Nearest Point Projection on Convex Set

Recall that a subset C of a geodesic metric space (X, d) is *convex* if for any geodesic segment $[p, q] \subset X$ we have

$$p, q \in C \quad \Rightarrow \quad [p, q] \subset C.$$

We say that the space has *strictly convex balls* if for any point $o \in X$ and any geodesic segment $[p, q] \subset X$, we have

$$x \in [p, q],\ x \neq p, x \neq q \quad \Rightarrow \quad d(p, x), \max\{d(o, p), d(o, q)\}.$$

In class we have proved the following:

Proposition 1. *Let (X, d) be a metric space with strictly convex balls. Then for any closed convex set $C \subset X$ there exists a map $\pi_C : X \to C$ such that for any points $x \in X$ and $y \in C$ we have*

$$d(x, \pi_C(x)) \leq d(x, y),$$

with equality if and only if $y = \pi_C(x)$.

The map π_C is called the *nearest point projection map* of X onto C.
It is obvious that any CAT(0) space has strictly convex balls. We will prove the following result:

Proposition 2. *In a CAT(0) space (X, d) the nearest point projection map onto a closed convex subset C is 1-Lipschitz.*

We will need the following lemma:

Lemma 3. *Let x, u, v be three points in a CAT(0) space X. Assume that $d(x, u) \leq d(x, w)$ for all point $w \in [u, v]$. Then the Alexandrov angle at u of the triangle xuv satisfies*

$$\angle_u(x, v) \geq \frac{\pi}{2}.$$

Proof. Observe first that for any point $z \in [x, u]$ and any $w \in [u, v]$ we have

$$d(z, u) = d(x, u) - d(x, z) \leq d(x, w) - d(x, z) \leq d(z, w). \tag{1}$$

We claim that

$$\widetilde{\angle}_u^0(z, w) \geq \frac{\pi}{2}.$$

Suppose otherwise, that is the Euclidean comparison triangle $\triangle' z'u'w'$ of $\triangle zuw$ has an angle $< \frac{\pi}{2}$ at u'. This would implies the existence of a point $\tilde{w}' \in [u', w']$ such that $\|z' - \tilde{w}'\| < \|z' - u'\|$ an the point $\tilde{w} \in [u, w] \subset [u, v]$ corresponding to $\tilde{w}'$ would satisfy:

$$d(z, \tilde{w}) \leq \|z' - \tilde{w}'\| < \|z' - u'\| = d(z, u),$$

contradicting (1). If we now let $z \in [y, u]$ and $w \in [u, v]$ converge to u we then have

$$\angle_u(x, v) = \lim_{z,w \to u} \widetilde{\angle}_u^0(z, w) \geq \frac{\pi}{2}.$$

□

Proof of the Proposition. We first prove the proposition in the case $X = \mathbb{R}^n$. Let us consider a closed convex subset $C \subset \mathbb{R}^n$ with non empty complement and let $x, y \in \mathbb{R}^n \setminus C$. Let us denote by $u = \pi_C(x)$ and $v = \pi_c(y)$ their projections on C. We observe that the points u and v satisfy the conditions

$$\langle x - u, w - u \rangle \leq 0, \quad \text{and} \quad \langle y - v, w - v \rangle \leq 0$$

for any $w \in C$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product. In particular we have

$$\delta = \langle (x - u) - (y - v), (u - v) \rangle \geq 0.$$

We therefore have

$$\begin{aligned} \|x - y\|^2 &= \|(x - u) - (y - v) + (u - v)\|^2 \\ &= \|(x - u) - (y - v)\|^2 + \|u - v\|^2 + 2\delta \\ &\geq \|u - v\|^2, \end{aligned}$$

that is $\|\pi_C(x) - \pi_C(y)\| \leq \|x - y\|$. The Corollary is proved in the Euclidean case.

Let us now consider a general CAT(0) space (X, d) and $C \subset X$ a closed convex subset. Given $x, y \in X \setminus C$ with projections $u = \pi_C(x)$ and $v = \pi_c(y)$. For any point $w \in [u, v]$ we have $d(x, w) \geq d(x, u)$ since $[u, v] \subset C$. Using previous Lemma we obtain:

$$\angle_u(x, v) \geq \frac{\pi}{2}, \tag{2}$$

where $\angle_u(x, v)$ is the Alexandrov angle at u of the triangle $\triangle xuv$. Likewise we have

$$\angle_v(y, u) \geq \frac{\pi}{2}. \tag{3}$$

We now construct a quadrilateral x', y', v', u' in $\mathbb{R}^2$ such that $\triangle' x'u'v'$ is a comparison triangle for $\triangle xuy$ and $\triangle' u'y'v'$ is a comparison triangle for $\triangle uyv$ with the line through u and y separating x from v. Using the CAT(0) condition we have

$$\angle_{v'}(y', u') \geq \angle_v(y, u) \geq \frac{\pi}{2}.$$

The proof now subdivides in two cases.

Case 1. The quadrilateral $x'u'v'y'$ is convex at u', that is $\angle_{u'}(x', y') + \angle_{u'}(y', v') \leq \pi$. In that case we have

$$\begin{array}{rcll} \angle_{u'}(x', v') & = & \angle_{u'}(x', y') + \angle_{u'}(y', v') & \\ & \geq & \angle_u(x, y) + \angle_u(y, v) & \text{from the CAT(0) condition} \\ & \geq & \angle_u(x, v) & \text{from the triangle inequality for Alexandrov's angles} \\ & \geq & \dfrac{\pi}{2} & \text{from (2).} \end{array}$$

We therefore have

$$\langle x' - u', v' - u' \rangle \leq 0, \quad \text{and} \quad \langle y' - v', u' - v' \rangle \leq 0,$$

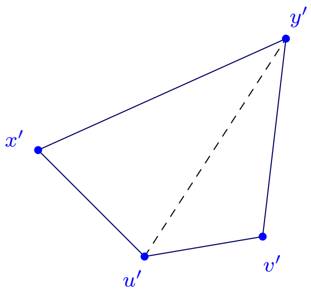

Figure 1: The configuration in case 1

and the previous proof implies that $\|u' - v'\| \leq \|x' - y'\|$. We then conclude that

$$d_X(u, v) = \|u' - v'\| \leq \|x' - y'\| = d_X(x, y).$$

Case 2. The quadrilateral $x'u'v'y'$ is not convex at u', that is $\angle_{u'}(x', y') + \angle_{u'}(y', v') > \pi$. Let us denote by $\triangle \bar{x}\bar{y}\bar{v}$ the Euclidean triangle with side length

$$\|\bar{x} - \bar{y}\| = \|x' - y'\|, \quad \|\bar{v} - \bar{y}\| = \|v' - y'\| \quad \text{and} \quad \|\bar{x} - \bar{v}\| = \|x' - u'\| + \|u' - v'\|.$$

By Alexandrov's lemma we have

$$\angle_{\bar{v}}(\bar{x}, \bar{y}) \geq \angle_{v'}(u', y') \geq \frac{\pi}{2}.$$

Therefore $\|\bar{x} - \bar{y}\| > \|\bar{x} - \bar{v}\|$ and we thus have

$$\|u' - v'\| \leq \|\bar{x} - \bar{v}\| \leq \|\bar{x} - \bar{y}\| \ = \|x' - y'\|,$$

which implies $d_X(u, v) \leq d_X(x, y)$.

□

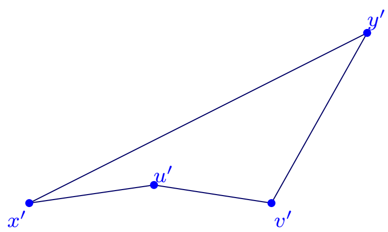

Figure 2: The configuration in case 2