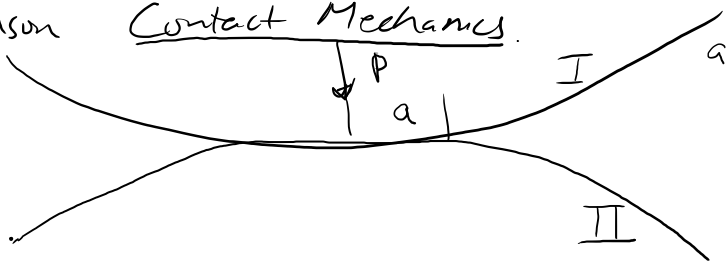
Aim of the module is to measure the real area of contact between two spheres. (note area varies w/ applied load). Solution is in ch's 3&4 of Johnson <u>Contact Mechanics</u>.



a : effective radius of contact

Hertz's Assumptions:

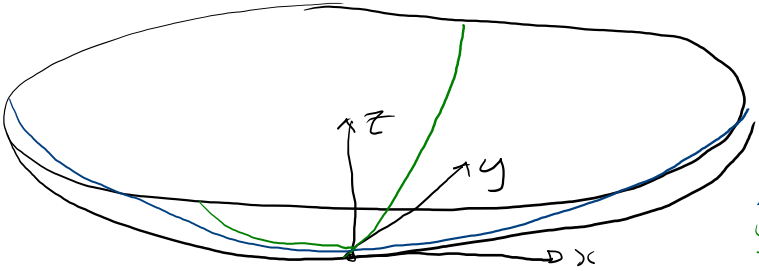
1. Surfaces are <u>continuous</u> & <u>non-conforming</u>

2. The strains are small: $a \ll R$

3. The bodies I & II are <u>elastic</u> <u>half-spaces</u>.

4. The surfaces are <u>frictionless</u>: $\boxed{q_x, q_y = 0.}$
(no tangential stresses).

The full solution is in Johnson's textbook; we'll do the dimensional analysis.



x - z curvature
y - z curvature

Coordinates of the surface:

$$z_1 = A_1 x^2 + B_1 y^2 \qquad \text{Spherical cap}$$

For the complementary side (body II):

$$z_2 = -(A_2 x^2 + B_2 y^2).$$

In lab reports, please explain equivalence b/t crossed cylinders & spherical cap contact

For spherical caps: $A = B = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$.

This implies that the contours of constant separation are circles centered on the loading axis.

Now, we'll apply a load, P to force the spherical caps into contact. P is applied along z-axis.

Distance between points on surface of bodies 1 & 2 is given by:

$$h = Ax^2 + By^2 = \frac{1}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right)$$

Upon increasing P, the bodies begin to deform.

This causes the bodies to displace toward one another of δ_1, & δ_2.

The displacements along z we'll designate as u_{z1}, u_{z2} (functions of x & y).

For points in contact:

$$u_{z1} + u_{z2} + h = \delta_1 + \delta_2$$

Now, define $\delta = \delta_1 + \delta_2$

we can re-write the above expression using the the known geometry from h:

$$u_{z1} + u_{z2} = \delta - Ax^2 - By^2.$$

For points outside the region of contact ($x^2+y^2 > a^2$)

$$u_{z1} + u_{z2} > \delta - Ax^2 - By^2.$$

At this point, we could do the full solution; here we'll do dimensional analysis.

<u>Dimensional Analysis:</u>

- Keep sphere-sphere contact geometry.

note that displacements of bodies I & II at $x, y = 0, 0$ are $u_{z1}(0) = \delta_1$, $u_{z2} = \delta_2$.

Thus within the surface of contact:

$$\left(\frac{u_{z1}(0) - u_{z1}(x)}{a}\right) + \left(\frac{u_{z2}(0) - u_{z2}(x)}{a}\right) = \frac{x^2}{2a}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

(from contact geometry)

Substitute $r=a$, and write $u_z(0)-u_z(a)=d$
we can re-write the deformation as:

$$\frac{d_1}{a}+\frac{d_2}{a}=\frac{a}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

Assume deformation is <u>small</u>: $d \ll a$
$\Rightarrow$ typical strain scale is d/a.

Because the bodies are linearly elastic, the strain will be proportional to the contact pressure divided by the elastic modulus.

Defining p_m as a characteristic contact pressure, we can substitute $\boxed{p_m/E}$ for $\boxed{\varepsilon = d/a}$.

$$\frac{p_m}{E_1} + \frac{p_m}{E_2} \propto a\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

$$\Rightarrow p_m \propto \frac{a\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\frac{1}{E_1} + \frac{1}{E_2}}.$$

$\Rightarrow$ Contact stresses increase <u>linearly</u> w/ area.

Now, we want to relate this to applied load, P.

$$P = \pi a^2 p_m$$

$$\Rightarrow a \propto \left[\frac{P\left(1/E_1 + 1/E_2\right)}{\left(1/R_1 + 1/R_2\right)}\right]^{1/3}.$$

Now we can write p_m as:

$$p_m = \frac{P}{\pi a^2} \propto \left[\frac{P\left(1/R_1 + \frac{1}{R_2}\right)^2}{\left(\frac{1}{E_1} + 1/E_2\right)^2}\right]^{1/3}$$

For a 3-D solid, compressions δ_1 & δ_2 are proportional to displacements d_1 & d_2.

$$\Rightarrow \delta \propto d_1 + d_2$$

$$= \frac{\delta}{a} = \frac{a}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \Rightarrow \delta \propto \frac{a^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \frac{P^{2/3}\left(1/E_1 + 1/E_2\right)^{2/3}}{\left(1/R_1 + \frac{1}{R_2}\right)^{1/3}}.$$

Experimental method of Tabor: (available on the wiki)

For conductors with specific conductivity λ

Maxwell: sphere-plane contact $R = \frac{1}{2a\lambda}$

For sphere-sphere contact: $R \simeq \frac{1}{2\lambda}\left(\frac{1}{a} - \frac{2}{\pi r}\right)$, r is spherical radius.

In the limit of a small contact radius $a \ll r$, junction conductance $\Lambda = 2a\lambda$

Conductance can be related to applied force P:

$$\boxed{\Lambda = K_1 \cdot P^{1/3}}$$

$$\Lambda = \frac{i}{v}$$ Read Tabor.

Passive components.

2 Fundamental laws for circuits:

Ohm's law; Kirchoffs law(s)

$$\hookrightarrow \boxed{V = IR}$$

Voltage/potential = Flow of electrons. Resistance to flow of electrons.

Hydrodynamics analogy:

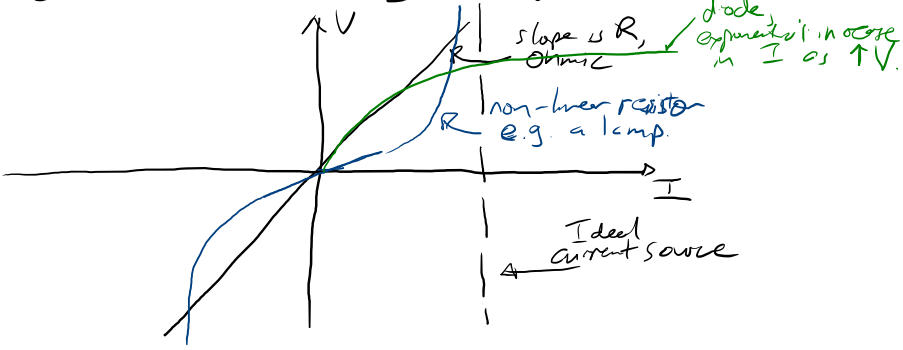
$$H = J \cdot R$$

Hydrostatic pressure — Fluid flux — hydrodynamic resistance.

"A Volt pushes an Amp through an Ohm"

Most often, we measure V relative to a reference (typ. ground)

Ohmic vs. non-Ohmic behavior:



Power: $P = IV = I^2R$.

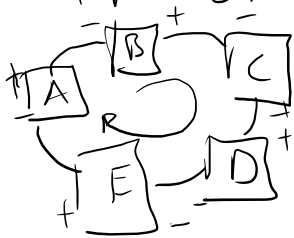
Dimensionally:

$$\left.\begin{array}{l} I: \text{current, charge/time} \\ V: \text{work/charge} \end{array}\right\} \Rightarrow IV = \frac{\text{work}}{\text{time}} = [P]$$

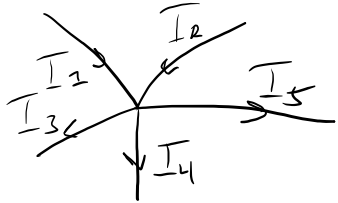
dimensions of power

Kirchoff's law(s):

I Sum of voltages around a closed loop is 0:



II: Sum of currents @ a node is 0.



$$I_1 + I_2 + I_3 + I_4 + I_5 = 0$$

Analysis of simple circuits w/ resistors:

Series resistance:

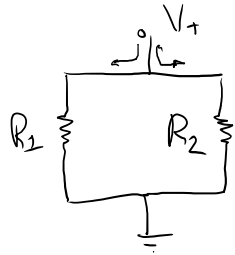
$$I_{tot} = I_1 = I_2$$

$$V_{tot} = V_1 + V_2$$

$$V_t = I_t \cdot R_{eq} = I_t R_1 + I_t \cdot R_2$$

$$\Rightarrow R_{eq} = R_1 + R_2$$

parallel resistors:



$$I_{tot} = I_1 + I_2$$

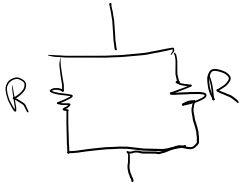
$$V_{tot} = V_1 = V_2$$

$$V_{tot} = I_{tot} \cdot R_{eq}$$

$$I_1 \cdot R_1 = I_2 \cdot R_2 = I_{tot} \cdot R_{eq}$$

$$\frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{V_{tot}}{R_{eq}} \Rightarrow \boxed{R_{eq} = \frac{R_1 R_2}{R_1 + R_2}}$$

Estimation of R values:



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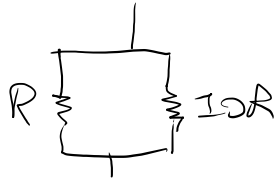

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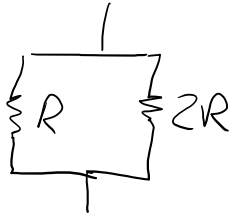


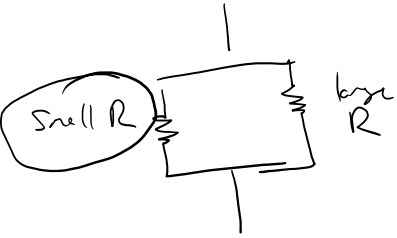
$$R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

1: $\frac{R^2}{2R} = \frac{R}{2}$

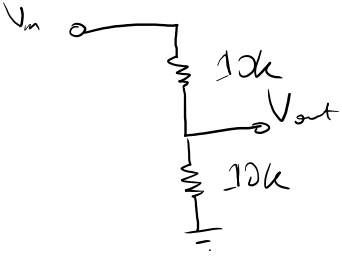
2: $\frac{10R^2}{11R} \cong 0.9\,R$

3: $\frac{2R^2}{3R} \simeq 0.67\,R$

Rule of dominance of resistors:



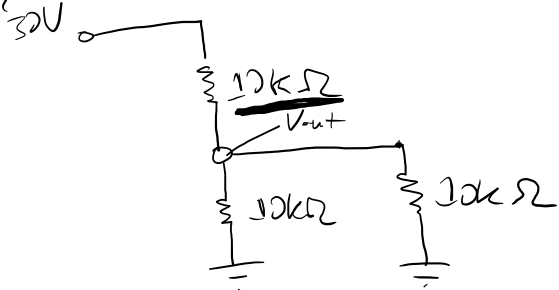
Device: V-divider.



find $V_{out}(V_{in})$.

$$I = \frac{V_{in}}{(R_1+R_2)} \Rightarrow V_{out} = I \cdot R_2 = \frac{V_{in} R_2}{R_1+R_2}$$

Say V_{out} is used to drive a resistive load:



viewed from the load, the "looks like" 10k || 10k.

What values of R_{load} don't modify V_{out}?

<u>Capacitors</u>: A static capacitor obeys the law:

$$Q = CV$$

charge = capacitance · voltage.

charge is very hard to quantify. So we move to current $= \frac{dQ}{dt} = I$.

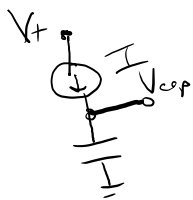
$$\Rightarrow \boxed{I = C\frac{dV}{dt}}$$

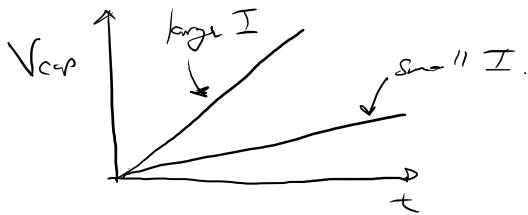
fixed parameter for our capacitor.

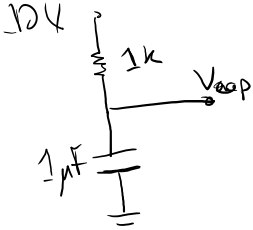
size is C.

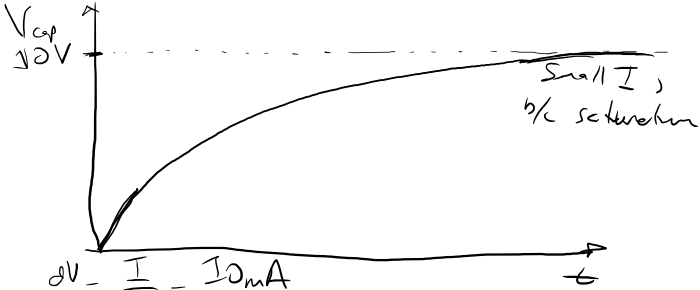
Q is volume of water

Constant I:







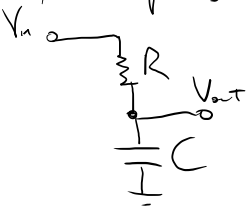


$$\frac{dV}{dt} = \frac{I}{C} = \frac{10mA}{10\mu F}$$

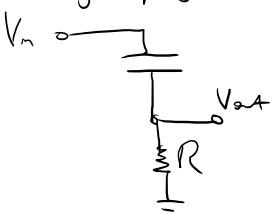
$$\boxed{V_{cap} = V_{cap} \cdot \left(1 - e^{-t/RC}\right)}$$

RC is our circuit's time constant.

Devices: low-pass filter



high-pass filter



$V_{in} = V_0 \sin(2\pi f t)$.

A capacitor will <u>not</u> conduct current at zero frequency.

At higher frequencies, current <u>will pass</u> through the capacitor.

A capacitor has a "reactance": $X_c = -\frac{j}{\omega C} = \frac{-j}{2\pi f C} = \frac{-\sqrt{-1}}{2\pi f C}$.

Reactance @ low & high f: high f, $X_c \sim 0$; low f, $X_c \sim \infty$.

low pass: $\uparrow f$: $X_c \simeq 0$; $\frac{V_{out}}{V_{in}} \simeq 0$

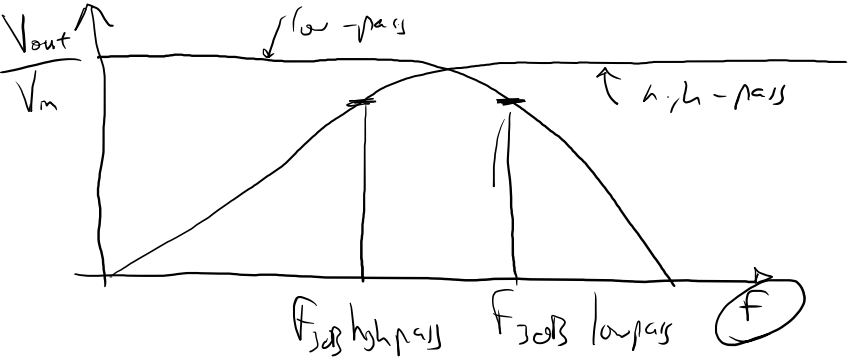
$\downarrow f$: $X_c \simeq \infty$; $\frac{V_{out}}{V_{in}} \simeq 1$.

high pass: $\uparrow f$: $X_C \simeq 0$, $\frac{V_{out}}{V_{in}} \simeq 1$

$\downarrow f$: $X_C \simeq \infty$, $\frac{V_{out}}{V_{in}} \simeq 0$.

Evaluate the circuit from Y_{out}: $R \| C$: $\boxed{\dfrac{\frac{-j}{\omega C} \cdot R}{\frac{-j}{\omega C} + R}}$

$\boxed{f_{3dB} : \frac{1}{2\pi RC}}$; $\boxed{\omega_{3dB} = \frac{1}{RC}}$



$$\frac{V_{out}}{V_m} = \frac{1}{\sqrt{2}}$$

Blocking Caps: