ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Pendules couplés

Mécanique, cours 27.2

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Pendules couplés

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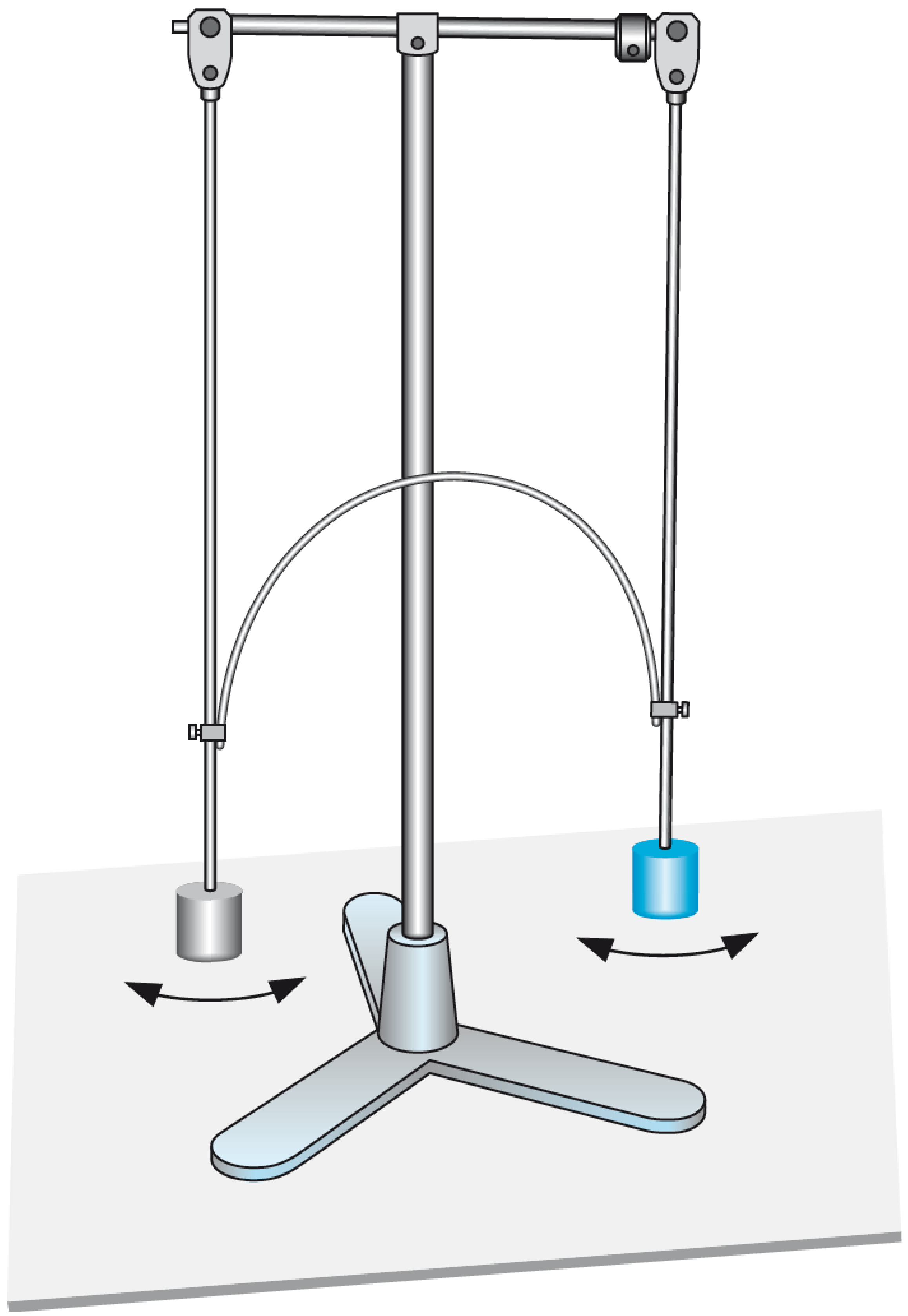
- 2 pendules couplés
- Equations du mouvement
- Modes et fréquences propres

Pendules couplés : Lagrange



Coordonnées : écarts à l'équilibre x_1 et x_2

$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$

Pesanteur, énergie potentielle d'un pendule :

$$V_1 = mg\ell(1 - \cos\theta_1) \approx mg\ell\left(1 - 1 + \frac{1}{2}\theta_1^2\right)$$

Petits angles : $\theta_1 \approx \frac{x_1}{\ell}$

Couplage, énergie potentielle (choix) :

$$\frac{1}{2} k(x_1 - x_2)^2$$

$$L = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_1^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_2^2 + kx_1x_2$$

Pendules couplés : équations du mouvement

$$L = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_1^2 - \frac{1}{2}\left(k + \frac{mg}{\ell}\right) x_2^2 + k x_1 x_2$$

$$m\ddot{x}_1 + \left(k + \frac{mg}{\ell}\right) x_1 - k x_2 = 0$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{\ell}\right) x_2 - k x_1 = 0$$

Fréquences propres

$$m\ddot{x}_1 + \left(k + \frac{mg}{\ell}\right) x_1 - kx_2 = 0$$

$$m\ddot{x}_2 + \left(k + \frac{mg}{\ell}\right) x_2 - kx_1 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$$

$$m\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k + \frac{mg}{\ell} & -k \\ -k & k + \frac{mg}{\ell} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{vmatrix} = 0$$

$$0 = \left(k + \frac{mg}{\ell} - m\omega^2\right)^2 - k^2 = \left(k + \frac{mg}{\ell} - m\omega^2 - k\right)\left(k + \frac{mg}{\ell} - m\omega^2 + k\right)$$

$$\omega_1 = \pm\sqrt{\frac{g}{\ell}} \qquad \omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

Modes propres

$$\begin{pmatrix} k + \frac{mg}{\ell} - m\omega^2 & -k \\ -k & k + \frac{mg}{\ell} - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\omega_1 = \pm\sqrt{\frac{g}{\ell}}$$

$$ka_1 - ka_2 = 0$$

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 = \pm\sqrt{\frac{g}{\ell} + \frac{2k}{m}}$$

$$-ka_1 - ka_2 = 0$$

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution générale et coordonnées propres

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_1 = A_{-1}^* \qquad A_2 = A_{-2}^*$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \cos(\omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cos(\omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$x_1 + x_2\ :\ \text{pulsation } \omega_1$ $\qquad$ $x_1 - x_2\ :\ \text{pulsation } \omega_2$

Conditions initiales, projections

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1\, e^{+i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_{-1}\, e^{-i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2\, e^{+i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_{-2}\, e^{-i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A_{-1} = A_1^*$$

$$A_{-2} = A_2^*$$

Position :

$$\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = (A_1 + A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_2 + A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(1\ 1)\cdot \implies x_{10} + x_{20} = 2(A_1 + A_{-1})$$

$$(1\ -1)\cdot \implies x_{10} - x_{20} = 2(A_2 + A_{-2})$$

Vitesse :

$$\begin{pmatrix} v_{10} \\ v_{20} \end{pmatrix} = +i\omega_1\,(A_1 - A_{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\omega_2\,(A_2 - A_{-2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(1\ 1)\cdot \implies v_{10} + v_{20} = 2i\omega_1 \big(A_1 - A_{-1}\big)$$

$$(1\ -1)\cdot \implies v_{10} - v_{20} = 2i\omega_2 \big(A_2 - A_{-2}\big)$$