1. Exercises on metric spaces

1.1 Prove that for a metric space (X, d) we have the following implications:

$$X \text{ is proper} \Rightarrow X \text{ is locally compact} \Rightarrow X \text{ is complete.}$$

1.2 Prove that every proper metric space is separable.

1.3 Find an example of a locally compact space that is not proper.

1.4 Find a bounded sequence in ℓ^∞ containing no convergent subsequence.

1.5 Le (X, d) be a metric space and $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function such that $h(0) = 0$. Prove that $\rho(x, y) = h(d(x, y))$ is again a metric on X.

1.6 Prove that for any metric space (X, d) there is a new metric ρ that induces the same topology as d on X and such that (X, ρ) is bounded (that is the ρ-diameter of X is finite).

1.7 Prove that in an intrinsic connected proper metric space any pair of points can be joined by a minimal geodesic.
(Hint. The proof is standard and can be found in many books on metric geometry. Consider first compact metric spaces and use Arzelà–Ascoli's Theorem).

2. BV functions and length of curves

2.1 Prove that a function $f \in BV[a, b]$ has at most countably many disconituities.

2.2 Let $f : [0, 1] \to [0, 1]$ be the Cantor-Vitalli function, and consider the graph of this function, that is the curve $\gamma : [0, 1] \to \mathbb{R}^2$ defined by $\gamma(t) = (t, f(t))$. Observe that γ is a continuous curve in the unit square joining $(0, 0)$ to $(1, 1)$. Prove that γ is rectifiable and

$$\int_0^1 \|\dot{\gamma}(t)\| dt = 1 < d((0, 0), (1, 1)) = \sqrt{2} < \ell(\gamma) = 1.$$

2.3 Give an exemple of a function $f \in BV([a, b])$ such that $\int_a^b f'(x)dx = f(b) - f(a)$, yet $f \notin AC([a, b])$.

2.4 Let $\gamma : [a, b] \to X$ be a rectifiable curve in a metric space X. The *arclength* function of γ is the function $s = s_\gamma : [a, b] \to \mathbb{R}$ defined by $s_\gamma(t) = \ell_a^t(\gamma)$. Prove that s_γ is a continuous function.
(Hint. For the case $X = \mathbb{R}$ a proof can be found in Taylor, Theorem 9.2.V).

2.5 Let γ be a curve in a metric space. Prove that γ is metrically differentiable at t_0 if and only if there exists $q \in \mathbb{R}$ such that

$$d(\gamma(t_1), \gamma(t_2)) - q \cdot |t_1 - t_2| = o(|t_1 - t_0| + |t_2 - t_0|)$$

as $t_1, t_2 \to t_0$. In that case $q = v_\gamma(t)$.

3. Hyperbolic Geometry

The first three exercices provides alternative formulas to compute the hyperbolic distances between two points

3.1 We denote by $\mathbb{H}^2$ the upper half-plane with the Poincaré metric. Consider two points p and q in $\mathbb{H}^2$ that are not vertically aligned (i.e. $\mathrm{Im}(p) \neq \mathrm{Im}(p)$) and let $a, b \in \mathbb{R}$ be the two ideal points (i.e. the "points at infinity") of the hyperbolic line through p and q. Prove that

$$d_{\mathbb{H}^2}(p,q) = |\log\tan(\angle_a pb) - \log\tan(\angle_a qb)|.$$

Where $\angle_a pb$ is the angle at a of the Euclidean triangle apb and likewise for $\angle_a qb$.

3.2 Using the same notations, prove that

$$d_{\mathbb{H}^2}(p,q) = |\log\tan(\frac{1}{2}\angle_c pb) - \log\tan(\frac{1}{2}\angle_c qb)|,$$

where $c = \frac{1}{2}(a+b)$ is the Euclidean center of the segment $[a,b]$.

3.3 The hyperbolic distance between $z, w \in \mathbb{H}^2$ is also given by

$$d_{\mathbb{H}^2}(z,w) = \log\left(\frac{|z-\overline{w}| + |z-w|}{|z-\overline{w}| - |z-w|}\right).$$

This formula is convenient because it does not involve the ideal points of the hyperbolic line through z and w.
(Hint. It is useful to check that the righthandside of this formula is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$).

3.4 Prove that the homography f_θ given by

$$f_\theta(z) = \frac{\cos(\theta)z - \sin(\theta)}{\sin(\theta)z + \cos(\theta)}$$

is a hyperbolic rotation of $\mathbb{H}^2$ centered at i (that is $f_\theta(i) = i$) *and rotation angle* 2θ.

3.5 Prove that the group of orientation preserving isometries of the Poincaré disk $\mathbb{D}^2$ is isomorphic to

$$PSU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\} / \{\pm 1\}$$

acting by homographies on the disk.