Solution to Exercise 1.5

We recall that h being concave means that for all $t \in [0,1]$ and for all $x, y \in \mathbb{R}_+$ we have

$$h(tx + (1-t)y) \geq th(x) + (1-t)h(y).$$

The proof is made in three steps.

1. First see that for all $x, y \in \mathbb{R}_+$, $h(x)+h(y) \geq h(x+y)$, indeed, suppose $t = \frac{x}{x+y}$, then $t(x+y) = x$ and $(1-t)(x+y) = y$, thus

   (a) $h(x) = h(t(x+y)+0) \geq th(x+y)$ since $h(0) = 0$.

   (b) $h(y) = h(0 + (1-t)(x+y)) \geq (1-t)h(x+y)$ since $h(0) = 0$.

   Adding (a) and (b) gives what we want.

2. We also need to prove that h is non-decreasing:
   Suppose it is not the case, then there would exist $t_0 \in \mathbb{R}_+$ where $h'(t)$ changes its sign, i.e. there would exist $t_1 > t_0$ such that
   $$h(t_0) > h(t_1) + \delta \tag{1}$$
   for $\delta > 0$.
   Let $t_2 > t_1$, then $t_1 = t_0 + \lambda(t_2 - t_0)$ where $\lambda = \frac{t_1 - t_0}{t_2 - t_0} \leq 1$. Thus:
   $$h(t_1) = h(\lambda t_2 + (1-\lambda)t_0) \geq \frac{t_1 - t_0}{t_2 - t_0} h(t_2) - (1-\lambda)h(t_0) \tag{2}$$
   but then, combining (1) and (2), we get
   $$\delta < -\lambda h(t_2) + \lambda h(t_0) \leq \lambda h(t_0).$$
   Because $h(t_2) \geq 0$ and $\lambda \geq 0$. But λ goes to 0 as t_2 goes to infinity, so we get the contradiction since δ is positive.

3. We now can prove that $\rho := h \circ d$ is indeed a metric.

   - Symmetry is clear.
   - $\rho(x,x) = h(d(x,x)) = h(0) = 0$.
   - Since h is non decreasing, $\rho(x,y) = h(d(x,y)) > 0$ if $x \neq y$.
   - Let $x, y, z \in X$, then, using points 1) and 2), we have
     $$\rho(x,z) = h(d(x,z)) \leq h(d(x,y) + d(y,z)) \leq h(d(x,y)) + h(d(y,z)) = \rho(x,y) + \rho(y,z).$$

   Which proves that ρ is a metric.