ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Méthode de Lagrange

Mécanique, cours 25.1

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Méthode de Lagrange

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- Contraintes
- Nombre de degrés de liberté
- Déplacements virtuels compatibles
- Forces généralisées
- Equations de Lagrange

Définition : contraintes

N points matériels

positions : $\boldsymbol{r}_\alpha,\ \alpha = 1, \ldots, N$

contraintes exprimables sous la forme d'un ensemble de k équations :

$$
\begin{aligned}
f_1(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\
f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0 \\
&\vdots \\
f_k(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) &= 0
\end{aligned}
$$

k : le nombre de contraintes

dites : "holonômes"

Ces contraintes peuvent dépendre explicitement du temps

Définition : nombre de degrés de liberté

Résoudre pour k des variables

en fonction des $3N - k$ autres.

$n = 3N - k$ variables indépendantes

n ***coordonnées généralisées***

n = nombre de degrés de liberté

Définition : coordonnées généralisées

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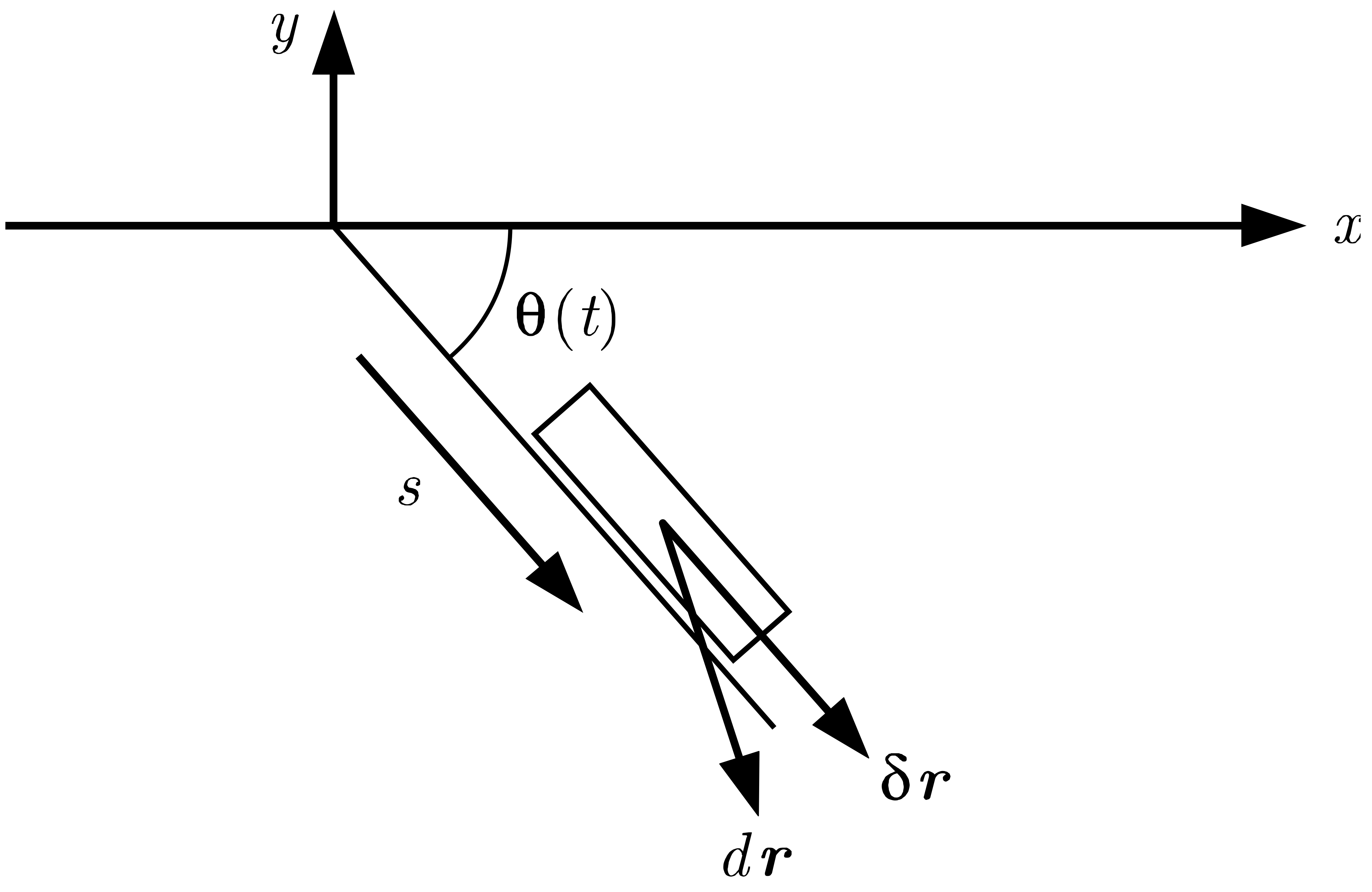
$$(q_1, \ldots, q_n)$$

$$\begin{aligned}
\boldsymbol{r}_1 &= \boldsymbol{r}_1(q_1, q_2, \ldots, q_n, t) \\
\boldsymbol{r}_2 &= \boldsymbol{r}_2(q_1, q_2, \ldots, q_n, t) \\
&\vdots \\
\boldsymbol{r}_N &= \boldsymbol{r}_N(q_1, q_2, \ldots, q_n, t)
\end{aligned}$$

Positions de N points matériels
données par n coordonnées généralisées

Exemple de contrainte dépendant du temps



déplacements virtuels compatibles avec les contraintes

$$\delta \boldsymbol{r}(t)$$

$$\neq \boldsymbol{v} dt$$

si les contraintes dépendent du temps

$$x = s\cos(at) \qquad \frac{\partial x}{\partial s} = \cos(at)$$

$$y = -s\sin(at) \qquad \frac{\partial y}{\partial s} = -\sin(at)$$

Définition : déplacements compatibles

$$\delta \boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(q_1 + \delta q_1, \ldots, q_n + \delta q_n, t) - \boldsymbol{r}_\alpha(q_1, \ldots, q_n, t)$$

$$\delta \boldsymbol{r}_\alpha = \sum_j \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \, \delta q_j$$

Principe de d'Alembert

décomposition $\quad \boldsymbol{F}^{tot} = \boldsymbol{F}^{cont} + \boldsymbol{F}$

d'abord : un seul point matériel

$$\boldsymbol{F} + \boldsymbol{F}^{cont} - m\frac{d\boldsymbol{v}}{dt} = 0$$

$\delta\boldsymbol{r}$ virtuel compatible avec les contraintes

$$\left(\boldsymbol{F} + \boldsymbol{F}^{cont}\right) \cdot \delta\boldsymbol{r} - m\frac{d\boldsymbol{v}}{dt} \cdot \delta\boldsymbol{r} = 0$$

hypothèse $\quad \boldsymbol{F}^{cont} \cdot \delta\boldsymbol{r} = 0$

$$\left(\boldsymbol{F} - m\frac{d\boldsymbol{v}}{dt}\right)\delta\boldsymbol{r} = 0$$

pour un système de points matériels

$$\sum_{\alpha}\left(\boldsymbol{F}_{\alpha} - m_{\alpha}\frac{d\boldsymbol{v}_{\alpha}}{dt}\right) \cdot \delta\boldsymbol{r}_{\alpha} = 0$$

Définitions : forces généralisées

$$\sum_\alpha \left(\boldsymbol{F}_\alpha - m_\alpha \frac{d\boldsymbol{v}_\alpha}{dt} \right) \cdot \delta \boldsymbol{r}_\alpha = 0$$

$$\boldsymbol{F}_\alpha \cdot \delta \boldsymbol{r}_\alpha$$

$$\sum_{j=1}^{n} \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \, \delta q_j \quad = \sum_{j=1}^{n} Q_j \, \delta q_j$$

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \qquad (j = 1 \cdots n)$$

Cinétique

$$\sum_\alpha \left(\boldsymbol{F}_\alpha - m_\alpha \frac{d\boldsymbol{v}_\alpha}{dt} \right) \cdot \delta \boldsymbol{r}_\alpha = 0$$

$$m \frac{d\boldsymbol{v}}{dt} \cdot \delta \boldsymbol{r} = \sum_j \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right] \delta q_j \qquad \text{A démontrer !}$$

$$m\ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} = \sum_j m\ddot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \delta q_j = \sum_j \left(\frac{d}{dt} \left(m\dot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} \right) - m\dot{\boldsymbol{r}} \cdot \frac{d}{dt} \left(\frac{\partial \boldsymbol{r}}{\partial q_j} \right) \right) \delta q_j$$

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_n, t) \qquad \boldsymbol{v} = \sum_j \frac{\partial \boldsymbol{r}}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}}{\partial t} \qquad \Longrightarrow \frac{\partial \boldsymbol{v}}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}}{\partial q_j}$$

$$\frac{d}{dt} \left(\frac{\partial \boldsymbol{r}(q_1 \ldots q_i \ldots q_n, t)}{\partial q_j} \right) = \sum_i \left\{ \frac{\partial^2 \boldsymbol{r}}{\partial q_i \partial q_j} \dot{q}_i \right\} + \frac{\partial^2 \boldsymbol{r}}{\partial q_j \partial t} = \frac{\partial}{\partial q_j} \left\{ \sum_i \frac{\partial \boldsymbol{r}}{\partial q_i} \dot{q}_i + \frac{\partial \boldsymbol{r}}{\partial t} \right\} = \frac{\partial \boldsymbol{v}}{\partial q_j}$$

Cinétique

$$m\ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r} = \sum_j m\ddot{\boldsymbol{r}}\cdot\frac{\partial\boldsymbol{r}}{\partial q_j}\,\delta q_j \quad = \sum_j \left(\frac{d}{dt}\left(m\dot{\boldsymbol{r}}\cdot\frac{\partial\boldsymbol{r}}{\partial q_j}\right) - m\dot{\boldsymbol{r}}\cdot\frac{d}{dt}\left(\frac{\partial\boldsymbol{r}}{\partial q_j}\right)\right)\delta q_j$$

$$m\dot{\boldsymbol{v}}\cdot\delta\boldsymbol{r} = \sum_j \left(\frac{d}{dt}\left(m\boldsymbol{v}\cdot\frac{\partial\boldsymbol{r}}{\partial q_j}\right) - m\boldsymbol{v}\cdot\frac{d}{dt}\left(\frac{\partial\boldsymbol{r}}{\partial q_j}\right)\right)\delta q_j$$

$$m\dot{\boldsymbol{v}}\cdot\delta\boldsymbol{r} = \sum_j \left[\frac{d}{dt}\left(m\boldsymbol{v}\frac{\partial\boldsymbol{v}}{\partial\dot{q}_j}\right) - m\boldsymbol{v}\frac{\partial\boldsymbol{v}}{\partial q_j}\right]\delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m\boldsymbol{v}^2\right)\right]\delta q_j \qquad \text{cqfd}$$

Equations de Lagrange (1)

Pour un point matériel :

$$m\dot{\boldsymbol{v}} \cdot \delta \boldsymbol{r} = \sum_j \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m \boldsymbol{v}^2 \right) \right] \delta q_j$$

$$\boldsymbol{F} \cdot \delta \boldsymbol{r} = \sum_{j=1}^{n} Q_j \, \delta q_j$$

Pour un système de points matériels :

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \boldsymbol{v}_\alpha^2$$

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j}$$

$$\sum_{j=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

q_j indépendants $\Longrightarrow$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

Equations de Lagrange (2)

Forces conservatives :

$$Q_j = \sum_{\alpha=1}^{N} \boldsymbol{F}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_j} \qquad Q_j^{pot} = -\sum_{\alpha=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial V_\alpha}{\partial x_\alpha^i}\frac{\partial x_\alpha^i}{\partial q_j}\right) = -\frac{\partial V(q_1,\ldots,q_n)}{\partial q_j} \qquad V = \sum_{\alpha=1}^{N} V_\alpha$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0 \qquad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

lagrangien : $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Avec des forces non-conservatives en plus :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NC}$$