Aim of the module is to measure the real area of contact between two spheres. (Note area varies w/ applied load). Solution is in Ch's 3&4 of Johnson Contact Mechanics. 

\[ \text{effective radius of contact} \]
Lecturer's Assumptions:

1. Surfaces are continuous and non-conforming

2. The strains are small: \( a \ll R \)

3. The bodies I & II are elastic half-spaces.

4. The surfaces are frictionless: \( q_x, q_y = 0 \) (no tangential stresses).

The full solution is in Johnson's textbook; we'll do the dimensional analysis.
Coordinates of the surface:
\[ Z_1 = A_1 x^2 + B_1 y^2 \quad \text{Spherical cap} \]

For the complementary side (body II):
\[ Z_2 = -(A_2 x^2 + B_2 y^2) \]

*In lab reports, please explain equivalence bit crossed cylinders & spherical cap contact.*
For spherical caps: \( A = B = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \).

This implies that the contours of constant separation are circles, centered on the loading axis.

Now, we'll apply a load, \( P \) to force the spherical caps into contact. \( P \) is exerted along \( z \)-axis.

Distance between points on surface of bodies 1 & 2 is given by:

\[
h = Ax^2 + By^2 = \frac{1}{2} \left( \frac{x^2}{R_1} + \frac{y^2}{R_2} \right)
\]

Upon increasing \( P \), the bodies begin to deform.
This causes the bodies to displace from one another at $\delta_1$ and $\delta_2$.

The displacements along $z$ we'll designate as $u_{z1}$, $u_{z2}$ (functions of $x$ and $y$).

For points in contact:

\[ u_{z1} + u_{z2} + h = \delta_1 + \delta_2. \]

Now, define $\delta = \delta_1 + \delta_2$.

We can rewrite the above expression using the known geometry from $h$:

\[ u_{z1} + u_{z2} = \delta - Ax^2 - By^2. \]
For points outside the region of contact \((x^2 + y^2 > a^2)\)
\[U_{21} + U_{22} > J - A x^2 - B y^2.\]

At this point, we could do the full solution; here we'll do dimensional analysis.

**Dimensional Analysis:**
- Keep sphere–sphere contact geometry.

Note that displacements of nodes I & II at \((x, y) = 0, 0\) are \(U_{21}(0) = \delta_1\), \(U_{22} = \delta_2\).
Thus within the surface of contact:
\[
\left(\frac{U_{21}(0) - U_{21}(x)}{a}\right) + \left(\frac{U_{22}(0) - U_{22}(x)}{a}\right) = \frac{x^2}{2a} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)
\]
(from contact geometry)
Substitute \( x = a \), and write \( u_x(0) - u_x(a) = 0 \)

We can re-write the deflection as:

\[
\frac{d_1}{a} + \frac{d_2}{a} = \frac{a}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

Assume deflection is small: \( d \ll a \)

\( \Rightarrow \) typical strain scale is \( d/a \).

Because the bodies are linearly elastic, the strain will be proportional to the contact pressure divided by the elastic modulus.

Defining \( P_m \) as a characteristic contact pressure, we can substitute \( \frac{P_m}{E} \) for \( \epsilon = \frac{d}{a} \).
\[ \frac{P_m}{E_1} + \frac{P_m}{E_2} \propto a \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

\[ \Rightarrow P_m \propto \frac{a}{\frac{1}{E_1} + \frac{1}{E_2}} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

\[ \Rightarrow \text{contact stresses increase linearly with area.} \]

Now, we want to relate this to applied load, \( P \).

\[ P = \pi a^2 P_m \]

\[ \Rightarrow a \propto \left[ P \left( \frac{\frac{1}{E_1} + \frac{1}{E_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) \right]^{\frac{1}{3}} \]
Now we can write \( p_m \) as:

\[
p_m = \frac{p}{\pi a^2} \propto \left[ \frac{1}{\left( \frac{1}{R_1} + \frac{1}{R_2} \right)^2} \right]^{\frac{3}{2}} \frac{1}{\left( \frac{1}{E_1} + \frac{1}{E_2} \right)^2}
\]

For a 3-D solid, compressing \( \delta_1 \) and \( \delta_2 \) are proportional to displacements \( d_1 \) and \( d_2 \):

\[
\Rightarrow \delta \propto d_1 + d_2
\]

\[
= \frac{\delta}{a} = \frac{9}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad \Rightarrow \delta \propto \frac{a^2}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

\[
= \rho^{2/3} \left( \frac{1}{E_1} + \frac{1}{E_2} \right)^{2/3} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{2/3}
\]
Experimental method of Tobar. (available on the wiki.)

For conductors with specific conductivity $\sigma$

I

II

Maxwell: spher–plane contact $R = \frac{1}{2\sigma a^2}$

For spher–spher contact: $R = \frac{1}{2\sigma a^2}\left(\frac{1}{a} - \frac{2}{\pi^{1/2}}\right)$, $\sigma$ is spherical radius.

In the limit of a small contact radius $a \ll r$, junction conductance $\Delta = 2\sigma a^2$
Conductance can be related to applied force \( F \):

\[
\Lambda = k_2 \cdot \rho^{4/3}
\]

\[
\Lambda = \frac{i}{V}
\]

Read Tabor.

Passive components.

2 Fundamental laws for circuits:

Ohm’s law; Kirchhoff’s law(s)

\[
V = IR
\]

Hydrodynamics analogy:

\[ H = J \cdot R \]

Hydraulic pressure\hspace{2cm} Fluid flux \hspace{2cm} Hydrodynamic resistance

"A Volt pushes an Amp through an Ohm."

Most often, we measure \( V \) relative to a reference (by ground).
Ohmic vs. non-Ohmic behavior:

- Ohmic: slope is $R$, ohmic

- Non-linear resistor, e.g., a lamp

Power: $P = IV = I^2R$

Dimensionally:

- $I$: current, charge/ time
- $V$: work/ charge

$IV = \frac{\text{work}}{\text{time}} = [P]$
Kirchhoff's Laws:

I: Sum of voltages around a closed loop is 0.

II: Sum of currents at a node is 0.
Analysis of simple circuits / series resistance:

\[ I_{tot} = I_1 = I_2 \]
\[ V_{tot} = V_1 + V_2 \]
\[ V_T = I_t \cdot R_{eq} = I_t \cdot R_1 + I_t \cdot R_2 \]
\[ \rightarrow R_{eq} = R_1 + R_2 \]

Parallel resistors:

\[ I_{tot} = I_1 + I_2 \]
\[ V_{tot} = V_1 = V_2 \]
\[ V_{tot} = I_{tot} \cdot R_{eq} \]
\[ I_1 \cdot R_1 = I_2 \cdot R_2 = I_{tot} \cdot R_{eq} \]
\[ \frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{V_{tot}}{R_{eq}} \]
\[ R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \]
Estimation of \( R \) values:

\[
R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2}
\]

1. \( \frac{R^2}{2R} = \frac{R}{2} \)

2. \( \frac{10R^2}{10R} = 0.9R \)

3. \( \frac{2R^2}{5R} \approx 0.67R \)
Rule of dominance of resistors:

Device: V-divider.

\[ I = \frac{V_m}{R_1 + R_2} \implies V_{out} = I \cdot \frac{R_2}{R_1 + R_2} \]
Say Vout is used to drive a resistive load.

What values of R load don't modify Vout?

**Capacitors**: A static capacitor obeys the law:

\[ Q = CV \]

charge = capacitance \times voltage.
A change is very hard to quantify. So we have to current \( \frac{dQ}{dt} = I \).

\[ I = C \frac{dU}{dt} \]

\( I \) is a fixed parameter for our capacitor.

\( V \) is voltage of water

**Constant I:**

- \( V_t \)
- \( V_{esp} \)

**Graph:**

- Large \( I \)
- Small \( I \)
\( V_{in} = V_0 \sin(2\omega ft) \).

A capacitor will \underline{not} conduct current at zero frequency.

At higher frequencies, current will \underline{pass} through the capacitor.

A capacitor has a "reactance": \( X_c = -\frac{j}{\omega C} = -\frac{j}{2\pi f C} = -\frac{j}{2\pi f C} \).

Reactance @ low & high f: high f, \( X_c \approx 0 \), low f, \( X_c \approx \infty \).

\( j \): \( X_c = 0 \) \( \frac{V_A}{V_{in}} = 0 \) \( V_{in} \)

\( j \): \( X_c = \infty \) \( \frac{V_{in}}{V_{in}} = 1 \).
High pass: \[ X_c \approx 0, \quad \frac{V_{out}}{V_{in}} \approx 1 \]

Low pass: \[ X_c \approx \infty, \quad \frac{V_{out}}{V_{in}} \approx 0. \]

Evaluate the circuit from \( V_{in} \): \( R || C \):

\[
\frac{-j\omega C \cdot R}{\frac{-j}{\omega C} + R}
\]

\( F_{3dB} \): \[ \frac{1}{2\pi RC} \]

\( \omega_{3dB} \): \[ \frac{1}{RC} \]
\[ V_{out} = \frac{1}{V_{m}} \]

\[ \left< V_m \right>_t = 0 \]