ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Pendule paramétrique

Mécanique, cours 28.2

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Pendule paramétrique

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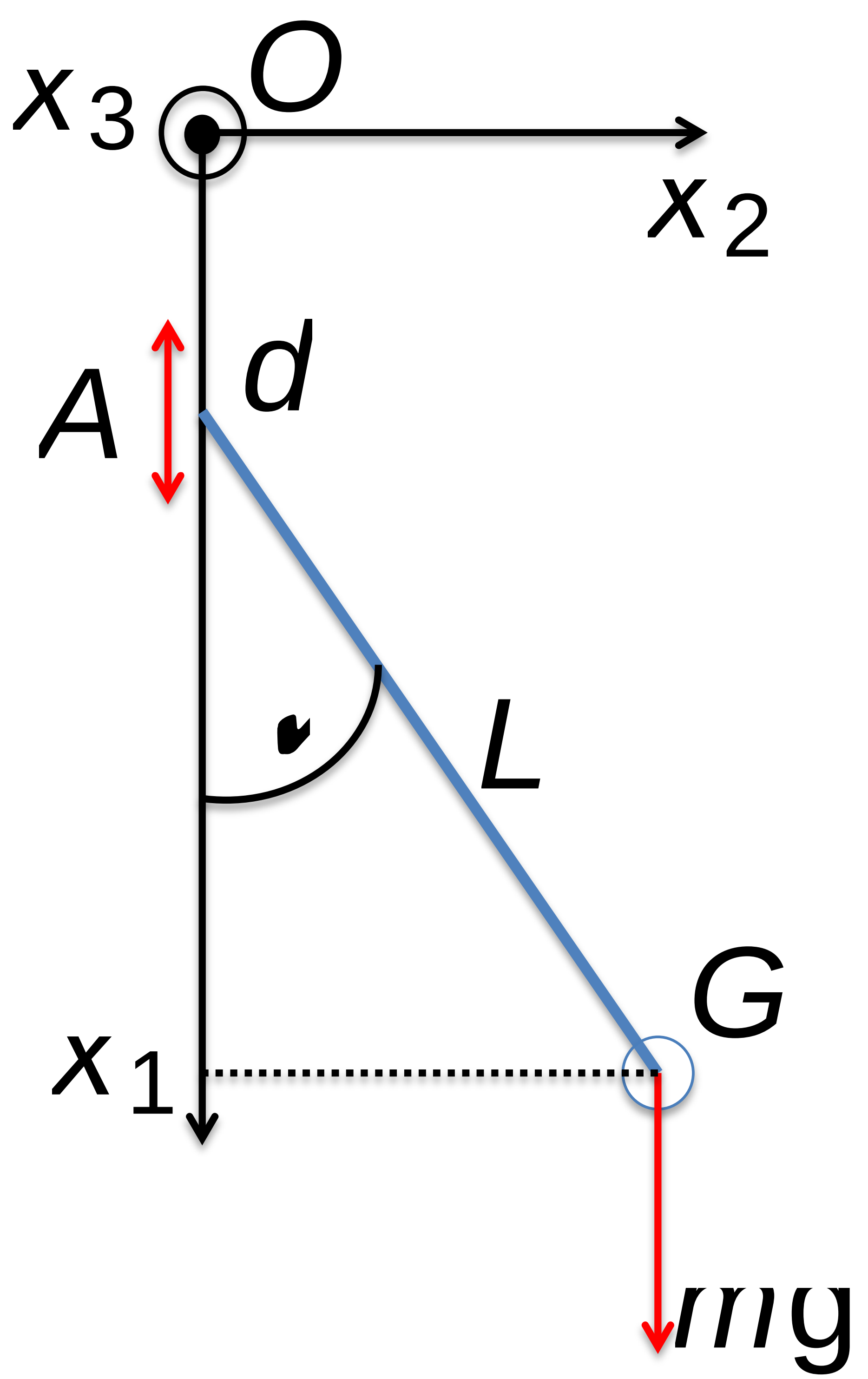
- Equation du mouvement par la méthode de Lagrange
- Equation de Mathieu
- Fonctions propres
- Domaines de stabilité

Pendule forcé : méthode de Lagrange



$$x_1 = d + L\cos\theta \qquad x_2 = L\sin\theta$$

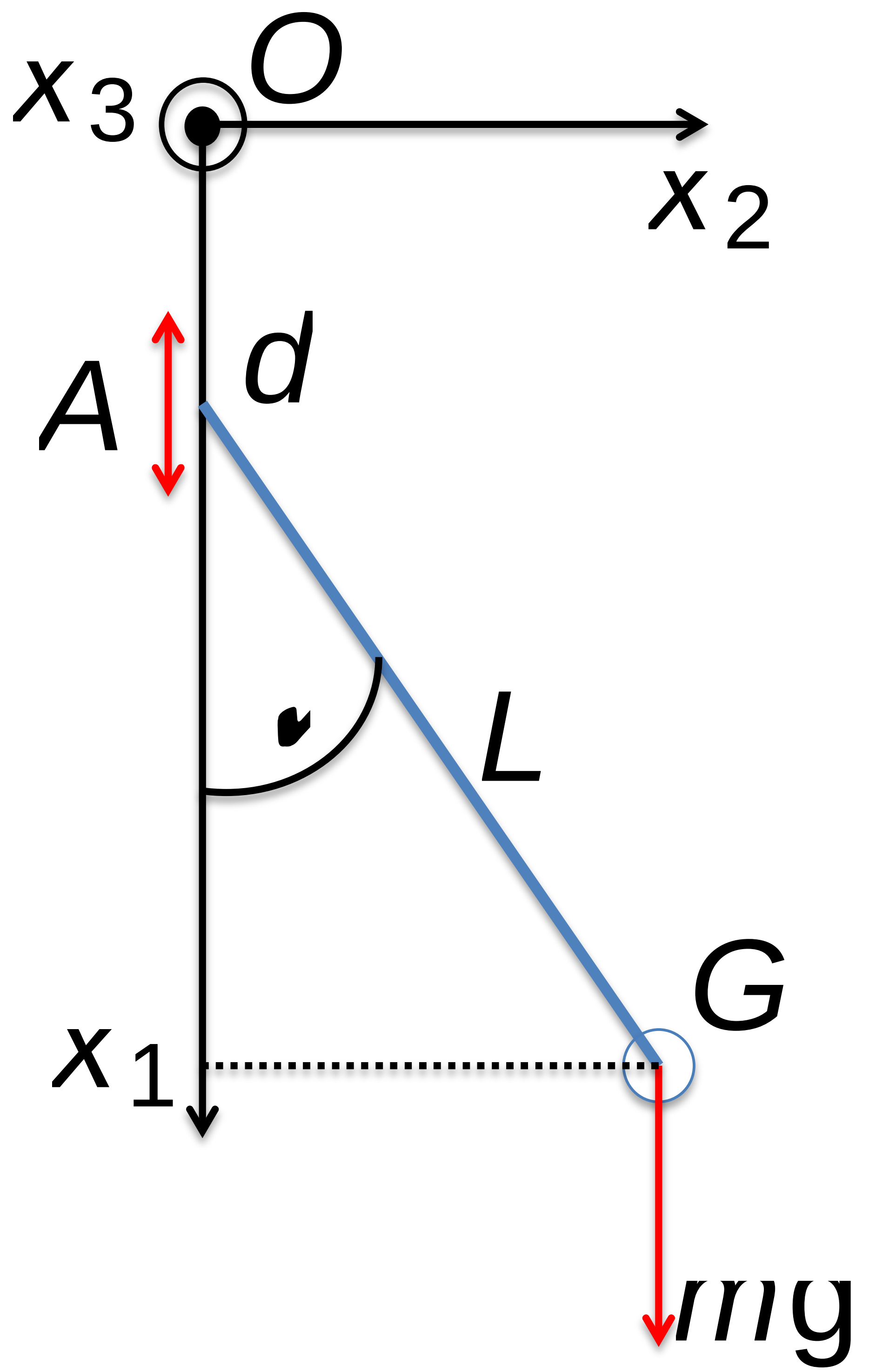
$$\dot{x}_1 = \dot{d} - L\dot{\theta}\sin\theta \qquad \dot{x}_2 = L\dot{\theta}\cos\theta$$

$$T = \frac{1}{2}m\left(\dot{d}^2 - 2\dot{d}L\dot{\theta}\sin\theta + L^2\dot{\theta}^2\right)$$

$$V = -mg\,(d + L\cos\theta)$$

$$L = \frac{1}{2}m\dot{d}^2 - m\dot{d}L\dot{\theta}\sin\theta + \frac{1}{2}mL^2\dot{\theta}^2 + mgd + mgL\cos\theta$$

Pendule forcé : équation du mouvement



$$L = \frac{1}{2}m\dot{d}^2 - m\dot{d}L\dot{\theta}\sin\theta + \frac{1}{2}mL^2\dot{\theta}^2 + mgd + mgL\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = -m\dot{d}L\sin\theta + mL^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = -m\ddot{d}L\sin\theta - m\dot{d}\dot{\theta}\cos\theta + mL^2\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -m\dot{d}L\dot{\theta}\cos\theta - mgL\sin\theta$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta - \frac{\ddot{d}}{L}\sin\theta = 0$$

Equation de Hill

$$\ddot{\theta} + \frac{g}{L}\sin\theta - \frac{\ddot{d}}{L}\sin\theta = 0$$

$$d = d_0\cos(2\Omega t)$$

$$\theta \to 0$$

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{4d_0\Omega^2}{L}\cos(2\Omega t)\right)\theta = 0$$

$$\ddot{x} + G(t)x = 0 \qquad\qquad x = L\theta$$

$$\text{période : } \tau = \frac{\pi}{\Omega}$$

Equation de Mathieu

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{4 d_0 \Omega^2}{L} \cos(2\Omega t) \right) \theta = 0$$

$$y'' + (p - 2q \cos 2\bar{t}\,)\, y = 0$$

$$\bar{t} = \Omega t$$

$$p = \frac{g}{L\Omega^2} \qquad q = \frac{-2 d_0}{L}$$

Méthode de Ince

$$y'' + (p - 2q\cos 2\bar{t}\,)\,y = 0$$

chercher une solution de période π :

$$\bar{e}_1(\bar{t}) = \sum_{r=0}^{\infty} A_{2r}\cos 2r\bar{t}$$

$$2\cos 2r\bar{t}\cos 2\bar{t} = \cos(2r+2)\bar{t} + \cos(2r-2)\bar{t}$$

$$2\sin 2r\bar{t}\cos 2\bar{t} = \sin(2r+2)\bar{t} + \sin(2r-2)\bar{t}$$

$$pA_0 - qA_2 = 0$$

$$(p-4)A_2 - q(2A_0 + A_4) = 0$$

$$\ldots\ldots\ldots\ldots$$

$$(p-4r^2)A_{2r} - q(A_{2r-2} + A_{2r+2}) = 0$$

$$-\frac{p}{2} = \cfrac{q^2}{4 - p - \cfrac{q^2}{16 - p - \cfrac{q^2}{36 - p - \cfrac{q^2}{(\ldots)}}}}$$

$$\frac{A_2}{A_0} = \frac{p}{q}$$

$$\frac{A_2}{A_0} = \frac{-q}{4 - p + q\frac{A_4}{A_2}}$$

$$\frac{A_{2r}}{A_{2r-2}} = \frac{-q}{4r^2 - p + q\frac{A_{2r+2}}{A_{2r}}}$$

Fonctions propres

autres solutions de période π :

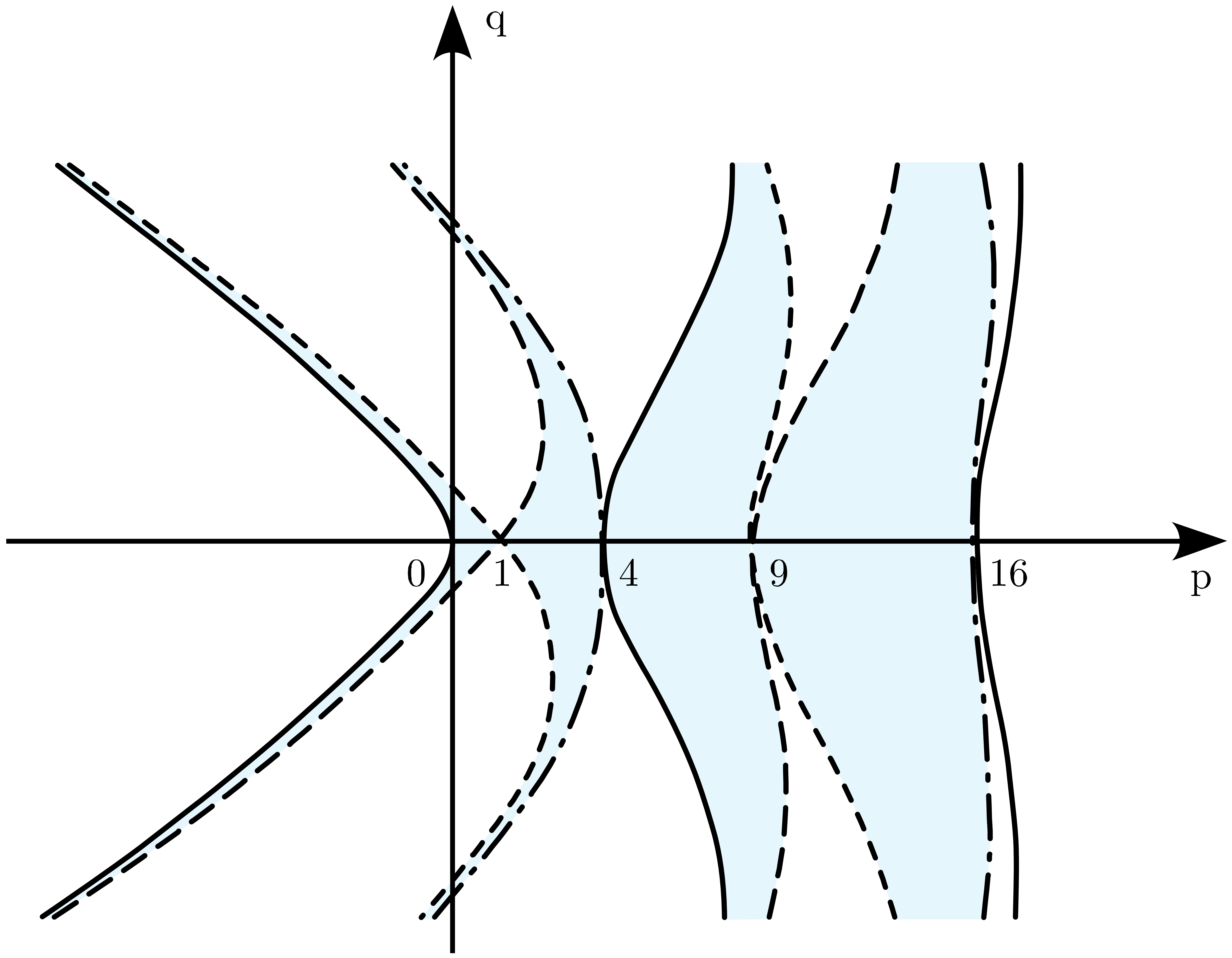
$$\bar{e}_2(\bar{t}) = \sum_{r=0}^{\infty} B_{2r+2} \sin(2r+2)\bar{t}$$

solutions de période 2π :

$$\bar{e}_3(\bar{t}) = \sum_{r=0}^{\infty} A_{2r+1} \cos(2r+1)\bar{t}$$

$$\bar{e}_4(\bar{t}) = \sum_{r=0}^{\infty} B_{2r+1} \sin(2r+1)\bar{t}$$

Relation de dispersion p(q) des fonctions propres



$$y'' + (p - 2q\cos 2\bar{t}\,)\,y = 0$$

$$q = \frac{-2d_0}{L}$$

$$p = \frac{g}{L\Omega^2}$$