

Symplectic Manifolds

Tamás Hausel

1 Symplectic Vector Spaces

Let us first fix a field k .

Definition 1.1 (Symplectic form, symplectic vector space). Given a finite dimensional k -vector space W , a *symplectic form* on W is a bilinear, alternating non degenerate form on W , i.e. an element $\omega \in \wedge^2 W^*$ such that if $\omega(v, w) = 0$ for all $w \in W$ then $v = 0$.

We call the pair (W, ω) a *symplectic vector space*.

Example 1.2 (Standard example). Consider a finite dimensional k -vector space V and define $W = V \oplus V^*$. We can introduce a symplectic form on W :

$$\begin{aligned} \omega : W \times W &\rightarrow k \\ ((v_1, \alpha_1), (v_2, \alpha_2)) &\mapsto \alpha_1(v_2) - \alpha_2(v_1) \end{aligned}$$

Definition 1.3 (Symplectomorphism). Given two symplectic vector spaces (W_1, ω_1) and (W_2, ω_2) , we say that a linear map $f : W_1 \rightarrow W_2$ is a *symplectomorphism* if it is an isomorphism of vector spaces and, furthermore, $f^*\omega_2 = \omega_1$.

We denote by $\text{Sp}(W)$ the group of symplectomorphism $W \rightarrow W$.

Let us consider a symplectic vector space (W, ω) with even dimension $2n$ (we will see soon that it is always the case). Then $\omega \wedge \cdots \wedge \omega \in \wedge^{2n} W^*$ is a volume form, i.e. a non-degenerate top form. For $f \in \text{Sp}(W)$ we must have $\omega^n = f^*\omega^n = \det f \cdot \omega^n$ so that $\det f = 1$ and, in particular, $\text{Sp}(W) \leq \text{SL}(W)$.

We also note that, for $n = 1$, we have $\text{Sp}(W) \cong \text{SL}(W)$.

Definition 1.4. Let U be a linear subspace of a symplectic vector space W . Then we define

$$U^\perp = \{v \in W \mid \omega(v, w) = 0 \ \forall u \in U\}.$$

We say that U is *isotropic* if $U \subseteq U^\perp$, *coisotropic* if $U^\perp \subseteq U$ and *Lagrangian* if $U^\perp = U$.

We can deduce some properties from the definitions above.

- It is clear that $\dim U + \dim U^\perp = \dim W$ if ω is non-degenerate and $U^{\perp\perp} = U$.
- We have $U \cap U^\perp = 0$ if and only if $\omega|_U$ is non-degenerate, i.e. if and only if U is a symplectic vector subspace.
- Since $\omega(v, v) = 0$, all one dimensional vector subspaces are isotropic.
- U is isotropic if and only if $\omega|_U = 0$. Notice that in this case U^\perp/U with the symplectic form induced from ω is a symplectic vector space.
- In the opposite way, all codimension one vector subspaces are coisotropic.
- A Lagrangian subspace U is a maximal subspace with respect to the condition $\omega|_U = 0$.

- If there exists a Lagrangian subspace U (we will see that it is always the case), then $\dim W = 2 \dim U$ so that W has even dimension. Notice that this also follows from the fact that, after choosing a basis for W , the bilinear form ω will be represented by a $k \times k$ matrix A , where $k = \dim W$. Moreover from the fact that ω is non-degenerate and antisymmetric we get $\det A = \det A^T = \det(-A) = (-1)^k \det A$ and since $\det A \neq 0$ we must have k even.

Proposition 1.5. Every symplectic vector space (W, ω) is symplectomorphic to $V \oplus V^*$ for some finite dimensional vector space V . In particular $\dim W = 2n$.

Dimostrazione. First of all we prove that there exist a Lagrangian subspace V . Start with any isotropic space E (there is always one, for example any one dimensional E will do), if $E = E^\perp$ we are done, otherwise pick $v \in E^\perp \setminus E$. Now $E' = E \oplus \langle v \rangle$ is strictly larger than E and $\omega|_{E'} = 0$ so it is also isotropic. This process has to stop at some point giving us a maximal isotropic subspace V , i.e. a Lagrangian subspace.

Let us take an isotropic subspace F such that $F \cap V = 0$. Again, if $F = F^\perp$ we stop, otherwise F^\perp/F is a non-trivial symplectic vector space properly containing $(F + V \cap F^\perp)/F$ as an isotropic subspace. In particular we can pick $v \in F^\perp \setminus (F + V)$ and build a bigger $F' = F \oplus \langle v \rangle$ with the same properties.

Summarizing, we proved that there exist two Lagrangian vector spaces V and V' such that $V \cap V' = 0$. Therefore $\dim(V \oplus V') = \dim W - \dim(V \oplus V')^\perp = \dim W$ so that $W = V \oplus V'$. We also have a map $V' \rightarrow V^*$ given by $v' \mapsto \omega(v', \cdot)$ which is injective by non-degeneracy of ω . Finally it is easy to check that W is also symplectomorphic to the standard structure on $V \oplus V^*$. \square

Example 1.6 (Standard example). Let us consider again the example of the symplectic space $W = V \oplus V^*$. We can choose a basis e_1, e_2, \dots, e_n of V and the corresponding dual basis f_1, f_2, \dots, f_n of V^* so that $f_i(e_j) = \delta_{ij}$. Therefore $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n$ is a basis of W and $\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$ while $\omega(e_i, f_j) = -\delta_{ij}$.

We can also introduce the basis of W^* dual to the one we have for W , call it $\epsilon_1, \dots, \epsilon_n, \phi_1, \dots, \phi_n$. Then it is easily verified that $\omega = \epsilon_1 \wedge \phi_1 + \dots + \epsilon_n \wedge \phi_n$ and also:

$$\omega \wedge \dots \wedge \omega = n! \epsilon_1 \wedge \phi_1 \wedge \dots \wedge \epsilon_n \wedge \phi_n.$$

2 Symplectic manifolds

In this section we will assume that $k = \mathbb{R}$.

Definition 2.7 (Almost Symplectic Manifold). A manifold M is said to be *almost symplectic* if there exists a 2-form $\omega \in \Omega^2(M) = \Gamma(M, \wedge^2 T^*M)$ which is symplectic at every point.

Example 2.8 (Flat Example). Let us consider $M = W$ a symplectic vector space. Then $\wedge^2 T^*W \cong W \times \wedge^2 W^*$ and we can endow it with a structure of an almost symplectic manifold by defining a 2-form ω equal to the standard one at every point. Referring to the previous example, we have a basis $e_1, \dots, e_n, f_1, \dots, f_n$ of W which gives us coordinate functions $p_i, q_i : W \rightarrow \mathbb{R}$. Then at every point ω is equal to $dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$. Note that in this case $d\omega = 0$.

Definition 2.9 (Symplectic manifold). An almost symplectic manifold is said to be *symplectic* if its associated 2-form is also closed. A *symplectomorphism* is a diffeomorphism f that pulls the symplectic form on the codomain back to the symplectic form on the domain.

Example 2.10. – Every symplectic vector space W can be regarded as a symplectic manifold.

- For every manifold N , the cotangent bundle T^*N has a canonical symplectic structure. Call $\pi : T^*N \rightarrow N$ the canonical projection and $T_\pi : T(T^*N) \rightarrow TN$ the tangent map to π . Let us consider the *Liouville form* $\alpha \in \Omega^1(T^*N) = \Gamma(T^*(T^*N))$. For $(p, \beta) \in T^*N$ and

$X \in T_{(p,\beta)}T^*N$ we define $\alpha(X) = \beta(T_\pi(X))$. Then $(T^*N, d\alpha)$ can be seen to be a symplectic manifold.

- If $\dim M = 2$ then ω is a symplectic form if and only if it is a volume form. In particular every oriented surface is a symplectic manifold.