2)^{1/2} \to 0 \qquad \text{as} \qquad \epsilon \to 0 \qquad \text{and} \qquad \delta = 0(\epsilon). \tag{20}$$

Let

$$\mathbf{F}(\epsilon) = \int_{S(\epsilon)} \mathbf{T}(\epsilon)\, d\sigma, \qquad \mathbf{M}(\epsilon) = \int_{S(\epsilon)} \mathbf{r} \times \mathbf{T}(\epsilon)\, d\sigma \qquad (0 < \epsilon \leq \epsilon_0). \tag{21}$$

Thus, $\mathbf{F}(\epsilon)$ and $\mathbf{M}(\epsilon)$ are the resultant force and the resultant moment in $0(\mathbf{r} = 0)$ of the tractions on $S(\epsilon)$. Equation (18), in conjunction with (19), (20), (21), now yields the following conclusions:

(a) $\mathbf{F}(\epsilon) \neq 0$ $(0 < \epsilon \leq \epsilon_0)$, *in general, implies* $\Delta^Q(\epsilon) = 0(\epsilon^2)$. Exceptions are possible, however. In the example of the sphere, we see from (4) that $\Delta^Q = 0$ whenever the surface tractions are purely tangential, provided Q is the center of the sphere.

(b) $\Delta^Q(\epsilon) = 0(\epsilon^3)$ *or smaller if* $\mathbf{F}(\epsilon) = 0 \qquad (0 < \epsilon \leq \epsilon_0)$.

(c) $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller if*

$$\mathbf{F}(\epsilon) = 0, \qquad \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma = 0, \qquad \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{22}$$

[21] This limited form of the definition is sufficient for our purposes.

[22] Recall that $S(\epsilon)$ stands for $S_k(\epsilon)$ $(k = 1, 2, \cdots m)$. The piecewise continuity of $\mathbf{T}$ does not refer to its dependence on ϵ for which no continuity requirements are imposed.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

Within terms of $0(\epsilon^3)$, Equations (22) are equivalent[23] to the 12 scalar conditions,

$$\int_{S(\epsilon)} X_i(\epsilon)\, d\sigma = 0, \qquad \int_{S(\epsilon)} X_i(\epsilon) x_j\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{23}$$

Hence, if the tractions on $S(\epsilon)$ *satisfy* (23) *then* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.*

(d) Equations (23) imply,

$$\mathbf{F}(\epsilon) = 0, \qquad \mathbf{M}(\epsilon) = 0 \qquad (0 < \epsilon \leq \epsilon_0), \tag{24}$$

but the converse is not true, in general. Within terms of $0(\epsilon^3)$ the equilibrium conditions (24), in view of (21), (11), are equivalent[23] to

$$\mathbf{F}(\epsilon) = 0, \qquad \mathbf{r}_\alpha^0 \times \int_{S(\epsilon)} \mathbf{T}(\epsilon)\alpha\, d\sigma + \mathbf{r}_\beta^0 \times \int_{S(\epsilon)} \mathbf{T}(\epsilon)\beta\, d\sigma = 0 \qquad (0 < \epsilon \leq \epsilon_0). \tag{25}$$

Again, these conditions are met if (22) hold, but (22) do not follow from (25). *Thus, if the tractions on* $S(\epsilon)$ *are self-equilibrated* $\Delta^Q(\epsilon) = 0(\epsilon^3)$ *or smaller.* This conclusion contradicts the interpretation of the traditional statement of Saint-Venant's principle cited in the Introduction, according to which the order of magnitude of $\Delta^Q(\epsilon)$ should always be smaller when the tractions on $S(\epsilon)$ are self-equilibrated than when they are not.

(e) Suppose, in particular, the tractions on $S(\epsilon)$ are parallel, so that

$$\mathbf{T}(\alpha, \beta; \epsilon) = \mathbf{k}T(\alpha, \beta; \epsilon) \qquad \text{in} \qquad \Sigma(\epsilon) \qquad (0 < \epsilon \leq \epsilon_0), \tag{26}$$

in which $\mathbf{k}$ is a fixed vector and T a scalar function. Here (23) are satisfied if and only if (24) hold for every choice of $\mathbf{k}$, that is, if and only if the system of parallel tractions remains in equilibrium under an arbitrary change of its direction, the magnitude and sense of the tractions being maintained ("astatic equilibrium"). *Therefore, in the event the tractions on* $S(\epsilon)$ *are parallel and in astatic equilibrium,* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.*

(f) Suppose (26) holds and, in addition,

$$\mathbf{k} \cdot (\mathbf{r}_\alpha^0 \times \mathbf{r}_\beta^0) \neq 0, \tag{27}$$

so that $\mathbf{k}$ is not parallel to the tangent plane of S^* at P_0. Then (25) imply (22). *Hence, if the tractions on* $S(\epsilon)$ *are parallel to each other, self-equilibrated, and not parallel to a tangent plane of* S^**, then* $\Delta^Q(\epsilon) = 0(\epsilon^4)$ *or smaller.* The analogous conditions for concentrated forces were satisfied in the special example investigated by Boussinesq [2] and described in the Introduction.

We have so far considered only the order of magnitude of the dilatation. According to Lauricella,[24] the strains e_{ij}^Q at an interior point Q of D, in the absence of body forces, admit the representation,

$$e_{ij}^Q = \int_B \mathbf{T} \cdot \mathbf{g}_{ij}\, dB, \tag{28}$$

which is analogous to that given by Betti's formula (1) for the dilatation Δ^Q. Here the $\mathbf{g}_{ij}$ are displacement fields which satisfy the equilibrium conditions inside D, with the

[23]In case S^* is plane, and for the parametrization (15), the equivalence is exact.

[24]See [3], p. 216.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

exception of the point Q where they have certain prescribed singularities,[25] and which give rise to vanishing surface tractions on B. Clearly, the $\mathbf{g}_{ij}$ are again regular on any regular portion of B, and the previous argument applies without modification. Therefore, the conclusions listed under (a) to (f) remain valid if $\Delta^Q(\epsilon)$ is replaced with $e_{ij}^Q(\epsilon)$, and hence with $\tau_{ij}^Q(\epsilon)$. If a rigid displacement of the whole body is excluded, the conclusions also apply to the displacement $\mathbf{u}^Q(\epsilon)$, as follows from Somigliana's representation of the displacement field.[26] Specific illustrations of these general conclusions were given by v. Mises in [4].[27]

Extension to the case of concentrated forces. Remarks. The theorem proved in the preceding section is readily extended to concentrated forces. With reference to (13), let the concentrated forces $\mathbf{T}_n = [X_1^{(n)}, X_2^{(n)}, X_3^{(n)}]$ $(n = 1, 2, \cdots N)$ be applied at the points A_n of S^*, the points of application together with P_0 lying within a sphere of radius ϵ_0. Let $\mathbf{r}^{(n)} = [x_1^{(n)}, x_2^{(n)}, x_3^{(n)}] = \mathbf{r}(\alpha_n, \beta_n)$ be the position vector of A_n. Now consider an S consisting of N non-intersecting closed subregions $S^{(n)}$ of S^* such that each $S^{(n)}$ is simply connected and contains A_n in its interior. Proceeding to the limit in (13) as $S^{(n)}$ is contracted to A_n while $\int_{S^{(n)}} \mathbf{T}\, d\sigma \to \mathbf{T}_n$, we obtain,[28]

$$c\Delta^Q[S] = \mathbf{g}^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n + \mathbf{g}_\alpha^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n\alpha_n + \mathbf{g}_\beta^0 \cdot \sum_{n=1}^{N} \mathbf{T}_n\beta_n + \cdots . \tag{29}$$

The conclusions previously reached, therefore, remain valid for concentrated forces provided the integrals in (18) to (25) are replaced with the corresponding finite sums, and provided $O(\epsilon^2)$, $O(\epsilon^3)$, $O(\epsilon^4)$ are replaced with $O(1)$, $O(\epsilon)$, $O(\epsilon^2)$, respectively.

At this place we discuss an example which may serve to clarify the implications of the theorem established earlier. Consider a bar of the general shape indicated in Figure 2. Let the bar be acted on by the two equal, opposite, and collinear concentrated loads, each of magnitude L, the region S consisting of the two points of application A_1 and A_2. According to conclusion[29] (d), in this instance $\tau_{ij}^Q(\epsilon) = O(\epsilon)$ or smaller, where Q is, say, the fixed interior point shown in Figure 2. Since we may choose the two ends of the bar as close together as we wish, a careless interpretation of (d) may lead to the absurd prediction that the stresses at Q, for arbitrarily large fixed magnitudes of the loads, can be kept as small as we wish. The paradox is resolved by observing that the shape of the bar is given once and for all, and that a definite gap, however small, exists between the two ends of the bar. The statement $\tau_{ij}^Q(\epsilon) = O(\epsilon)$, in view of the definition (17), merely implies $|\tau_{ij}^Q(\epsilon)| < M\epsilon$ as $\epsilon \to 0$, where M is a positive number independent of ϵ. Thus, by contracting the load region sufficiently, say toward A_1, while maintaining the loading within the ϵ-sphere in equilibrium, $|\tau_{ij}^Q(\epsilon)|$ can be made arbitrarily small. In this process of contraction, however, the end of the bar which carries the point A_2, will eventually cease to lie within the contracting ϵ-sphere so that the entire character

[25] $\mathbf{g}_{11}$, for example, at Q has the singularity appropriate to a force-doublet the axis of which is parallel to the x_1-axis.

[26] See [3], p. 245.

[27] These examples refer to concentrated forces (see the next section of the present paper).

[28] Note that S here consists of the points $A_n (n = 1, 2, \cdots N)$.

[29] Modified for the case of concentrated forces.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

of the loading changes, that end being now free from loading. This example provides additional evidence for the vagueness of the traditional statements of Saint-Venant's principle, quoted in the Introduction.

Hoff [17] pointed out important limitations inherent in certain conventional engineering approximations which are usually based on an appeal to Saint-Venant's principle. These observations are consistent with the results established here. For Saint-Venant's

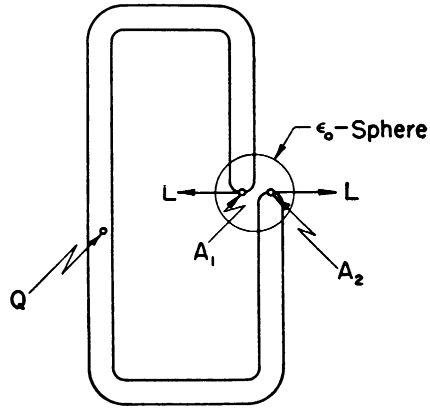

FIG. 2.

principle is a statement about relative orders of magnitude and does not tell us anything about the extent of the region within which a self-equilibrated system of tractions, applied to a portion of the surface of an elastic body, "materially" influences the stress distribution in the body.

In conclusion, it may be well to repeat v. Mises' remark [4] to the effect that the present theorem does not preclude the validity of a stronger Saint-Venant principle for special classes of bodies, such as "thin" plates or shells and "long" cylinders.

BIBLIOGRAPHY

1. B. de Saint-Venant, "Mémoire sur la torsion des prismes," Mém. Savants étrangers, Paris, 1855.
2. M. J. Boussinesq, "Application des potentiels," Gauthier-Villars, Paris, 1885.
3. A. E. H. Love, "A treatise on the mathematical theory of elasticity," 4th ed., Dover Publications, New York, 1944.
4. R. v. Mises, "On Saint-Venant's principle," Bull. Amer. Math. Soc., **51**, 555 (1945).
5. C. B. Biezeno and R. Grammel, "Technische Dynamik," J. W. Edwards, Ann Arbor, Michigan, 1944.
6. R. Southwell, "On Castigliano's theorem of least work and the principle of Saint-Venant," Phil. Mag. (6) **45**, 193 (1923).
7. J. N. Goodier, "A general proof of Saint-Venant's principle," Phil. Mag. (7) **23**, 607 (1937).
8. J. N. Goodier, "Supplementary note on 'A general proof of Saint-Venant's principle'," Phil. Mag. (7) **24**, 325 (1937).
9. O. Zanaboni, "Dimostrazione generale del principio del De Saint-Venant" Atti Acc. Naz. Lincei **25**, 117 (1937).
10. O. Zanaboni, "Valutazione dell' errore massimo cui dà luogo l'applicazione del principio del De Saint-Venant," Atti Acc. Naz. Lincei **25**, 595 (1937).
11. J. N. Goodier, "An extension of Saint-Venant's principle, with applications," J. Appl. Phys. **13**, 167, 1942.

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms

12. K. Erim, "Sur le principe de Saint-Venant," Proc., Seventh Int. Cong. for Appl. Mech., London, 1948.
13. O. D. Kellogg, "Foundations of potential theory," The Murray Printing Company, New York, 1929.
14. J. L. Synge, "Upper and lower bounds for the solution of problems in elasticity," Proc., Royal Irish Ac. (A) **53**, 41 (1950).
15. C. Weber, "Kugel mit normalgerichteten Einzelkräften," Z. Angew. Math. Mech. **32**, 186 (1952).
16. E. Sternberg and F. Rosenthal, "The elastic sphere under concentrated loads," J. Appl. Mech. **19**, 413 (1952).
17. N. J. Hoff, "The applicability of Saint-Venant's principle to airplane structures," J. Aero. Sci. **12**, 455 (1945).
18. P. Locatelli, "Extensione del principio di St. Venant a corpi non perfettamente elastici," Atti R. Acc. delle Sci. Torino **75**, 502 (1940).

This content downloaded from
2.37.166.172 on Sun, 31 Oct 2021 10:55:37 UTC
All use subject to https://about.jstor.org/terms