ON SAINT VENANT'S PRINCIPLE

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The so-called principle of statically equivalent loads, due to Saint Venant, has been referred to for the last fifty years in almost all texts on elasticity. The statements in different books vary only slightly. Let us quote A. E. H. Love's *Treatise on the mathematical theory of elasticity* (4th ed., p. 132): "According to this principle the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." In this form the statement is not very clear. Forces applied to a body at rest must be in equilibrium in any case. It would not make sense to speak of adding or subtracting a system of forces that is not an equilibrium system. What is meant may be correctly expressed in this way: If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not. If this statement is true, it must be capable of a mathematical proof, that is, it must be a consequence of the fundamental differential equations of elasticity theory. But no attempt is made in the usual textbooks to supply a demonstration. Most texts give Boussinesq as a reference for the proof. What Boussinesq really dealt with was the infinite body filling the half space $z>0$ and subjected to normal forces at its boundary $z=0$. If the forces are applied to points ξ, η, 0 where $\xi^2+\eta^2\leqq\epsilon^2$, Boussinesq proved that the stress at a point x, y, z is of order ϵ when the sum of forces is zero and of order ϵ^2 when their moments also vanish. It will be shown in the following that this is not the case, in general, if tangential components of the forces at $z=0$ are admitted. Moreover we shall consider a body of finite dimensions and see that there too Saint Venant's principle in its traditional form does not hold true. The main result, from a practical point of view, is that Saint Venant's principle can be applied if all forces involved are parallel and not tangential to the surface of the body, but not under more general conditions. No objection is raised in the present paper against using the principle in the case of bodies with one or two infinitesimal dimensions, like thin plates, shells or beams, although a proof of its

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validity in these cases or even a precise statement has not yet been given.

1. **The infinite half space.** If the force X, Y, Z is applied in the origin to the body $z>0$, the displacements u, v, w at a point x, y, z can be expressed, as it follows from Boussinesq's results,[1] in the following way. We denote by U and V the harmonic functions

$$U = \frac{xX + yY}{r + z}, \qquad V = Z \log (r + z) \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$, by κ the value $1 - 2\sigma$ where σ is Poisson's ratio, and by μ the shear modulus. Then the displacements are given by

$$\begin{aligned}
4\pi\mu u &= \frac{2}{r} X + \left(\kappa - 1 + z \frac{\partial}{\partial z}\right)\frac{\partial U}{\partial x} - \left(\kappa + z \frac{\partial}{\partial z}\right)\frac{\partial V}{\partial x}, \\
4\pi\mu v &= \frac{2}{r} Y + \left(\kappa - 1 + z \frac{\partial}{\partial z}\right)\frac{\partial U}{\partial y} - \left(\kappa + z \frac{\partial}{\partial z}\right)\frac{\partial V}{\partial y}, \\
4\pi\mu w &= \frac{2}{r} Z + \left(- \kappa + z \frac{\partial}{\partial z}\right)\frac{\partial U}{\partial z} - \left(1 - \kappa + z \frac{\partial}{\partial z}\right)\frac{\partial V}{\partial z}.
\end{aligned} \tag{2}$$

It can easily be shown that these values satisfy the differential equations of elasticity theory:

$$\begin{gathered}
\Delta u + \frac{1}{\kappa}\frac{\partial \theta}{\partial x} = 0, \qquad \Delta v + \frac{1}{\kappa}\frac{\partial \theta}{\partial y} = 0, \qquad \Delta w + \frac{1}{\kappa}\frac{\partial \theta}{\partial z} = 0; \\
\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.
\end{gathered} \tag{3}$$

From (2) we derive by differentiation

$$\theta = - \frac{\kappa}{2\pi\mu}\frac{xX + yY + zZ}{r^3}. \tag{4}$$

On the other hand, it follows from (2) if the strain-stress relations are introduced that the stress vector for an element parallel to the x-y-plane is directed toward the origin and has the magnitude

[1] Boussinesq, J., *Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*, Paris, 1885. A. E. H. Love, *Treatise on the mathematical theory of elasticity*, 4th ed., Cambridge, 1934, p. 191, gives only the results for a force acting normal to the boundary. The complete expressions can be derived from the formulae developed by E. Trefftz in: Frank-Mises, *Differential- und Integralgleichungen der Mechanik und Physik*, 2nd ed., vol. 2, Braunschweig, 1935 (Reprint, New York, 1944), p. 303.

$$\text{(5)} \qquad \frac{3z}{2\pi} \, \frac{xX + yY + zZ}{r^4} \, .$$

Thus the stress is zero in all points $z=0$, $x^2+y^2 \neq 0$ and across each plane $z=\text{const.}>0$ the stresses are equivalent to a single force $-X$, $-Y$, $-Z$ passing through the origin.

The mean normal stress is given by

$$\text{(6)} \qquad \bar{\sigma} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{3-\kappa}{3\kappa}\mu\theta = \frac{\kappa - 3}{6\pi} \, \frac{xX + yY + zZ}{r^3} \, .$$

If the force attacks at ξ, η, 0 instead of at the origin, the same formulas can be used, where only x, y have to be replaced by $x-\xi$, $y-\eta$, and r^2 by $(x-\xi)^2+(y-\eta)^2+z^2$. Developing with respect to ξ, η we obtain for the mean normal stress due to a system of forces X_ν, Y_ν, Z_ν attacking at points ξ_ν, η_ν, 0 ($\nu = 1, 2, 3, \cdots$):

$$\text{(7)} \qquad \begin{aligned} \frac{6\pi}{\kappa - 3} r^3 \bar{\sigma} = {} & x \sum X_\nu + y \sum Y_\nu + z \sum Z_\nu \\ & + \frac{1}{r^2} \left[(3x^2 - r^2) \sum \xi_\nu X_\nu + 3xy \sum \xi_\nu Y_\nu \right. \\ & + 3xz \sum \xi_\nu Z_\nu + 3xy \sum \eta_\nu X_\nu \\ & \left. + (3y^2 - r^2) \sum \eta_\nu Y_\nu + 3yz \sum \eta_\nu Z_\nu \right] + \cdots \end{aligned}$$

and similar expressions for other stress values.

If all ξ_ν and η_ν are of the order of magnitude ϵ, we can conclude: The stresses (and strains) in a point x, y, z are of the order ϵ, if the sums of force components $\sum X_\nu$, $\sum Y_\nu$, $\sum Z_\nu$ are zero; they are of the order ϵ^2, if and only if the six linear moments $\sum \xi_\nu X_\nu$, $\sum \xi_\nu Y_\nu$, $\sum \xi_\nu Z_\nu$ and $\sum \eta_\nu X_\nu$, $\sum \eta_\nu Y_\nu$, $\sum \eta_\nu Z_\nu$ also vanish. The case of a system in equilibrium, that is, $\sum \xi_\nu Z_\nu = \sum \eta_\nu Z_\nu = \sum (\xi_\nu Y_\nu - \eta_\nu Z_\nu) = 0$, is, in general, in no way distinguished. Only if all forces are parallel to each other, either normal to the boundary surface or inclined under an angle different from zero, the three equilibrium conditions entail the other three conditions. In general, the order of magnitude of the inner strains is reduced to ϵ^2 if and only if the external forces acting upon a small part of the surface are such as to remain in equilibrium when turned through an arbitrary angle (astatic equilibrium).

(a) finite stress; (b), (c) stress of order of magnitude ϵ; (d) stress of order ϵ^2

FIG. 1

The results are illustrated in the four simple examples of Fig. 1. All forces are here parallel to the x-direction ($Y=Z=0$). In the case (a) we have a single force which provides a finite stress value according to the first term of (7). In both cases (b) and (c) the sum of forces is zero, but in (b) the sum of $\eta_\nu X_\nu$ and in (c) the sum of $\xi_\nu X_\nu$ is different from zero. It follows from (7) that in either case the stress has the order of magnitude ϵ. If the Saint Venant principle were correct, the stress should be small of higher order in the case (c) where all equilibrium conditions are fulfilled. In fact, in the case (d) only where the three forces form a system in astatic equilibrium with all linear moments zero the stress has the order of magnitude ϵ^2.

2. **Circular disk.** It may be doubted whether the infinite half space gives an adequate picture of what happens in the case of a finite body. Here one cannot speak of applying a single force to the body since the forces as a whole have to form an equilibrium system. It will be sufficient, however, for the present purpose, to analyze one particular shape of finite boundary.

Let us consider the two-dimensional stress distribution in a circular disk of radius R subjected to external forces $F_1, F_2, \cdots, F_n$ which

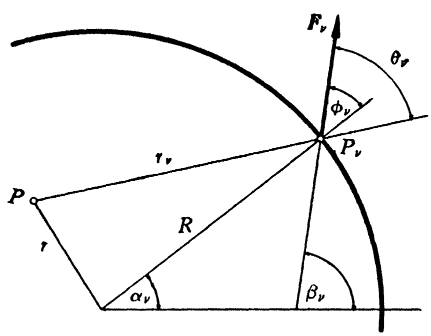


FIG. 2

attack at points $P_1, P_2, \cdots, P_n$ of the circumference. The stresses in a point P are determined by Airy's stress function $S(P)$. If (Fig. 2) r is the distance of P from the center C, r_ν the distance between P and P_ν, ϕ_ν the angle of F_ν with CP_ν, and θ_ν the angle of F_ν with PP_ν, the stress function is given by[2]

$$S(P) = \frac{1}{\pi} \sum_{\nu=1}^{n} F_\nu \left(r_\nu \theta_\nu \sin \theta_\nu - \frac{r^2}{4R} \cos \phi_\nu \right). \tag{8}$$

[2] Cf. Love, loc. cit. p. 217. H. Hertz, *Gesammelte Werke*, vol. 1, p. 283.

The stress components with respect to an x-y-coordinate system can be expressed as follows, if β_ν designates the angle of $\boldsymbol{F}_\nu$ with the x-axis:

$$
\begin{aligned}
\sigma_x &= \frac{1}{\pi}\sum F_\nu\left[\frac{2}{r_\nu}\cos\theta_\nu\cos^2(\theta_\nu-\beta_\nu)-\frac{\cos\phi_\nu}{2R}\right],\\
\sigma_y &= \frac{1}{\pi}\sum F_\nu\left[\frac{2}{r_\nu}\cos\theta_\nu\sin^2(\theta_\nu-\beta_\nu)-\frac{\cos\phi_\nu}{2R}\right],\\
\tau &= -\frac{1}{\pi}\sum F_\nu\frac{2}{r_\nu}\cos\theta_\nu\cos(\theta_\nu-\beta_\nu)\sin(\theta_\nu-\beta_\nu).
\end{aligned}
\tag{9}
$$

It can easily be shown that $S(P)$ fulfills the differential equation $\Delta\Delta S=0$ and that the stresses on the boundary $r=R$ are zero, except in the points $P_\nu(r_\nu=0)$. In these points the stresses become infinite and combine so as to balance $\boldsymbol{F}_\nu$.

In the center of the circle we have $r_\nu=R$ and $\theta_\nu=\phi_\nu$ and introducing $\alpha_\nu=\beta_\nu-\phi_\nu$, the polar angle of P_ν, we find for the stress components in C

$$
\begin{aligned}
\sigma_x &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)(1/2+\cos 2\alpha_\nu),\\
\sigma_y &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)(1/2-\cos 2\alpha_\nu),\\
\tau &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)\sin 2\alpha_\nu.
\end{aligned}
\tag{10}
$$

Let us now assume that there are two groups of forces $\boldsymbol{F}_\nu$ and $\boldsymbol{F}_\nu'$, forming angles β_ν and β_ν' with the x-axis. The points of attack P_ν of the first group may lie close to a point P_0 with polar angle α_0 and the points P_ν' of the second group near to P_0' with polar angle α_0' so as to have $\alpha_\nu=\alpha_0+\xi_\nu$, $\alpha_\nu'=\alpha_0'+\xi_\nu'$ (ξ, ξ' small). If we develop the expression for σ_x with respect to ξ and ξ', we find

$$
\begin{aligned}
\pi R\sigma_x = {} & (1/2+\cos 2\alpha_0)\sum F_\nu\cos(\beta_\nu-\alpha_0)\\
&+(1/2+\cos 2\alpha_0')\sum F_\nu'\cos(\beta_\nu'-\alpha_0')\\
&-2\sin 2\alpha_0\sum F_\nu\xi_\nu\cos(\beta_\nu-\alpha_0)\\
&-2\sin 2\alpha_0'\sum F_\nu'\xi_\nu'\cos(\beta_\nu'-\alpha_0')\\
&+(1/2+\cos 2\alpha_0)\sum F_\nu\xi_\nu\sin(\beta_\nu-\alpha_0)\\
&+(1/2+\cos 2\alpha_0')\sum F_\nu'\xi_\nu'\sin(\beta_\nu'-\alpha_0')+\cdots.
\end{aligned}
\tag{11}
$$

The first two terms vanish if in each group the sum of forces is zero, since $\beta_\nu-\alpha_0$ and $\beta_\nu'-\alpha_0'$ are the angles the forces form with CP_0 and

CP_0' respectively. In this case, therefore, the highest terms in the expression for σ_x are of the order ξ, ξ'.

If the sums of moments in each group are also zero, we have

$$(12)\qquad \begin{aligned} \sum F_\nu R \sin (\beta_\nu - \alpha_\nu) &= R \sum F_\nu \sin (\beta_\nu - \alpha_0 - \xi_\nu) = 0, \\ \sum F_\nu' R \sin (\beta_\nu' - \alpha_\nu') &= R \sum F_\nu' \sin (\beta_\nu' - \alpha_0' - \xi_\nu') = 0. \end{aligned}$$

If these expressions are developed with respect to ξ and ξ', it is seen that with the resultant forces and moments vanishing the sums

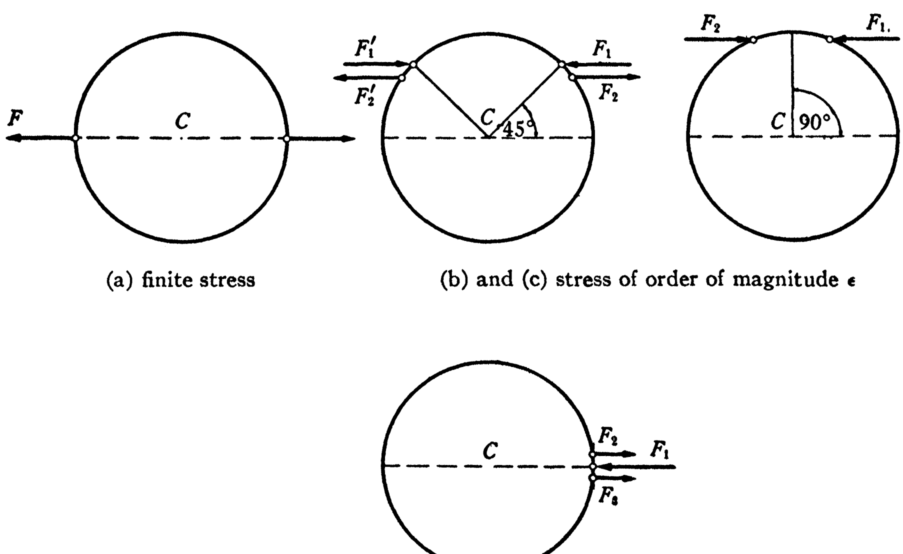
$$\sum F_\nu \xi_\nu \cos (\beta_\nu - \alpha_\nu) \quad \text{and} \quad \sum F_\nu' \xi_\nu' \cos (\beta_\nu' - \alpha_\nu')$$

are zero except for quantities of higher order. In this case the third and fourth terms in (11) vanish, but the fifth and sixth still supply a quantity of the order of magnitude ξ, ξ'. Only if the two additional conditions are fulfilled that

$$\sum F_\nu \xi_\nu \sin (\beta_\nu - \alpha_0) \quad \text{and} \quad \sum F_\nu' \xi_\nu' \sin (\beta_\nu' - \alpha_0')$$

vanish, except for terms of higher order, the development of σ_x will start with terms of the order ξ^2, $\xi\xi'$, ξ'^2.

Again the results may be illustrated on four simple examples shown in Fig. 3. In the case (a) the two single forces of magnitude F produce



(a) finite stress (b) and (c) stress of order of magnitude ϵ

(d) stress of order ϵ^2

FIG. 3

in the center a finite normal stress $\sigma_x = 3F/\pi R$ according to (11). In (b) we have two groups of parallel forces with $\alpha_0 = 45°$, $\alpha_0' = 135°$. If the angular distance of the two neighboring points of application is called ϵ, one has $\xi_1 = \xi_1' = 0$, $\xi_2 = -\epsilon$, $\xi_2' = \epsilon$ and with $\beta_1 = \beta_2' = 180°$, $\beta_2 = \beta_1' = 0$, equation (11) supplies $\sigma_x = 5F\epsilon/2^{1/2}\pi R$. In the case (c) the two forces attacking in the points $\alpha_1 = 90° - \epsilon$, $\alpha_2 = 90° + \epsilon$ form an equilibrium system. Equation (11) gives here the value $\sigma_x = F\epsilon/\pi R$ which is of the same order as in (b). Were Saint Venant's principle correct, the stresses in (c) should be small of a higher order. In fact, this is the case with the three forces in example (d) only which form an astatic system (up to terms of higher order). All terms written down on the right-hand side of (11) vanish in the case (d).

3. **Conclusions.** In order to obtain a precise and sufficiently general statement let us consider a finite simply connected body, supported at an adequate number of distinct surface points $S_1, S_2, S_3, \cdots$. Let P_1 be a point of the surface where the load $\boldsymbol{F}_1$ is applied and P an inner point of the body at finite distances from P_1 and from $S_1, S_2, S_3, \cdots$. Let, finally, σ be some well defined strain or stress quantity in P, for instance, the normal stress in x-direction, or any component of the distortion. Then, with constant $\boldsymbol{F}_1$, this σ will be a function of the coordinates of P_1. If P_1 is a regular surface point (tangential plane, finite curvature) the function will have finite derivatives. That means, if P_1 moves through a small distance ϵ the change in σ will be of the order of magnitude ϵ. Consequently, two equal and opposite forces attacking at points of distance ϵ will produce a σ-value of the order ϵ. On the other hand, the load $\boldsymbol{F}_1$ can be replaced by several loads that have the vector sum $\boldsymbol{F}_1$, all attacking at P_1. Each of them can be shifted to the neighborhood and then reversed. The system of these reversed forces combined with the original $\boldsymbol{F}_1$ will still produce a σ-value of order ϵ. Thus our first statement reads:

(a) *If a system of loads on an adequately supported body, all applied at surface points within a sphere of diameter ϵ, have the vector sum zero, they produce in an inner point P of the body a strain or stress value σ of the order of magnitude ϵ.*

To this statement we add the results reached in the preceding sections by way of direct computation for two particular cases, the infinite half space and the circular disk. The general proof following the same lines can be given without difficulty.

(b) *If the loads, in addition to having the vector sum zero, fulfill three* *further conditions so as to form an equilibrium system within the sphere of diameter* ϵ, *the* σ*-value produced in P will, in general, still be of the order of magnitude* ϵ.

(c) *If the loads, in addition to being an equilibrium system, satisfy three more conditions so as to form a system in astatic equilibrium, then the* σ*-value produced in P will be of the order of magnitude* ϵ^2 *or smaller. In particular, if loads applied to a small area are parallel to each other and not tangential to the surface and if they form an equilibrium system, they are also in astatic equilibrium and thus lead to a* σ *of the order* ϵ^2.

In this whole argument the loads as well as the supporting reactions were supposed to be concentrated, finite forces acting at distinct points of the surface. No difficulty arises if, instead, continually distributed surface stresses are assumed with the provision that all integrals of such stresses over finite regions (and the regions that tend to zero) remain finite.

A final remark is in order about the legitimate application of St. Venant's principle (or some equivalent statement) in cases of thin rods, shells, and so on. The only precise and consistent way to deal with thin elastic rods is the theory of the so-called one-dimensional elastica. In this theory the forces acting on the ends of the rod enter the computation only with their resultant vector and resultant moment. This implies, evidently, a principle of "statically equivalent loads." What Saint Venant originally had in mind was doubtlessly the case of a long cylinder with infinite ratio of length to diameter. The purpose of the present paper was to show that an extension of the principle to bodies of finite dimensions is not legitimate.

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