Introductory Topics in Finsler and Metric Geometry

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1 A brief review on metric spaces

Recall that a *metric space* is a set X together with a function

$$d : X \times X \to \mathbb{R}$$

satisfying the following properties for any x, y and z in X:

i) $d(x, y) = d(x, y)$;

ii) $d(x, z) \leq d(x, y) + d(y, z)$;

iii) $d(x, y) = 0$ if and only if $x = y$.

Such a function is called a *distance function*, or a *metric* on X and the second condition is the *triangle inequality*. Note that distances are always non negative since we have

$$2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0.$$

This definition is due to Maurice Frechet who introduced it in his famous 1906 paper named *Sur quelques points du calcul fonctionnel* (*On a few points in Functional Calculus*). This simple definition makes metric spaces one of the most elementary mathematical concepts, yet it leads to a rich array of concepts, problems and applications within all parts of mathematics.

A subset $U \subset X$ in a metric space is said to be *open* if for any point $x \in U$ and every $\varepsilon > 0$ the open ball centered at x with radius ε is contained in U:

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\} \subset U.$$

A subset $C \subset X$ is *closed* if its complement $X \setminus C$ is open. The open subsets of a metric spaces (X, d) form a topology on X. Recall that this means the following three properties hold:

a) The whole space X and the empty set $\emptyset$ are open sets.

b) The intersection of a finite collection of open set is an open set.

c) The union of an arbitrary collection of open set is an open set.

Therefore all the usual topological notions apply to metric spaces. We now introduce some more definitions.

Definitions. Let (X, d) be a metric space.

1. The *diameter* of a subset $A \subset X$ is defined as
$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$
2. The subset A is *bounded* if its diameter is finite.
3. A sequence $\{x_k\} \subset X$ *converges* to the point x if $d(x_k, x) \to 0$ as $k \to \infty$. In that case on writes $\lim\limits_{k\to\infty} x_k = x$.
4. A *Cauchy sequence* in X is a sequence $\{x_k\} \subset X$ such that
$$\lim_{n\to\infty} \operatorname{diam}\{x_k \mid k \geq n\} = 0.$$
It is easy to prove that every convergent sequence is Cauchy. The converse is not true in general and we define:
5. The metric space X is *complete* if every Cauchy sequence converges.
6. A subset $K \subset X$ is *compact* if every sequence in K contains a convergent subsequence. A subset $A \subset X$ is *relatively compact* if it is contained in a compact subset.
7. X is *locally compact* if every point x in X admits a relatively compact neighborhood, that is x is contained in a relatively compact open subset.
8. X is *separable* if it contains a countable dense subset: there is a countable set $S \subset X$ such that every point in X is the limit of a sequence contained in S.
9. The metric space X is *proper* (or *boundedly compact*) if every closed bounded set is compact. Equivalently it satisfies the Bolzano-Weirstrass property: *Every bounded sequence contains a converging subsequence.*

Proposition 1.1. *A proper metric space is separable.*

We live the proof as an exercise.

2 Banach spaces

Definitions 1. Let E be a vector space over the field of real numbers. A *norm* on E is a function $\nu : E \to \mathbb{R}$ such that

(i) $\nu(\lambda x) = |\lambda|\nu(x)$ for any $x \in E$ and $\lambda \in \mathbb{R}$.

(ii) $\nu(x) > 0$ for any $x \in E \setminus \{0\}$.

(iii) $\nu(x + y) \leq \nu(x) + \nu(y)$ for any $x, y \in E$.

We conveniently write the norm of a vector as $\|x\| = \nu(x)$. Note that a normed vector space $(E, \|\cdot\|)$ is naturally a metric space for the distance defined by $d(x, y) = \|y - x\|$.

2. A *Banach space* is a normed vector space that is complete for the above metric.
Let us give a few examples to illustrate the previous definitions on Banach spaces.

Examples.

(a) Every finite dimensional normed real vector space is a Banach space (by the classical Bolzano-Weierstrass Theorem). It is also proper and separable (points with rational coordinates in a given basis form a countable dense subspace).

(b) The vector space of all bounded sequences in $\mathbb{R}$ is a Banach space for the sup norm

$$\|(x_k)\|_\infty = \sup_k |x_k|.$$

We denote this Banach space by ℓ^∞ or $\ell^\infty(\mathbb{N})$. It is not locally compact.

(c) The space $C^0([a, b])$ of bounded continuous functions on the interval $[a, b]$ is a Banach spaces for the sup norm

$$\|f\|_{L^\infty} = \sup_{a \leq x \leq b} |f(x)| = \max_{a \leq x \leq b} |f(x)|,$$

this follows from the fact that a uniformly convergent sequence of continuous function is continuous.
This space is separable (proof: By Stone-Weierstrass Theorem real polynomials form a dense subset in $C^0([a, b])$, then polynomials with rational coefficients clearly form a countable dense subset).

(d) A measurable function $f : [a, b] \to \mathbb{R}$ is *essentially bounded* if there is a real number $a \in \mathbb{R}$ such that $\{x \in [a, b] \mid f(x) > a\}$ has zero Lebesgue measure. Such a number a is an *essential upper bound* for f. The vector space of all essentially bounded measurable function on the interval $[a, b]$ is a Banach space for the norm

$$\begin{aligned} \|f\|_{L^\infty} &= \text{ess sup}_{a \leq x \leq b} |f(x)| \\ &= \inf\{a \in \mathbb{R} \mid a \text{ is an essential upper bound for } f\}. \end{aligned}$$

This Banach space is denoted by $L^\infty([a, b])$, it is not separable. Observe that $C^0([a, b]) \subset L^\infty([a, b])$ is a closed subset.

(e) $C^0([a,b])$ is *not* a Banach spaces for the L^1-norm:

$$\|f\|_{L^1} = \int_a^b |f(x)|dx.$$

The following theorem characterizes finite dimensional Banach spaces.

Theorem 2.1. *For a Banach space* $(E, \| \|)$*, the following conditions are equivalent:*

(i) E *is a proper metric space.*

(ii) E *is locally compact.*

(iii) The closed unit ball $\bar{B} = \{x \in E \mid \|x\| \leq 1\}$ *is compact.*

(iv) $\dim(E) < \infty$.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. Assume E to be locally compact, then there exists $\epsilon > 0$ such that the closed ball $\bar{B}_\epsilon = \{x \in E \mid \|x\| \leq \epsilon\}$ is compact. Therefore $(ii) \Rightarrow (iii)$ since the unit ball is homeomorphic to $\bar{B}_\epsilon$.
We prove $(iii) \Rightarrow (iv)$ by contraposition. Assuming $\dim(E) = \infty$, we shall construct a sequence with no convergent subsequence in $S = \{x \in E \mid \|x\| = 1\} = \partial B$.
Suppose $\{x_1, x_2, \dots x_m\} \subset S$ is a family of unit vectors such that $\|x_i - x_j\| \geq 1$ for all $1 \leq i, j \leq m$. Such a family certainly exists for $m = 1$ (chose an arbitrary unit vector x_1). Denote by $F_m \subset E$ the vector subspace generated by these vectors. This is a finite dimensional subspace and it is therefore closed.
Chose a point $y \in E \setminus F_m$. Because F_m is proper one can find a point $z \in F_m$ minimizing the distance to y, that is

$$\|z - y\| = \min\{\|w - y\| : w \in F_m\}.$$

Set $x = \dfrac{z - y}{\|z - y\|}$, then for any $u \in F_m$ we have

$$\|u - x\| = \frac{\|\|z - y\| \cdot u - (z - y)\|}{\|z - y\|} = \frac{\|w - y\|}{\|z - y\|} \geq 1$$

where we have set $w = z - \|z - y\| \cdot u \in F_m$.

We now set $x_{m+1} = x$, we then have $\|x_i - x_{m+1}\| \geq 1$ for each $1 \leq i \leq m$. Therefore the $(m + 1)$ points $\{x_1, x_2, \dots x_{m+1}\} \subset S$ have pairwise distances ≥ 1.
Repeating the argument, we construct an infinite sequence of unit vectors with pairwise distances ≥ 1. Such a sequence contains no convergent subsequence. It follows that the ball $\bar{B}$ is not compact.
$(iv) \Rightarrow (i)$ is Bolzano-Weierstrass Theorem. □

Proposition 2.2. *The Banach space* $\ell^\infty(\mathbb{N})$ *is not separable.*

Proof. The characteristic function $\mathbf{1}_A$ of an arbitrary subset $A \subset \mathbb{N}$ defines an element in $\ell^\infty(\mathbb{N})$ and if A and B are distinct subsets then $\|\mathbf{1}_A - \mathbf{1}_B\|_\infty = 1$. If $S \subset \ell^\infty(\mathbb{N})$ is a dense subset, it contains for any non empty subset $A \subset \mathbb{N}$ an element $s_A \in S$ such that $\|\mathbf{1}_A - s_A\|_\infty < \frac{1}{2}$. It follows from the triangle inequality that the s_A are pairwise distinct, therefore S is uncountable (it contains at least $\mathrm{Card}(\mathcal{P}(\mathbb{N}))$ elements). □

Banach spaces play an important role in metric geometry. In fact every separable metric space can be seen as a subset of a Banach space.

Theorem 2.3. *Every separable metric space (X, d) admits an isometric embedding into $\ell^\infty(\mathbb{N})$, that is there is a map $\psi : X \to \ell^\infty(\mathbb{N})$ such that*

$$\|\psi(x) - \psi(y)\|_\infty = d(x, y)$$

for any $x, y \in X$.

Proof. Let us fix a base point s_0 and chose a countable dense subset $S = \{s_k\}_{k\in\mathbb{N}} \subset X$. Now to any point $x \in X$ we associate the real sequence $\psi(x) = (\psi_k(x))_{k\in\mathbb{N}}$ where

$$\psi_k(x) = d(x, s_k) - d(s_0, s_k) \in \mathbb{R}.$$

This sequence is bounded since by the triangle inequality we have for all $k \in \mathbb{N}$

$$|\psi_k(x)| \leq |d(x, s_k) - d(s_0, s_k)| \leq d(x, s_0).$$

We therefore have defined map $\psi : X \to \ell^\infty$; we need to prove this maps preserves distances. Let us fix two points x and y in X, using again the triangle inequality we have

$$\begin{aligned}|\psi_k(x) - \psi_k(y)| &= |(d(x, s_k) - d(s_0, s_k)) - (d(y, s_k) - d(s_0, s_k))| \\ &= |d(x, s_k) - (d(y, s_k)| \\ &\leq d(x, y).\end{aligned}$$

This inequality holds for any $k \in \mathbb{N}$, therefore

$$\|\psi(x) - \psi(y)\|_\infty = \sup_k |\psi_k(x) - \psi_k(y)| \leq d(x, y).$$

To prove the reverse inequality we fix $\epsilon > 0$ and chose a point $s_m \in S$ such that $d(x, s_m) \leq \epsilon$ (here we use the density of $S \subset X$). We then have

$$\begin{aligned}|\psi_m(x) - \psi_m(y)| &= |d(x, s_m) - (d(y, s_m)| \\ &\geq d(x, s_m) + d(y, s_m) - 2\epsilon \\ &\geq d(x, y) - 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have

$$\|\psi(x) - \psi(y)\|_\infty \geq d(x, y).$$

The theorem is proved.

□

Remarks 1. The maps $\psi : X \to \ell^\infty$ we constructed in this proof is called the *Kuratovski* embedding of X.

2. If the metric space X is bounded one can use instead the simpler map $\phi : X \to \ell^\infty$ defined by $\phi(x) = (\phi_k(x))_{k\in\mathbb{N}}$ where $\phi_k(x) = d(x, s_k) \in \mathbb{R}$.

3. A non separable metric space (X, d) can also be isometrically embedded in a metric space. Namely the map that associates to any point x the function $f_x : X \to \mathbb{R}$ defined by $f_x(y) = d(x, y) - d(x_0, y)$ is an isometric embedding of X into the Banach space $C_b(X)$ of bounded continuous functions on X with the sup norm. This is also called the Kuratowski embedding. However we find it convenient to embed a (proper) metric space on ℓ^∞ since it provides us with "coordinates" on X.

3 BV functions and Rectifiables Curves

Definitions Let (X, d) be a metric space and $[a, b]$ be a closed interval. A function $f : [a, b] \to X$ has *bounded variation* if

$$V_a^b(f) = \sup_{\sigma} \sum_{i=1}^{m-1} d(f(t_i), f(t_{i+1})) < \infty, \tag{1}$$

where the supremum is taken over all the subdiviion $\sigma = [a = t_1 < t_2 < \cdots < t_m = b]$ of the interval $[a, b]$. The quantity $V_a^b(f)$ is then called the *total variation* of f on the interval $[a, b]$. A *rectifiable curve* in X is a continuous function $\gamma : [a, b] \to X$ of bounded variation. In that case, the total variation is called the *length* of the curve and is denoted by

$$\ell(\gamma) = \ell_a^b(\gamma) = V_a^b(\gamma).$$

Each sum in equation (1) is sometimes called a *discrete approximation*, or a *polygonal approximation* of the length of γ. If the continuous curve γ is not rectifiable, one notes $\ell(\gamma) = \infty$. The notion of length leads to some additional definitions.

a) The metric space is *rectifiably connected* if any pair of points p, q in X can be joined by a rectifiable curve.

b) The space X is *totaly unrectifiable* if it contains no non constant rectifiable curve (equivalently no pair of distinct points in X can be joined by a rectifiable curve). A classical example of a totally unrectifiable curve is the Von Koch snowflake curve.

c) The curve $\gamma : [a, b] \to X$ is a (minimal) *geodesic* if it is rectifiable and its length is the distance between its endpoints:

$$\ell(\gamma) = d(\gamma(a), \gamma(b)).$$

(Note that this definition differs slightly from the notion of geodesics in Riemannian geometry).

d) The distance d on X is *intrinsic* if for any pair of points p, q in X we have

$$d(p, q) = \inf\{\ell(\gamma) \mid \gamma \text{ is a rectifiable curve joining } p \text{ to } q\}.$$

e) The metric space X is *geodesic* if any pair of points p, q in X can be joined by a geodesic.

Observe the following obvious implications

$$(X, d) \text{ is geodesic } \Rightarrow (X, d) \text{ is intrinsic } \Rightarrow (X, d) \text{ is rectifiably connected.}$$

Exercice. Give examples showing that the converse implications fail.

Proposition 3.1. *Let (X, d) be a rectifiably conneced metric space. Define a new function $\tilde{d} : X \times X \to \mathbb{R}$ by*

$$\tilde{d}(p, q) = \inf\{\ell(\gamma) \mid \gamma \textit{ is a rectifiable curve joining } p \textit{ to } q\}.$$

Then $\tilde{d}$ is a new metric on X. Furthermore $\tilde{d}$ is intrinsic and $\tilde{d}(p, q) \geq d(p, q)$ for any $p, q \in X$.

The metric $\tilde{d}$ is called the *intrinsic* metric on X associated to d. Note that d is intrinsic if and only if $d = \tilde{d}$.

Exercice. Prove the Proposition (hint: use the fact that $d(p,q) \leq \ell(\gamma)$ for any curve joining p to q).

Proposition 3.2. *An intrinsic connected proper metric space is geodesic.*

We live the proof as an exercise. This result is sometimes called the *Hopf-Rinow Theorem for metric spaces.*

4 Real valued functions with bounded variation.

In this section we cover some basic facts on BV functions. Let us denote by $BV([a,b])$ the set of functions $f : [a,b] \to \mathbb{R}$ with bounded variation. We start with the following

Lemma 4.1 (Jordan decomposition)**.** *A function $f : [a,b] \to \mathbb{R}$ has bounded variation if and only if it is the difference of two monotone functions.*

Proof. Assume $f = g - h$ where $g, h : [a,b] \to \mathbb{R}$ are monotone non decreasing. We then have for any subdivision σ of $[a,b]$:

$$\begin{aligned}
\sum_{i=1}^{m-1} |f(t_{i+1}) - f(t_i)| &= \sum_{i=1}^{m-1} |(g(t_{i+1}) - h(t_{i+1})) - (g(t_i) - h(t_i))| \\
&\leq \sum_{i=1}^{m-1} |g(t_{i+1}) - g(t_i)| + \sum_{i=1}^{m-1} |h(t_{i+1}) - h(t_i)| \\
&\leq \sum_{i=1}^{m-1} (g(t_{i+1}) - g(t_i)) + \sum_{i=1}^{m-1} (h(t_{i+1}) - h(t_i)) \\
&= (g(b) - g(a)) + (h(b) - h(a)).
\end{aligned}$$

It follows that f has bounded variation with

$$V_a^b(f) \leq (g(b) - g(a)) + (h(b) - h(a)).$$

Assume conversely that $f \in BV([a,b])$, then we can write this function as

$$f(x) = V_a^x(f) - (V_a^x(f) - f(x))$$

where $V_a^x(f)$ is the variation of the restriction $f|_{[a,x]}$. Clearly $x \mapsto V_a^x(f)$ is monotone non decreasing and the function $(V_a^x(f) - f(x))$ is also non decreasing since for $x < y$ we have

$$(V_a^y(f) - f(y)) - (V_a^x(f) - f(x)) = V_x^y(f) - (f(y) - f(x)) \geq 0.$$

□

As an immediate consequence, we observe that the set of functions with bounded variation on $[a,b]$ is a real vector space. We shall denote it by $BV[a,b]$.

Corollary 4.2. *A BV function has at most countably many discontinuities.*

We leave the proof as an exercice.

Theorem 4.3. *A monotone function $f : [a, b] \to \mathbb{R}$ is almost everywhere differentiable. Furthermore the derivative f' is Lebesgues integrable on $[a, b]$ and*

$$\left| \int_a^b f'(x)dx \right| \leq |f(b) - f(a)|.$$

The proof of this theorem is quite involved, see e.g. [2, §9.7]

Corollary 4.4. *Any function $f \in BV[a, b]$ is almost everywhere differentiable and the derivative f' is Lebesgues integrable.*

The strict inequality is possible. The classical example is the *Cantor-Vitalli function* (also known as the *devil staircase*). This is a continuous surjective monotone function $f : [0, 1] \to [0, 1]$ such that $f'(x) = 0$ on the complement of the cantor set. In particuler $f'(x) = 0$ almost everywhere, yet we have

$$0 = \left| \int_0^1 f'(x)dx \right| = |f(1) - f(0)| = 1.$$

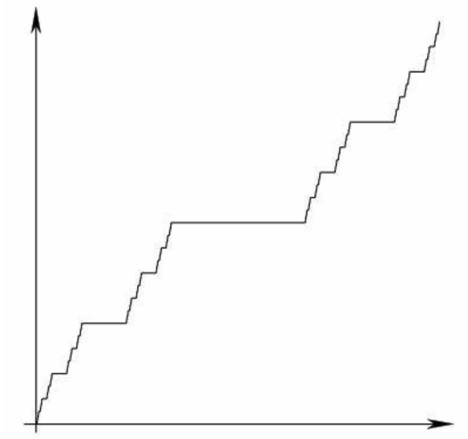

Figure 1: The Cantor-Vitali function

The next result relates rectifiable curves in $\mathbb{R}^n$ to BV functions:

Theorem 4.5. *A continuous curve $\gamma(t) = (x_1(t), \ldots, x_n(t))$ in $\mathbb{R}^n$ is rectifiable if and only if every componant $x_j : [a, b] \to \mathbb{R}$ is a function of bounded variation. Furthermore we have*

$$\int_a^b \|\dot{\gamma}\| dt \leq \ell(\gamma).$$

This inequality can again be a strict inequality.

Returning to absolutely continuous curves, we state the following

Proposition 4.6. *Let $\gamma : [a, b] \to X$ be a rectifiable curve in a metric space X then the* arclength function

$$t \mapsto \ell_a^t(\gamma)$$

is continuous on $[a, b]$.

5 Absolutely continuous function

Theorem 3 leads us to the following natural question: *for which class of function the fundamental theorem of calculus hold?* This question has been answered by Henri Lebesgue and is given by the class of absolutely continuous functions.

Definition. The function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any finite set of pairwise disjoint intervals $(a_1, b_1), (a_2, b_2), \dots (a_m, b_m)$ contained in $[a, b]$ we have

$$\sum_{i=1}^{m} (b_i - a_i) < \delta \implies \sum_{i=1}^{m} |f(b_i) - f(a_i)| < \epsilon.$$

We denote by $AC[a, b])$ the set of absolutely continuous functions on $[a, b]$. It is not difficult to check that this is a real vector space. Observe that $AC[a, b] \subset BV[a, b]$: every absolutely continuous function has bounded variation.

Examples. 1. Any Lipschitz function f is absolutely continuous.
2. The Cantor-Vitally function is *not* absolutely continuous.

Theorem 5.1. *Let f be an arbitrary function defined on the interval $[a, b]$, then the following conditions are equivalent:*

i) f is absolutely continuous.

ii) There exists a function $g \in L^1([a, b])$ such that for any $a \le x_1 \le x_2 \le b$ we have

$$|f(x_2) - f(x_1)| \le \int_{x_1}^{x_2} g(x)dx$$

iii) f is almost everywhere differentiable, with derivative $f' \in L^1([a, b])$ and

$$\int_{x_1}^{x_2} f'(x)dx = f(x_2) - f(x_1)$$

for any $a \le x_1 \le x_2 \le b$.

iv) The function f has bounded variation and it maps sets of zero Lebesgue measure to sets of zero Lebesgue measure.

The proof is delicate. The last condition is known as the *Banach-Zaredski Theorem.*

Theorem 5.2. *Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ in $\mathbb{R}^n$ be a rectifiable curve. Then*

$$\int_a^b \|\dot{\gamma}\| dt = \ell(\gamma).$$

if and only if each component $x_j : [a, b] \to \mathbb{R}$ is an absolutely continuous function.

A more general version of this result later will be given in Theorem 6.4 below.

6 Metric derivative of curves in metric spaces

In this section we consider the following notion:

Definition. Let $\gamma : [0,1] \to X$ be an arbitrary curve in the metric space (X,d). On says that γ is *metrically differentiable* at $t \in [a,b]$ if the following limit exists

$$v_\gamma(t) = \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|}$$

This limit is then called the *metric derivative* or the *speed* of γ at t.

Remark. If $(E, \| \, \|)$ is a Banach space and $\gamma : [0,1] \to E$ is a curve that is differentiable at $t \in [a,b]$, then it is metrically differentiable at t and

$$v_\gamma(t) = \|\dot\gamma(t)\|.$$

Indeed we have

$$\left| \frac{\|\gamma(t+\epsilon) - \gamma(t)\|}{|\epsilon|} - \|\dot\gamma(t)\| \right| \leq \left\| \frac{\gamma(t) - \gamma(t+\epsilon)}{\epsilon} - \dot\gamma(t) \right\| \to 0$$

as $\epsilon \to 0$.

The following Stepanov type theorem is a useful criterion for the existence of metric derivative (see [1, Theorem 2.5]):

Proposition 6.1. *Let $\gamma : [a,b] \to X$ be a curve in a separable metric space. Assume that for a.e. $t \in [a,b]$ we have*

$$\limsup_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} < \infty,$$

then γ is almost everywhere metrically differentiable.

Corollary 6.2. *A rectifiable curve in an arbitrary metric space is a.e. metrically differentiable.*

Proof. Let X be a metric space and $\gamma : [a,b] \to X$ be a rectifiable curve. The function $s(t) = \ell_a^t(\gamma)$ is continuous and monotonous, therefore it is a.e. differentiable. The corollary then follows immediately from the Proposition since for any point $t \in [a,b]$ of differentiability of $s(t)$ we have

$$\limsup_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} \leq \limsup_{\epsilon \to 0} \frac{\ell_t^{t+\epsilon}(\gamma)}{|\epsilon|} = \limsup_{\epsilon \to 0} \frac{|s(t+\epsilon) - s(t)|}{|\epsilon|} = |s'(t)| < \infty.$$

□

Definition. The curve $\gamma : [a,b] \to X$ is *absolutely continuous* if there exists $g \in L^1([a,b])$ such that

$$d(\gamma(t_1), \gamma(t_2)) \leq \int_{t_1}^{t_2} g(t)dt \tag{2}$$

for any $a \leq t_1 \leq t_2 \leq b$. Any function g satisfying the above inequality is called a *dominating function* for γ.

Proposition 6.3. *Let $\gamma : [0,1] \to X$ be an absolutely continuous curve in a separable metric space (X, d), then it is metrically differentiable almost everywhere. Furthermore the function $t \to v_\gamma(t)$ is integrable and it is the smallest dominating function for γ.*

Proof. It is easy to check that if the curve γ is absolutely continuous with dominating function g, then it is rectifiable and

$$\ell_{t_1}^{t_2}(\gamma) \leq \int_{t_1}^{t_2} g(t)dt$$

for any $a \leq t_1 \leq t_2 \leq b$. From the Corollary 6.2, γ is a.e. metrically differentiable and we have at any Lebesgue point of g:

$$v_\gamma(t) = \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} \leq \lim_{\epsilon \to 0} \left| \frac{1}{\epsilon} \int_t^{t+\epsilon} g(u)du \right| = g(t).$$

We now assume X to be separable. By Kuratowski's embedding, we may assume $X = \ell^\infty(\mathbb{N})$. Then $\gamma(t) = (\gamma_k(t))_{k \in \mathbb{N}}$ where $\gamma_k : [a,b] \to \mathbb{R}$ is the k^{th} component of γ. We clearly have for any $a \leq s < t \leq b$:

$$|\gamma_k(t) - \gamma_k(s)| \leq \sup_{j \in \mathbb{N}} |\gamma_j(t) - \gamma_j(s)| = d(\gamma(s), \gamma(t)) \leq \int_t^s g(r)dr$$

where g is a dominating function for γ. This implies in particular that γ_k is absolutely continuous and $|\dot{\gamma}_k(t)| \leq g(t)$ almost everywhere. Let us denote by $w : [a,b] \to \mathbb{R}$ the function defined by

$$w(t) = \begin{cases} \sup_k |\dot{\gamma}_k(t)| & \text{if } \gamma_k \text{ is differentiable at } t \text{ for all } k, \\ 0 & \text{else.} \end{cases}$$

We then have almost everywhere

$$w(t) = \sup_k \lim_{\epsilon \to 0} \frac{|\gamma_k(t+\epsilon) - \gamma_k(t)|}{|\epsilon|} \leq \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t+\epsilon))}{|\epsilon|} = v_\gamma(t) \leq g(t),$$

in particular $w(t)$ is integrable. It is a dominating function for γ because

$$d(\gamma(t), \gamma(t+\epsilon)) = \sup_{k \in \mathbb{N}} |\gamma_k(t+\epsilon) - \gamma_k(t)| \leq \int_t^{t+\epsilon} w(u)du.$$

We have thus proved the following facts:

(i) v_γ is smaller or equal than any dominating function for γ.

(ii) $w \leq v_\gamma$.

(iii) w is dominating function for γ.

It follows that $v_\gamma = w$ and is therefore the smallest dominating function. $\square$

Remark. The above proposition says that the metric derivative of an absolutely continuous curve in a separable metric space exists a.e., is an integrable function and satisfies

$$\ell_{t_1}^{t_2}(\gamma) \leq \int_{t_1}^{t_2} v_\gamma(t)dt \tag{3}$$

for any $a \leq t_1 \leq t_2 \leq b$. Furthermore, if $X = \ell^\infty$, then we have a.e.

$$v_\gamma(t) = \sup_{k\in\mathbb{N}} |\dot{\gamma}_k(t)|.$$

Theorem 6.4. *Let $\gamma : [0,1] \to X$ be a continuous curve in a separable metric space (X,d). Then it is absolutely continuous if and only if it is metrically differentiable almost everywhere, the function v_γ is integrable and we have*

$$\ell_{t_1}^{t_2}(\gamma) = \int_{t_1}^{t_2} v_\gamma(t)dt$$

for any $a \leq t_1 \leq t_2 \leq b$.

Proof. Fix $m \in \mathbb{N}$ and consider the uniform subdivision $\sigma = [a = t_1 < t_2, \dots, , < t_m = b]$ of the interval $[a,b]$ defined by $t_i = a + (b-a)\epsilon$, where $\epsilon = \frac{i-1}{m-1}$. We then have

$$\begin{aligned} \frac{1}{\epsilon}\int_b^{a-\epsilon} d(\gamma(t),\gamma(t+\epsilon))dt &= \frac{1}{\epsilon}\sum_{i=1}^{m-2}\int_0^\epsilon d(\gamma(t_i+s),\gamma(t_{i+1}+s))ds \\ &\leq \frac{1}{\epsilon}\int_0^\epsilon \ell(\gamma)dt = \ell(\gamma). \end{aligned}$$

We therefore have

$$\int_b^a v_\gamma(t)dt = \int_b^{a-\epsilon} \lim_{\epsilon\to 0}\frac{d(\gamma(t),\gamma(t+\epsilon))}{\epsilon}dt \leq \liminf_{\epsilon\to 0}\int_b^{a-\epsilon}\frac{d(\gamma(t),\gamma(t+\epsilon))}{\epsilon}dt \leq \ell(\gamma).$$

The reverse inequality has been proved above in (3), which concludes the proof of the Theorem. $\square$

7 Reparametrization and natural parametrization of a rectifiable curve

Definition 7.1. *One says that a curve $\tilde{\gamma} : [\tilde{a},\tilde{b}] \to X$ in a metric space (X,d) is a* reparametrization *of the curve $\gamma : [a,b] \to X$ if there exists a function $g : [a,b] \to [\tilde{a},\tilde{b}]$ such as*

(i) g is continuous, monotone and surjective.

(ii) If $g(t_1) = g(t_2)$ for some $t_1, t_2 \in [a,b]$, then $\gamma(t_1) = \gamma(t_2)$.

(iii) $\gamma = \tilde{\gamma} \circ g$.

Unsurprisingly the length of a curve is invariant under reparametrization.

Proposition 7.2. *Let $\tilde{\gamma}$ be a reparametrization of the curve γ. Then $\tilde{\gamma}$ is rectifiable if and only if so is γ. In that case both curves have the same length.*

Proof. Let us assume $\gamma = \tilde{\gamma} \circ g$ where the function g is as in the above definition.
Fix an arbitrary partition $\sigma = [a = t_0 \leq t_1 \leq \cdots \leq t_m = b]$ of the interval $[a, b]$ and set $\tilde{t}_j = g(t_j)$, then $\tilde{\sigma} = [\tilde{a} = \tilde{t}_0 \leq \tilde{t}_1 \leq \cdots \leq \tilde{t}_m = \tilde{b}]$ is a partition of $[\tilde{a}, \tilde{b}]$ and we clearly have

$$\sum_{k=1}^{m} d(\gamma(t_{k-1}), \gamma(t_k)) = \sum_{k=1}^{m} d(\tilde{\gamma}(\tilde{t}_{k-1}), \tilde{\gamma}(\tilde{t}_k)) \leq \ell(\tilde{\gamma}).$$

Taking the supremum over all partitions of $[a, b]$ gives us the inequality $\ell(\gamma) \leq \ell(\tilde{\gamma})$.
The proof of the converse inequality is almost the same but we have to take care of the fact that g might be non injective. Namely let us consider an arbitrary partition $\tilde{\sigma} = [\tilde{a} = u_0 \leq u_1 \leq \cdots \leq u_m = \tilde{b}]$ of $[\tilde{a}, \tilde{b}]$. For each k chose an element $t_k \in g^{-1}(u_k)$ with the convention that $t_0 = a$ and $t_m = b$. Then $\sigma = [a = t_0 \leq t_1 \leq \cdots \leq t_m = b]$ is a partition of $[a, b]$ which is mapped by g to σ and we have

$$\sum_{k=1}^{m} d(\tilde{\gamma}(u_{k-1}), \tilde{\gamma}(u_k)) = \sum_{k=1}^{m} d(\gamma(t_{k-1}), \gamma(t_k)) \leq \ell(\gamma).$$

Taking now the supremum over all partitions of $[\tilde{a}, \tilde{b}]$ yields $\ell(\tilde{\gamma}) \leq \ell(\gamma)$.

□

Definition. A rectifiable curve $\gamma : [a, b] \to X$ in a metric space is said to be *naturally parametrized*, or *parametrized by its arc length* if for any $a \leq s_1 < s_2 \leq b$ we have

$$\ell_{s_1}^{s_2}(\gamma) = s_2 - s_1.$$

Remark. A curve that is naturally parametrized is clearly 1-Lipschitz. In particular it is absolutely continuous. Furthermore it follows from Theorem 6.4 that γ has almost everywhere unit speed:

$$v_\gamma(t) = \lim_{\epsilon \to 0} \frac{d(\gamma(t), \gamma(t + \epsilon))}{|\epsilon|} = 1 \quad \text{for a.e. } t \in [a, b].$$

In particular it is not constant on any (non trivial) interval. Observe also that $\ell(\gamma) = b - a$.

Theorem 7.3. *Let $\gamma : [a, b] \to X$ be an arbitrary rectifiable curve. Then γ can be reparametrized by its arc length.*

Proof. Let us set $L = \ell(\gamma)$ and denote by $g : [a, b] \to [0, L]$ the arclength function of γ. Recall that $g(t) = \ell_a^t(\gamma)$ and g is continuous by Proposition 4.6. It is clearly monotone and since $g(a) = 0$ and $g(b) = L$ the function g is also surjective. Furthermore, if $g(t_1) = g(t_2)$ with $a \leq t_1 \leq t_2 \leq b$, then $\gamma(t_1) = \gamma(t_2)$ because

$$d(\gamma(t_1), \gamma(t_2)) \leq \ell_{t_1}^{t_2}(\gamma) = \ell_a^{t_2}(\gamma) - \ell_a^{t_1}(\gamma) = g(t_2) - g(t_1) = 0.$$

The function g is generally not injective, but it has a left inverse defined by

$$h(s) = \inf\{t \in [a,b] \mid \ell_a^t(\gamma) = s\},$$

we indeed clearly have $g(h(s)) = s$ for any $s \in [a,b]$. We claim that the function $\tilde{\gamma} = \gamma \circ h : [0,L] \to X$ is 1-Lipschitz. Indeed we have for any $0 \leq s_1 \leq s_2 \leq L$

$$d(\tilde{\gamma}(s_1), \tilde{\gamma}(s_2)) = d(\gamma(h(s_1)), \gamma(h(s_2)) \leq \ell_{h(s_1)}^{h(s_2)}(\gamma) = g(h(s_2)) - g(h(s_1)) = s_2 - s_1.$$

In particular $\tilde{\gamma}$ is a continuous curve and it is a reparametrization of γ since $\gamma = \tilde{\gamma} \circ g$ where g satisfies the three conditions in Definition 7.1. Using the definition of h and the previous Proposition we have for any $s \in [0,L]$

$$\ell_0^s(\tilde{\gamma}) = \ell_a^{h(s)}(\gamma) = s,$$

and therefore

$$\ell_{s_1}^{s_2}(\tilde{\gamma}) = \ell_0^{s_2}(\tilde{\gamma}) - \ell_0^{s_1}(\tilde{\gamma}) = s_2 - s_1$$

for any $a \leq s_1 < s_2 \leq b$. □

Remark. Using Theorem 6.4 we observe that if the curve γ is absolutely continuous, then the reparametrization function can be written as

$$g(t) = \int_a^t v_\gamma(u)du.$$

References

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