DIOPHANTINE CONSEQUENCES OF DELIGNE'S WEIL II MAIN THEOREM

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(1) I strongly recommend the reading of N. Katz lectures on Weil II [Kat01].
(2) Thanks are due to Paul Nelson for re-reading the notes and correctiing them. All remaining errors are mine.

1. Local systems and representations of the étale fundamental group [Del77]

Let $k = \mathbb{F}_p$, $\overline{k}$ a choice of an algebraic closure, $X = X_k$ a scheme of finite type of dimension d defined over k, connected normal and which we assume geometrically irreducible ($X_{\overline{k}}$ is irreducible). Let $K = k(X)$ be its field of functions and and $\overline{K}$ a separable closure. We note $\eta = \mathrm{Spec}(K)$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$ the geometric point above it and $\overline{X} = X_{\overline{k}}$. We will note k'/k a finite extension of k, $q = |k'| = p^{[k':k]}$.

1.1. **The arithmetic/geometric fundamental group.** The etale fundamental group $\pi_1^{et}(X, \overline{\eta})$ is the quotient of $\mathrm{Gal}(\overline{K}/K)$ obtained as the limit

$$\pi_1^{et}(X, \overline{\eta}) = \lim_{\leftarrow} \mathrm{Gal}(L/K)$$

for L/K ranging of the finite extensions of K such that the normalization of X in L is etale over X.

The group maps surjectively on $\mathrm{Gal}(\overline{k}/k) = \pi_1^{et}(\mathrm{Spec}(k), \mathrm{Spec}(\overline{k}))$ and its quotient is the *geometric etale fundamental group*

$$\pi_1^{et}(\overline{X}, \overline{\eta}) = \pi_1^{geom}(X, \overline{\eta}).$$

We also call $\pi_1^{et}(X, \overline{\eta})$ the *arithmetic fundamental group*, and note it

$$\pi_1^{et}(X, \overline{\eta}) = \pi_1^{arith}(X, \overline{\eta}).$$

These sit into the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\overline{K}/\overline{k}(X)) \arrow[r] \arrow[d, two heads] & \mathrm{Gal}(\overline{K}/K) \arrow[r] \arrow[d, two heads] & \mathrm{Gal}(\overline{k}/k) \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & \pi_1^{geom}(X, \overline{\eta}) \arrow[r] & \pi_1^{arith}(X, \overline{\eta}) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] & 1
\end{tikzcd}$$

Let $\ell \neq p$ and $E_\lambda/\mathbb{Q}_\ell$ a finite extension.

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Definition 1.1. *A lisse E_λ-sheaf $\mathcal{F}$ on X (or a E_λ-local system) is a finite dimensional continuous E_λ-representation*

$$\rho_{\mathcal{F}} : \pi_1^{arith}(X, \overline{\eta}) \to \mathrm{GL}(V_{\mathcal{F}})$$

(here $V_{\mathcal{F}}$ is a finite dim. E_λ-vector space). In terms of sheaves, the vector space

$$V = \mathcal{F}_{\overline{\eta}}$$

is the fiber of the sheaf $\mathcal{F}$ over $\overline{\eta}$. The rank of $\mathcal{F}$ is

$$\mathrm{rk}\,\mathcal{F} = \dim V_\lambda.$$

The restriction of $\rho_{\mathcal{F}}$ to the geometric fundamental group

$$\rho_{\mathcal{F}|\pi_1^{geom}} =: \rho_{\mathcal{F}}^{geom}$$

is the geometric lisse sheaf (or geometric representation) associated to $\mathcal{F}$.

The dual sheaf is noted (E_λ the trivial representation)

$$\check{\mathcal{F}} = \mathrm{Hom}_{E_\lambda[\pi_1^{arith}]}(\mathcal{F}, E_\lambda)$$

It is lisse of the same rank as $\mathcal{F}$.

1.2. **Weil-Deligne group and representation.** The Galois group

$$\mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}}$$

is topologically generated by the (geometric) Frobenius Fr_k, which is the inverse of the arithmetic Frobenius on $\overline{k}$

$$\mathrm{Fr}_k^{-1} : u \in \overline{k} \mapsto u^p.$$

The Weil-Deligne group is the semidirect product

$$WD(X, \overline{\eta}) = \pi_1^{geom}(X, \overline{\eta}) \ltimes \mathrm{Fr}_k^{\mathbb{Z}} \subset \pi_1^{arith}(X, \overline{\eta}).$$

It will be useful (for the purpose of normalization) to consider representations of the Weil-Deligne group instead of the arithmetic fundamental group. We will speak instead of Weil-Deligne sheaf or Weil-Deligne representation.

Example 1.1 (Tate twists I)**.** Let $\mu(\ell^\infty)$ be the group of ℓ-power root of 1 in $\overline{k}$ and

$$\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}.$$

This group is a free $\mathbb{Z}_\ell$-module of rank 1 and the geometric Frobenius Fr_k acts on it by multiplication by $1/p \in \mathbb{Z}_\ell^\times$. We denote by $\mathbb{Q}_\ell(1)$ the tensor product of this $\mathbb{Z}_\ell$ module by $\mathbb{Q}_\ell$, by $\mathbb{Q}_\ell(-1)$ its dual and $\mathbb{Q}_\ell(\pm m) = \mathbb{Q}_\ell(\pm 1)^{\otimes m}$ for $m \in \mathbb{Z}$; we also write $E_\lambda(m) = \mathbb{Q}_\ell(m) \otimes E_\lambda$. These are $\mathrm{Gal}(\overline{k}/k)$-modules on which Fr_k acts by multiplication by p^{-m} and which we extend to $\pi_1^{arith}(X_{\overline{k}}, \overline{\eta})$ (by trivial action to $\pi_1^{geom}(X_{\overline{k}}, \overline{\eta})$) and use the same notation for these sheaves over X.

If $\mathcal{F}$ is a lisse sheaf we denote its *Tate twist* by

$$\mathcal{F}(m) := \mathcal{F} \otimes E_\lambda(m).$$

Example 1.2 (Tate twists II)**.** Given $\alpha \in E_\lambda^\times$ let $E_\lambda^{(\alpha)}$ be the Weil-Deligne sheaf which is trivial on $\pi_1^{geom}(X_{\overline{k}}, \overline{\eta})$ and whose action by Fr_k is multiplication by α. This is a priori only a Weil-Deligne sheaf excepted when $\alpha = p^m \in p^{\mathbb{Z}}$ in which case

$$E_\lambda^{(\alpha)} = E_\lambda(-m).$$

If E_λ contains the square root of p we will extend the notation of Tate twist $\mathcal{F}(m)$ to $m \in \frac{1}{2}\mathbb{Z}$ (for the corresponding Weil-Deligne sheaf).

2. The trace formula

Let $x \in |X|$ be a closed point with field of functions $k(x)$ of degree d_x over k and geometric point above it $\overline{x}$

$$x = \mathrm{Spec}(k(x)) \hookrightarrow X.$$

By functoriality we have a map

$$\pi_1^{et}(x, \overline{x}) = \mathrm{Gal}(\overline{k}/k(x)) \to \pi_1^{arith}(X, \overline{x}) \simeq \pi_1^{arith}(X, \overline{\eta}).$$

The group $\mathrm{Gal}(\overline{k}/k(x))$ is topologically generated by the (geometric) Frobenius $\mathrm{Fr}_x^{-1} : u \to u^{|k(x)|}$ and the isomorphism $\pi_1^{arith}(X, \overline{x}) \simeq \pi_1^{arith}(X, \overline{\eta})$ is canonically defined up to inner automorphisms, therefore the above map defines a conjugacy class

$$\mathrm{Fr}_x^{\natural} \subset \pi_1^{arith}(X, \overline{\eta})$$

which "acts" on $V_{\mathcal{F}}$ and one can speak unambiguously of the "trace" of $\rho(\mathrm{Fr}_x)$ or of its determinant or its eigenvalues.

2.1. **The Grothendieck-Lefschetz trace formula.**

In particular we will be interested in the sums

$$\sum_{x \in X(k')} \mathrm{tr}(\mathrm{Fr}_x | \mathcal{F}_{\overline{\eta}})$$

for k'/k ranging over the finite extensions of k. One has the following formula

$$\sum_{x \in X(k')} \mathrm{tr}(\mathrm{Fr}_x | \mathcal{F}_{\overline{\eta}}) = \sum_{i=0}^{2d} (-1)^i \,\mathrm{tr}(\mathrm{Fr}_{k'} | H_c^i(X_{\overline{k}}, \mathcal{F})).$$

Here $H_c^i(X_{\overline{k}}, \mathcal{F})$ is the i-th cohomology group with compact support: by a Theorem of Nagata, X admits a compactification $j : X \hookrightarrow \widehat{X}$ and let $j_!\mathcal{F}$ be the sheaf on $\widehat{X}$ obtained from $\mathcal{F}$ by extension by 0 on $\widehat{X} - X$; then

$$H_c^i(X_{\overline{k}}, \mathcal{F}) = R^i j_! \mathcal{F}$$

is the i-th right derived functor of the functor $j_!$ applied to $\mathcal{F}$. These are finite dimensional E_λ-vector spaces on which the Galois group $\mathrm{Gal}(\overline{k}/k) = \overline{\langle \mathrm{Fr}_k \rangle}$ acts by transport of structure (alternatively $H_c^i(X_{\overline{k}}, \mathcal{F})$ can be seen as a lisse sheaf on $\mathrm{Spec}(k)$) and therefore

$$\mathrm{tr}(\mathrm{Fr}_{k'} | H_c^i(X_{\overline{k}}, \mathcal{F})) = \mathrm{tr}(\mathrm{Fr}_k^{[k':k]} | H_c^i(X_{\overline{k}}, \mathcal{F}))$$

is well defined.

2.2. **Cohomology groups.** The cohomology groups satisfy the following

Theorem 2.1 (Poincaré duality). *For $i = 0, \cdots, 2d$ one has an perfect pairing of $E_\lambda[\mathrm{Gal}(\overline{k}/k)]$-modules*

$$H_c^i(X_{\overline{k}}, \mathcal{F}) \times H^{2d-i}(X_{\overline{k}}, \check{\mathcal{F}}) \to E_\lambda(-d).$$

Here $\check{\mathcal{F}} = \mathrm{Hom}(\mathcal{F}, E_\lambda)$ is the dual of $\mathcal{F}$.

In particular if X is affine

$$H^{2d-i}(X_{\overline{k}}, \check{\mathcal{F}}) = 0,\ i = 0 \cdots d-1$$

and therefore

$$H_c^i(X_{\overline{k}}, \mathcal{F}) = 0,\ i = 0 \cdots d-1.$$

Moreover

$$H^0(X_{\overline{k}}, \check{\mathcal{F}}) = \check{\mathcal{F}}^{\pi_1^{geom}}$$

is the subspace of π_1^{geom}-invariant vectors. By duality

$$H_c^{2d}(X_{\overline{k}}, \mathcal{F}) = \mathcal{F}_{\pi_1^{geom}}(-d), \tag{2.1}$$

the Tate twist of the co-invariants of $\mathcal{F}$.

2.3. ***L*-functions.** Let us recall that if $A \in \mathrm{End}(V)$ is an endomorphism of a finite dimensional vector space over a field F of characteristic 0 one has the identity of power series in $F[[T]]$

$$\exp(\sum_{n\geq 1} \frac{\mathrm{tr}(A^n)}{n} T^n) = \det(\mathrm{Id} - AT|V)^{-1}.$$

Combining this formula with the Grothendieck-Lefschetz trace formula we obtain

(2.2)

$$\prod_{x\in|X|} \det(\mathrm{Id} - \mathrm{Fr}_x T^{\deg(x)}|\mathcal{F}_{\overline{\eta}})^{-1} = \prod_{i=0}^{2d} \det(\mathrm{Id} - \mathrm{Fr}_k T|H_c^i(X_{\overline{k}}, \mathcal{F}))^{(-1)^{i+1}}.$$

The infinite product

$$\prod_{x\in|X|} \det(\mathrm{Id} - \mathrm{Fr}_x T^{\deg(x)}|\mathcal{F}_{\overline{\eta}})^{-1} = \mathscr{L}(T, \mathcal{F})$$

is (a version of) the Artin L-function associated to the sheaf $\mathcal{F}$. The G-L trace formula states precisely that this formal series is in fact a rational function whose coefficients are in the field E_λ.

3. Deligne main theorem [Del80]

The Lefschetz formula and the identity (2.2) are a priori identities in the field E_λ and in the formal sense. In the most interesting cases this can be turned into identities over the complex numbers (equipped with the usual topology). For this we need to be able to speak of the "size" of the traces of Frobenius involved. Observe however that when $\mathcal{F} = \mathbb{Q}_\ell$ is the trivial sheaf

$$\mathscr{L}(T, \mathbb{Q}_\ell) = \zeta_X(s) = \prod_{x \in |X|} (1 - T^{\deg(x)})^{-1}$$

(the zeta function of the scheme X itself) is a well defined power series with coefficients in $\mathbb{Q}$ with positive radius of convergence. By the G-L trace formula, this equals the rational fraction (with ℓ-adic coefficients):

$$\zeta_X(s) = \prod_{i=0}^{2d} \det(\mathrm{Id} - \mathrm{Fr}_k T | H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

A simple argument shows then that, in fact, the factors $\det(\mathrm{Id}-\mathrm{Fr}_k T|H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))$ are polynomials with rational coefficients.

We wish to extend this to not necessarily trivial sheaves $\mathcal{F}$.

3.1. **Pure sheaves.**

Definition 3.1. *A Weil number is a $\mathbb{Q}$-algebraic number all whose complex conjugates have modulus $p^{w/2}$ for $w \in \mathbb{Z}$ (w is the weight of the Weil number).*

Definition 3.2. *Let $w \in \mathbb{Z}$, a lisse sheaf $\mathcal{F}$ is pure of weight w if for any $x \in |X|$, the eigenvalues of $\rho(\mathrm{Fr}_x)$ (which a priori belong to $\overline{\mathbb{Q}_\ell}$) are Weil numbers of weight w.*

Remark 3.1. If $\mathcal{F}$ is pure of weight w, its dual $\check{\mathcal{F}}$ is also pure of weight $-w$ because of the equivariant perfect pairing

$$\mathcal{F} \times \check{\mathcal{F}} \to E_\lambda.$$

Definition 3.3. *Let $w \in \mathbb{Z}$, a lisse sheaf $\mathcal{F}$ is mixed of weight w if $\mathcal{F}$ is an iterated extension of pure sheaves of weight $\leq w$.*

Example 3.1. The Tate twist $E_\lambda(m),\ m \in \frac{1}{2}\mathbb{Z}$ has weight $-2m$.

Example 3.2. E/k an elliptic curve over k, the Tate module

$$T_\ell(E) = \varprojlim_n E[\ell^n]$$

is a lisse sheaf of rank 2 over $\mathrm{Spec}(k)$ pure of weight 1 (Hecke).

Example 3.3 (The Artin-Schreier/Kummer sheaves [Del77])**.** Let $\psi : (\mathbb{F}_p, +) \to \mathbb{C}^\times$ be a non-trivial additive character, there exist a sheaf $\mathcal{L}_\psi$, lisse on $A^1_{\mathbb{F}_p}$, of rank 1 such that

$$\mathrm{tr}(\mathrm{Fr}_x|\mathcal{L}_\psi) = \psi(\mathrm{tr}_{k(x)/k}(x)) \in \mu_p.$$

Let $\chi : (\mathbb{F}_p{}^\times, \times) \to \mathbb{C}^\times$ be a non-trivial multiplicative character, there exist a sheaf $\mathcal{L}_\chi$, lisse on $\mathbb{G}_{m\mathbb{F}_p}$, of rank 1 such that

$$\mathrm{tr}(\mathrm{Fr}_x|\mathcal{L}_\chi) = \chi(\mathrm{Nr}_{k(x)/k}(x)) \in \mu_{p-1}.$$

Both sheaves are pure of weight 0.

Example 3.4 (Generalization of the above examples)**.** Let C_k be a curve and $f, g : C \to P^1_k$ be rational maps. Let

$$X = C - \{\text{poles of } f \text{ and zero and poles of } g\}$$

the sheaf

$$\mathcal{L}_{\psi(f)\chi(g)} = f^*\mathcal{L}_\psi \otimes g^*\mathcal{L}_\chi$$

is lisse on X, pure of weight 0 and for $x \in |X|$

$$\mathrm{tr}(\mathrm{Fr}_x|\mathcal{L}_{\psi(f)\chi(g)}) = \psi(\mathrm{tr}_{k(x)/k}(f(x)))\chi(\mathrm{Nr}_{k(x)/k}(g(x)x)) \in \mu_{p(p-1)}$$

3.2. **L-functions and their zeros.** Let $\mathcal{F}$ be a mixed sheaf of some weight w. Fixing an(y) embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$ we may then view $\det(\mathrm{Id} - \mathrm{Fr}_x T|\mathcal{F}_{\overline{\eta}})$ as a polynomial in T with complex coefficient which factors as

$$\prod_{\lambda_x \in \mathrm{Spec}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})} (1 - \lambda_x T),\ |\lambda_x| \leq p^{w/2}.$$

Because for any extention k'/k the number

$$|X(k')| \ll |k'|^d$$

we see that for $T \in \mathbb{C}$ small enough ($|T| < p^{-(w/2+d)}$), the infinite product

$$\mathscr{L}(T; \mathcal{F}) = \prod_{x \in |X|} \det(\mathrm{Id} - \mathrm{Fr}_x T^{\deg(x)}|\mathcal{F}_{\overline{\eta}})^{-1}$$

is converging and defines an holomorphic function in a small disk centered at the origin. By the G-L trace formula, this function is in fact a complex rational function of T and therefore extend to a meromorphic function to the whole complex plane. This is analogous to the fact that the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}$$

initially defined in the half-plane $\Re s > 1$ extend meromorphically to $\mathbb{C}$. In order to make this analogy more striking we make the change of variable

$$T = p^{-s},\ s \in \mathbb{C}$$

and put

$$L(s, \mathcal{F}) = \mathscr{L}(p^{-s}, \mathcal{F}) = \prod_{x \in |X|} \det(\mathrm{Id} - \mathrm{Fr}_x p^{-s\deg(x)}|\mathcal{F}_{\overline{\eta}})^{-1}.$$

This infinite product converge in the half-plane $\Re s > w/2 + d$ and admits meromorphic continuation to $\mathbb{C}$ with poles contained in the set of zeros of the product

$$\prod_{\substack{i=0\cdots 2d\\ i\ even}} \det(\mathrm{Id} - \mathrm{Fr}_k p^{-s} | H^i_c(X_{\overline{k}}, \mathcal{F}))$$

and with zeros contained in the set of zeros of the product

$$\prod_{\substack{i=0\cdots 2d\\ i\ odd}} \det(\mathrm{Id} - \mathrm{Fr}_k p^{-s} | H^i_c(X_{\overline{k}}, \mathcal{F})).$$

Observe that

$$L(s, \mathcal{F}) = L(s + 2\pi i/\log p, \mathcal{F})$$

so the set of zeros poles is invariant under translation

$$\rho \mapsto \rho + 2\pi i/\log p,$$

we may therefore restrict to the ones of imaginary part contained in $\frac{1}{\log p}[-\pi, \pi[$. The following theorem gives a precise localization of the zeros and poles of $L(s, \mathcal{F})$ in this fundamental domain.

Theorem 3.1 (Deligne, [Del80]). *Let $\mathcal{F}$ be pure of weight w, the $H^i_c(X_{\overline{k}}, \mathcal{F})$ are (sheave over* $\mathrm{Spec}(k)$*) mixed of weight $\leq w + i$. In other terms, the eigenvalues of* Fr_p *acting on $H^i_c(X_{\overline{k}}, \mathcal{F})$ are* Weil numbers *of weight*

$$w' \in \mathbb{Z},\ w' \leq i + w.$$

Remark 3.2. By Poincaré duality (Thm 2.1) and applying Thm. 3.1 to the $H^{2d-i}(X_{\overline{k}}, \check{\mathcal{F}})$ (after relating the later to $H^{2d-i}_c(X_{\overline{k}}, \check{\mathcal{F}})$), Deligne also obtained *lower bounds* on the weights appearing in $H^i_c(X_{\overline{k}}, \mathcal{F})$.

For the top dimension we have something more precise: let

$$q^{-w/2}\alpha_1^{-1}(\mathcal{F}), \cdots, q^{-w/2}\alpha_{r'}^{-1}(\mathcal{F}),\ r' = \dim \mathcal{F}_{\pi_1^{geom}} = \dim \check{\mathcal{F}}^{\pi_1^{geom}}$$

be the eigenvalues of Fr_k acting on $\check{\mathcal{F}}^{\pi_1^{geom}}$; the $\alpha_i(\mathcal{F})$ are Weil numbers of weight 0 and by (2.1) have

$$\begin{aligned}(3.1)\quad & \mathrm{tr}(\mathrm{Fr}_{k'} | H^{2d}_c(X_{\overline{k}}, \mathcal{F})) \\ & = \mathrm{tr}(\mathrm{Fr}_{k'} | \mathcal{F}_{\pi_1^{geom}}(-2d)) = q^{d+w/2}(\alpha_1(\mathcal{F})^{[k':k]} + \cdots + \alpha_{r'}(\mathcal{F})^{[k':k]}).\end{aligned}$$

In terms of L-functions, this theorem states that the real parts of the zero/poles of the rational function $L(s, \mathcal{F})$ are *quantized*: satisfy

$$\Re \rho = k/2 \text{ for } k \in \mathbb{Z},\ k \leq d + w/2.$$

Moreover, in view of (2.1), on the line $\Re s = d + w/2$ there are exactly

$$r' = \dim(\mathcal{F}_{\pi_1^{geom}})$$

poles (counted with multiplicity) with imaginary part contained in $\frac{1}{\log p}[-\pi, \pi[$.

3.3. **Deligne's main theorem as a Riemann type hypothesis.** Let us pursue the analogy with the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}, \ \Re s > 1.$$

The later should correspond to the L-function of the trivial $\mathbb{Q}_\ell$ sheaf on a curve): $\zeta(s)$ can be continued analytically to $\mathbb{C}$ and has its only one pole at $s = 1$; the *Riemann hypothesis* predicts that all its zeros have real part $\leq 1/2$ (there are also so called "trivial" zeros at the negative even integers). Currently, what is know about the location of its zeros is by comparison much weaker: there exist an explicit positive valued function

$$C : \mathbb{R} \to \mathbb{R}_{>0} \text{ satisfying } \lim_{t\to\infty} C(t) = 0,$$

such that any zero γ satisfies the the bound

$$\Re\gamma < 1 - C(\Im\gamma).$$

The first such function was exhibited by Hadamard and dela Vallée-Poussin and was of the shape

$$C(t) = \frac{C}{\log(2 + |t|)}$$

for some absolute constant $C > 0$.

Remarkably the starting point of Deligne's proof was the fact that $L(\mathcal{F}, s)$ has no zeros on the line $\Re s = d + w/2$ and this was proven by adapting the Hadamard-de la Vallée-Poussin method. Since the set of zeros is finite up to translation by $2\pi i/\log p$, this implies that the real part of any zero satisfies

$$\Re\gamma \leq d + w/2 - \eta$$

for some $\eta > 0$. Using beautiful geometric ideas, Deligne was able to obtain successive improvements of this initial estimate by considering (external) tensor powers $\mathcal{F} \boxtimes \mathcal{F}$ on $X \times X$ and by fibering that surface over a suitable (rational) curve $X' \subset \mathbb{P}^1$ and by analysing geometrically the sheaf of the fibers of $\mathcal{F} \boxtimes \mathcal{F}$ over that curve

4. Diophantine consequences of Deligne Theorem

In particular for any finite extension k'/k we have (writing $q = |k'| = p^{[k':k]}$)

$$|\operatorname{tr}(\mathrm{Fr}_{k'}|H^i_c(X_{\overline{k}}, \mathcal{F}))| \leq \dim(H^i_c(X_{\overline{k}}, \mathcal{F}))q^{\frac{i+w}{2}}.$$

By the G-L trace formula we obtain that

$$|\sum_{x\in X(k')} \frac{\operatorname{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})}{q^{w/2}} - \frac{\operatorname{tr}(\mathrm{Fr}_{k'}|H^{2d}_c(X_{\overline{k}}, \mathcal{F}))}{q^{w/2}}| \leq \sum_{i=0}^{2d-1} \dim(H^i_c(X_{\overline{k}}, \mathcal{F}))q^{i/2}.$$

Recall that

$$\frac{\operatorname{tr}(\mathrm{Fr}_{k'}|H^{2d}_c(X_{\overline{k}}, \mathcal{F}))}{q^{w/2}} = q^d(\alpha_1(\mathcal{F})^{[k':k]} + \cdots + \alpha_{r'}(\mathcal{F})^{[k':k]}),$$

where the $\alpha_i(\mathcal{F}),\ i = 1 \cdots r'$ are algebraic numbers of modulus 1.

Observe now that the complex numbers $\frac{\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})}{q^{w/2}}$ are bounded in absolute value by $\mathrm{rk}(\mathcal{F}) = \dim(\mathcal{F}_{\overline{\eta}})$ and the summation over $X(k')$ has

$$|X(k')| = q^d + O(q^{d-1/2})$$

terms (Lang-Weil) we therefore obtain the estimate, for $q \to \infty$

$$\frac{1}{|X(k')|}\sum_{x\in X(k')}\frac{\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})}{q^{w/2}} - (\alpha_1(\mathcal{F})^{[k':k]}+\cdots+\alpha_{r'}(\mathcal{F})^{[k':k]}) = O_{\mathcal{F}}(q^{-1/2}) \to 0.$$

This estimate is either

– an asymptotic formula if $r' = \dim \mathcal{F}_{\pi_1^{geom}} \neq \{0\}$ since

$$\limsup_{[k':k]\to\infty} \alpha_1(\mathcal{F})^{[k':k]} + \cdots + \alpha_{r'}(\mathcal{F})^{[k':k]} = r',$$

– or a non-trivial upper bound if $r' = \dim \mathcal{F}_{\pi_1^{geom}} = \{0\}$.

The later situation $r' = 0$ is relatively easy to decide: for instance it occurs if $\mathcal{F}$ is *geometrically irreducible and non trivial.*

4.1. **Application to representation theory.** Let $\mathcal{F}$ and $\mathcal{G}$ be pure sheaves; up to Tate twisting, we may assume (possibly up to restricting the representation to the WD group) that both are pure of weight 0: we have

Corollary 4.1. *If $\mathcal{F}$ is geometrically irreducible, one has*

$$\frac{1}{|X(k')|}\sum_{x\in X(k')}\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}}) = \alpha(\mathcal{F})^{[k':k]} + O_{\mathcal{F}}(q^{-1/2})$$

for $\alpha_1(\mathcal{F})$ of modulus 1 if $\mathcal{F}$ is geometrically trivial and equal to 0 otherwise.

A more general and interesting statement is the following

Corollary 4.2. *If $\mathcal{F}$ and $\mathcal{G}$ are geometrically irreducible, one has*

$$\frac{1}{|X(k')|}\sum_{x\in X(k')}\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})\overline{\mathrm{tr}(\mathrm{Fr}_x|\mathcal{G}_{\overline{\eta}})} = \alpha(\mathcal{F},\mathcal{G})^{[k':k]} + O_{\mathcal{F},\mathcal{G}}(q^{-1/2})$$

with

– $|\alpha(\mathcal{F},\mathcal{G})| = 1$ *if $\mathcal{F}$ and $\mathcal{G}$ are geometrically isomorphic,*
– $\alpha(\mathcal{F},\mathcal{G}) = 0$ *otherwise.*

Proof. Consider the weight 0 sheaf $\mathcal{F}\otimes\check{\mathcal{G}}$: we have

$$\mathrm{tr}(\mathrm{Fr}_x|(\mathcal{F}\otimes\check{\mathcal{G}}_{\overline{\eta}})) = \mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})\overline{\mathrm{tr}(\mathrm{Fr}_x|\mathcal{G}_{\overline{\eta}})}$$

and by Schur's Lemma the space of coinvariants is either 1 or 0 dimensional. □

In the above estimate the error term satisfies the bound

$$|O_{\mathcal{F},\mathcal{G}}(q^{-1/2})| \leq q^{-1/2}\sum_{i=0}^{2d}\dim H^i_c(X_{\overline{k}},\mathcal{F}\otimes\check{\mathcal{G}})$$

Thus it is possible to detect whether two sheaves are geometrically isomorphic by an arithmetic computation by letting q get large enough by comparison with the geometric invariants of $\mathcal{F}$ and $\mathcal{G}$ (ie. the sum

$$\sum_{i=0}^{2d} \dim H_c^i(X_{\overline{k}}, \mathcal{F} \otimes \check{\mathcal{G}}) \text{ .)}$$

This formula ressemble very much the orthogonality relation for characters of a finite group and will be called quasi-orthogonality.

4.2. **Deligne's equidistribution theorem.** Let

$$\mathcal{F} \simeq \rho : WD(X, \overline{\eta}) \to \mathrm{GL}(V)$$

a general lisse sheaf pure of some weight, Deligne obtained the following structural result by a similar argument as above (by considering Extension sheaves $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$, the proof however did not require the full strength of his main result but only the Hadamard-de la Vallée-Poussin argument).

Theorem 4.1. *$\mathcal{F}$ is geometrically semi-simple: the representation ρ^{geom} of π_1^{geom} can be decomposed into a sum of irreducible ones.*

This does not provide a decomposition of ρ^{arith} into irreducible components, but at least a decomposition in a direct sum of the *geometrically isotypic components*: the sum of copies of a given isomorphism class of an irreductible representation of ρ^{geom} form a representation of π_1^{arith}.

In particular we have the following

Corollary 4.3. *Let $\mathcal{F}$ be a pure sheaf, then*

$$\limsup_{q \to \infty} \frac{1}{|X(k')|} \sum_{x \in X(\mathbb{F}_q)} |\operatorname{tr}(\mathrm{Fr}_x | \mathcal{F})|^2 = \sum_{\rho'} m(\rho'; \mathcal{F})^2$$

where ρ' ranges over the irreductible components of ρ^{geom} occuring in the decomposition of ρ^{geom} and $m(\rho'; \mathcal{F})$ denote their multiplicity.

Another consequence:

Theorem 4.2. *Let*

$$G = \overline{\rho(\pi_1^{geom})}^{Zar} \subset \mathrm{GL}(V)$$

be the Zariski closure of the image of ρ^{geom} in $\mathrm{GL}(V)$; the connected component G^0 is a semisimple algebraic group.

Assume (up to a tate twist) that $\mathcal{F}$ has weight 0 and that

$$\rho(\pi_1^{arith}(X, \overline{\eta})) \subset G(\overline{\mathbb{Q}_\ell}).$$

Upon choosing an embedding $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$, we may consider the complex Lie group $G(\mathbb{C})$; fix $K \subset G(\mathbb{C})$ a maximal compact subgroup. The semi-simplification of the Frobenius conjugacy classes $\rho(\mathrm{Fr}_x)^{ss}$, $x \in |X|$ define semi-simple conjugacy classes in $G(\mathbb{C})$ which intersect K since the eigenvalues have modulus 1; in fact, $\rho(\mathrm{Fr}_x)^{ss}$ is a well defined K-conjugacy class in K (because the characters of K are the restriction of the characters of $G(\mathbb{C})$ which are also the characters of the algebraic representation of G): we denote this conjugacy class by

$$\theta(x) \in K^{\natural}.$$

Theorem 4.3 (Deligne equidistribution theorem)**.** *The set of conjugacy classes $\{\theta(x),\ x \in |X|\}$ is equidistributed on $K^{\natural}$ wrt μ_{ST} the direct image of the Haar measure on K: for any $f \in \mathcal{C}(K^{\natural})$*

$$\frac{1}{|X(k')|}\sum_{x\in X(k')} f(\theta(x)) \to \int_{K^{\natural}} \theta(\theta)d\mu_{ST}(\theta).$$

Proof. It is sufficient to prove this when $f(\theta) = \mathrm{tr}(r(\theta))$ for $r : G \to \mathrm{GL}(V_r)$ a non-trivial irreductible representation of G (hence of K). Let $r(\mathcal{F})$ be the lisse sheaf

$$r \circ \rho : \pi_1^{arith}(X, \overline{\eta}) \to G \to \mathrm{GL}(V_r).$$

It is pure of weight 0 and geometrically irreductible non-trivial, and

$$\mathrm{tr}(\mathrm{Fr}_x|V_r) = \mathrm{tr}(r(\rho(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}}))$$

therefore

$$\frac{1}{|X(k')|}\sum_{x\in X(k')} f(\theta(x)) = \frac{1}{|X(k')|}\sum_{x\in X(k')} \mathrm{tr}(\mathrm{Fr}_x|V_r) = O_{\mathcal{F},r}(q^{-1/2}) \to 0.$$

□

The measure μ_{ST} on $K^{\natural}$ (the direct image of the Haar probability measure) is called the *Sato-Tate measure* after the work of Sato and Tate on the case of families elliptic curves (see example below). Therefore sometimes versions of Deligne-Equidistribution theorem are called *sato-tate laws.*

4.2.1. *Example.* Suppose $p \neq 2, 3$ and $\sqrt{p} \subset E_\lambda$. Let

$$\pi : E \to \mathbb{P}^1_{\mathbb{F}_p}$$

be be a family of elliptic curves over the projective line, ie. a surface whose generic π-fiber is a non-trivial elliptic curve: say

$$E_t : zy^2 = x^3 + a(t)xz^2 + b(t)z^3 \in \mathbb{P}^2_{\mathbb{F}_p},\ a, b \in k(t).$$

We assume that the family is not *geometrically isotrivial* which means that the j-invariant $j(t)$ is not constant. Consider the sheaf $R^1\pi_*\mathbb{Q}_\ell$; this is a rank 2 sheaf which is lisse outside the set of zeros of the Discriminant

$$\Delta(t) = -16(4a^3(t) + 27b^2(t)),\ p \neq 2, 3,$$

and pure of weight 1: for $t \in \mathbb{F}_q$ such that $\Delta(t) \neq 0$ we have

$$(R^1\pi_*\mathbb{Q}_\ell)_t = H^1(E_t, \mathbb{Q}_\ell)$$

and

$$\mathrm{tr}(\mathrm{Fr}_t|R^1\pi_*\mathbb{Q}_\ell) = a(E_t) = |E_t(\mathbb{F}_q)| - q - 1 = 2q^{1/2}\cos(\theta_t),\ \theta_t \in [0, \pi[.$$

In that case one can show[1] that $G^0 = \mathrm{SL}_2$, $K = \mathrm{SU}_2(\mathbb{C})$, $K^\natural \simeq \mathbb{R}/\pi\mathbb{Z}$, $\mu_{ST} = \frac{2}{\pi}\sin^2(\theta)d\theta$ and the Frobenius conjugacy class is the class of the diagonal matrix

$$diag(e^{i\theta_t}, e^{-i\theta_t}) \in \mathrm{SU}_2(\mathbb{C}).$$

4.3. **Further vanishing: purity.** The consequence get stronger if more cohomology group vanish: let us recall that by Poincare duality, if X is affine

$$H_c^i(\cdot) = 0 \text{ for } i = 0, \cdots, d-1$$

but this is not so interesting since the Frobenius eigenvalues in these groups have the smallest weights and so contribute less.

On the other hand there exist very nice situations (a phenomenon sometimes called *purity*) for which

$$H_c^i(\cdot) = 0 \text{ for } i = d+1, \cdots, 2d$$

in which case one obtains

$$\frac{1}{q^d}\sum_{x\in X(k')}\frac{\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}_{\overline{\eta}})}{q^{w/2}} = O_{\mathcal{F}}(q^{d/2}).$$

Recall that in such situation we are summing over $q^d + O(q^{d-1/2})$ terms which are bounded by $\mathrm{rk}\,\mathcal{F}$ and that the resulting sum is bounded by $O_{\mathcal{F}}(q^{d/2})$ which is about square-root of the number of terms.

The lectures of Katz [Kat80] exhibit examples of purity situations. The work of Katz-Laumon [KL85] on Fourier transform exhibits further examples: in particular of families of sheaves (indexed by an affine space) for which purity occurs generically.

4.3.1. *Hyper-Kloosterman sums.* [Del77] An example of such a nice situation is given by the folowing d-dimensional variety: for $a \in \mathbb{F}_p^\times$ let

$$X_a = \{(x_1, \cdots, x_{d+1}) \in A^{d+1}_{\mathbb{F}_p}, x_1.\cdots.x_{d+1} = a\}$$

and let

$$\mathcal{F}_a = \mathcal{L}_\psi(f) = f^*\mathcal{L}_\psi,\ f(x_1, \cdots, x_{d+1}) = x_1 + \cdots + x_{d+1}.$$

This is a sheaf pure of weight 0 and rank 1 and the resulting sums are the d-dimensional hyper Kloosterman sums

$$\mathrm{Kl}(a; k') = \sum_{\substack{(x_1,\cdots,x_{d+1})\in(k')^d \\ x_1.\cdots.x_{d+1}=a}} \psi(\mathrm{tr}_{k'/k}(x_1 + \cdots + x_{d+1})).$$

[1] The way Deligne showed this was by saying that this representation is the restriction to a finite index subgroup of the representation of the fundamental group of the projective line minus finitely many points of a sheaf (obtained by considering the moduli space of elliptic curves with level structure) for which one knows by transcendental methods that its geometric monodromy group is SL_2; one can find in the works of Katz other somewhat more direct methods of proving this using the structure of the fundamental group of a curve minus finitely many points (see below.)

Deligne has shown that all the cohomology groups except the middle cohomology are 0 and that the later has dimension $d+1$ therefore

$$|\mathrm{Kl}_d(a;k')| \leq (d+1)q^{d/2};$$

this is to be compare with the fact that X_a is a ${\mathbb{G}_m}^d$-torsor and therefore

$$|X_a(k')| = |\mathbb{G}_m(k')^d| = (q-1)^d \simeq q^d.$$

4.4. **Equidistribution of Gauss sums.** Given $\chi : {\mathbb{F}_q}^\times \to \mathbb{C}^\times$ a non-trivial character the normalized Gauss sum is

$$g(\chi) = \frac{1}{q^{1/2}} \sum_{x \in {\mathbb{F}_q}^\times} \chi(x)\psi(\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)) \in \mathbb{C}^1.$$

Corollary 4.4. *The set*

$$\{g(\chi),\ \chi \neq 1\}$$

is ed on $\mathbb{C}^1$ *wrt the Lebesgue measure*

Proof. For $d \in \mathbb{N}_{>0}$

$$\frac{1}{q-2}\sum_{\chi\neq 1} g(\chi)^d = \frac{1}{q-2}\sum_{\chi} g(\chi)^d + O(1/q)$$

and

$$\begin{aligned}\frac{1}{q-1}\sum_{\chi} g(\chi)^d &= \frac{1}{(q-1)q^{d/2}}\sum_{\chi}\sum_{x_1\cdots x_d} \chi(x_1.\cdots x_d)\psi(x_1+\cdots+x_d)\\ &= \frac{1}{q^{d/2}}\mathrm{Kl}_{d-1}(1;k') = O(q^{-1/2}) \to 0.\end{aligned}$$

by Deligne's bound. □

5. The case of curves

Let us assume that $X \hookrightarrow C$ is a non-empty affine subset of a smooth proper geometrically connected curve.

Because we are in dimension 1, the situation is much simpler than in the general case; for instance there are only three cohomology groups to consider, two of which are "easy": $H^0_c(.) = 0$ since X is affine, $H^c_1(.)$ the most interesting part and $H^2_c(.)$ which can be computed in terms of co-invariants (and made equal to 0 under simple assumptions). In fact this simple case is the fundamental one for the proof of Deligne's main theorem.

Here, we would like to provide a more detailed description of the ℓ-adic sheaves on curves.

5.1. **The fundamental group of a curve.** [Kat80,Kat88] Let $K = k(X) = k(C)$ the function field of the curve. K is a global field and the set of its valuations is indexed by $|X|$. Explictely, if $C = \mathbb{P}^1$, $K = k(x)$, the closed point are either, ∞ and the corresponding valuation is

$$v_\infty(P/Q) = -\deg P + \deg Q$$

or the set of roots of an irreducible polynomial P and the valuation v_P is the order of divisibility by that polynomial.

We denote by k_x the residue field and $\deg(x)$ its degree; one defines an absolute value by

$$|f|_x = |k_x|^{-v_x(f)}.$$

We denote by $K_{\{x\}}$ the completion of K wrt $|.|_x$. The absolute value $|.|_x$ extend to an absolute value $|.|_{\overline{x}}$ on $\overline{K}$ and all the extensions are conjugate by $\mathrm{Gal}(\overline{K}/K)$. The decomposition subgroup is

$$D_{\overline{x}} = \{\sigma \in \mathrm{Gal}(\overline{K}/K),\ \sigma(\overline{x}) = \overline{x}\} \simeq \mathrm{Gal}(\overline{K}_{\{\overline{x}\}}/K_{\{x\}})$$

we have a surjective reduction map

$$1 \longrightarrow I_{\overline{x}} \longrightarrow D_{\overline{x}} \xrightarrow{red_{\overline{x}}} \mathrm{Gal}(\overline{k}/k_x) \longrightarrow 1$$

whose kernel is the *inertia subgroup at* $\overline{x}$ One has

$$\pi_1^{arith}(X, \overline{\eta}) \simeq \mathrm{Gal}(\overline{K}/K)/\langle I_{\overline{x}},\ \overline{x}|x \in |X|\rangle.$$

$$\pi_1^{geom}(X, \overline{\eta}) \simeq \mathrm{Gal}(\overline{K}/\overline{k}K)/\langle I_{\overline{x}},\ \overline{x}|x \in |X|\rangle.$$

In particular a lisse sheaf $\mathcal{F}$ is a representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$$

such that for any $\overline{x}|x \in |X|$, $I_{\overline{x}}$ acts trivially and hence one has an action of the quotient

$$I_{\overline{x}} \backslash D_{\overline{x}} \simeq \mathrm{Gal}(\overline{k}/k_x) \to \mathrm{GL}(V).$$

This defines the action the generator of that group $\mathrm{Fr}_{\overline{x}}$; therefore the trace and more generally characteristic polynomial of the corresponding Frobenius conjugacy class Fr_x are well defined.

Remark 5.1. When $x \notin |X|$, we have a well defined action of the Frobenius $\mathrm{Fr}_{\overline{x}}$ but only on the subspace

$$V^{I_{\overline{x}}}.$$

5.2. **Ramification.** The maximal open subset X of C for which

$$V^{I_{\overline{x}}} = V.$$

for all $x \in |X|$ is the "ouvert de lissite" of $\mathcal{F}$. For the x outside this open set the representation is said *ramified*

$$V^{I_{\overline{x}}} \subsetneq V.$$

Fix $\overline{x}$ and write I for $I_{\overline{x}}$. Let P be the p-sylow subgroup of the profinite group I one has the exact sequence

$$1 \longrightarrow P \longrightarrow I \longrightarrow \prod_{\ell\neq 0} \mathbb{Z}_\ell(1) \longrightarrow 1$$

The quotient $P\backslash I$ is the *tame inertia group* and if P act trivially on V one says that $\mathcal{F}$ is tamely ramified at $\overline{x}$. P is called the wild inertia subgroup.

5.3. **Swan conductors.** On I, one has (see. Serre [Ser79]) a decreasing "upper numbering" filtration of I by open normal subgroups

$$(I^\varepsilon)_{\varepsilon\in[0,\infty[},\ I^0 = I \supsetneq I^\varepsilon \supsetneq I^{\varepsilon'},\ 0<\varepsilon<\varepsilon'.$$

One has

$$P = \overline{\bigcup_{\varepsilon>0} I^\varepsilon},\ \bigcap_{\varepsilon\geq 0} I^\varepsilon = \{1\},\ \varepsilon>0\ I^\varepsilon = \bigcap_{0<\varepsilon'<\varepsilon} I^{\varepsilon'}.$$

If V is acted on by I, one has a decomposition

$$V = \bigoplus_\varepsilon V(\varepsilon)$$

with

$$V(0) = V^P,\ V(\varepsilon)^{I^\varepsilon} = 0,\ V(\varepsilon)^{I^{\varepsilon'}} = V(\varepsilon),\ \varepsilon' > \varepsilon.$$

the ε for which $V(\varepsilon)\neq 0$ are the breaks of the I-representation. The Swan conductor is

$$\mathrm{swan}_{\overline{x}}(V) = \sum_\varepsilon \varepsilon \dim(V(\varepsilon)) \in \mathbb{N}.$$

Observe that since the Galois group acts transitively on the $\overline{x}|x$,

$$\mathrm{swan}_{\overline{x}}(V) = \mathrm{swan}_x(V)$$

depends only on x. The Swan conductor account for the complexity of the local representation restricted to the Wild inertia subgroup P: the larger it is the less trivial it is on smaller subgroups of P. It is also not too hard to control the variation of the Swan conductor under operations such as tensor products or composition with another representation.

Example 5.1. The Kummer sheaves $\mathcal{L}_\chi$ are tamely ramified at 0, ∞. The Artin-Schreier sheaves are wildly ramified at ∞ with

$$\mathrm{swan}_\infty(\mathcal{L}_\psi) = 1.$$

5.4. **The Euler-Poincare formula.** We return to the global setting and have

$$\sum_{i=0}^{2}(-1)^i \dim(H^i_c(X_{\overline{k}})) = \mathrm{rk}(\mathcal{F})\chi_c(X_{\overline{k}}) - \sum_{x\in|C-X|} \deg(x)\mathrm{swan}_x(\mathcal{F})$$

where

$$\chi_c(X_{\overline{k}}) = \sum_{i=0}^{2}(-1)^i \dim(H^i_c(X_{\overline{k}},\mathbb{Q}_\ell)) = 2-2g-\sum_{x\in|C-X|}\deg(x).$$

In particular if X is affine

$$\dim(\mathcal{F}_{\pi_1^{geom}}) - \dim(H^1_c(X_{\overline{k}})) = \mathrm{rk}(\mathcal{F})\chi_c(X_{\overline{k}}) - \sum_{x\in|C-X|}\deg(x)\mathrm{swan}_x(\mathcal{F}).$$

Example 5.2. Let $f\in k[t]$ be a polynomial of degree $d\geq 1$ coprime with p and $\mathcal{F}=f^*\mathcal{L}_\psi$:

$$\mathrm{tr}(\mathrm{Fr}_x|\mathcal{F}) = \psi(f(x)).$$

$\mathcal{F}$ is lisse on $\mathbb{A}^1_{\mathbb{F}_p}$ of rank 1 and wild at ∞ with a unique break at d and swan conductor d. We have $H^0_c=H^2_c=0$ and

$$|\sum_{x\in k'}\psi(f(x))|\leq (d-1)q^{1/2}.$$

This was proven by A. Weil.

Example 5.3. There exist a Kloosterman sheaf, lisse on $X=\mathbb{G}_{m,\mathbb{F}_p}$ such that

$$\mathrm{tr}(\mathrm{Fr}_x|\mathcal{KL}) = -\mathrm{Kl}(x;k_x)\in\mathbb{R}.$$

This sheaf has rank 2, is pure of weight 1, moreover

$$\sum_{x\in\mathbb{F}_q{}^\times}|\mathrm{Kl}(x;k_x)|^2 = \sum_{y,y'}\sum_x\psi(x(y-y')+1/y-1/y')) = q^2+O(q^{3/2})$$

so this is irreducible. It is tame at 0 and wild at ∞ with $\mathrm{swan}_\infty=2$ (a single break at 1). One has

$$\det(\mathcal{KL}) = \mathbb{Q}_\ell(-1)$$

and $G^0=\mathrm{SL}_2$. This implies that the angles θ_x defined via

$$2\cos(\theta_x) = -\frac{\mathrm{Kl}(x;k_x)}{q^{1/2}}$$

are equidistributed wrt μ_{ST}.

5.5. **The conductor of a sheaf.** We define the "analytic conductor" of $\mathcal{F}$ by measuring the complexity of the local representations and by aggregating these:

$$C(\mathcal{F}) = g(C)+\mathrm{rk}(\mathcal{F})+\sum_{x\in|C-X|}\deg(x)\max(1,\mathrm{swan}_x(\mathcal{F})).$$

From the Euler-Pojncare formula we see that

$$\dim(H^1_c(.))\ll C(\mathcal{F})$$

5.5.1. *Application to counting sheaves.* Using the quasi-orthogonality relation Cor. 4.2 we have proven in [FKM12] that

Theorem 5.1. *The number of geometrically irreducible sheaves up to Tate twists of conductor $\leq C$ is bounded, more precisely it is bounded by*

$$\ll p^{O(C^6)}.$$

Here the implied constant are absolute.

Proof. The quasi-orthogonality relation and the fact that $C(\mathcal{F} \otimes \check{\mathcal{G}}) \ll C(\mathcal{F})C(\mathcal{G})$ shows that if p is large enough wrt C the map

$$\mathcal{F} \text{ geom. irr. sheaf with } C(\mathcal{F}) \leq C \mapsto \mathbb{C}[x \in X(\mathbb{F}_p) \to \mathrm{tr}(\mathrm{Fr}_x|\mathcal{F})] \in \mathbb{P}(\mathbb{C}^{X(\mathbb{F}_p)})$$

is injective, in particular the number of such sheaves is the number of such lines in this $|X(\mathbb{F}_p)|$-dimensional complex vector space. Each such line defines a point on the corresponding unit sphere and the cosine of the angles between two distinct points is $O(C^2p^{-1/2})$ or equivalently the angle is within $O(C^2p^{-1/2})$ is $\pi/2$ (if p is much larger than C the vectors are almost orthogonal). There are upper general bounds (due mainly to Kabatjanskii-Levenshtein in the context of spherical codes) on the maximal number of nearly orthogonal points that one can put on a sphere (possibly of high dimension). this yield the result. □

6. ℓ-ADIC FOURIER TRANSFORM [KL85, Lau87]

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