An introduction to ℓ-adic sheaves and the function-sheaf dictionary

Michael Groechenig
Lausanne, March 2013

1 The quest for the right topology

The underlying (Zariski) topological space $|X|$ of an algebraic variety/scheme X does not capture our intuition stemming from complex analysis. It is straightforward to show that two $\mathbb{C}$-curves X and Y (closed or not) have homeomorphic Zariski topologies, regardless of genus and singularities, etc.

In the first part this talk we will discuss purely algebraic means of recovering algebraic topological invariants of varieties, and the reader is encouraged to think of our objects of study being defined over the field of complex numbers. In a broad outline there are two theories, allowing purely algebraic treatments, which capture the algebraic topological properties of manifolds:

- differential equations (differential forms, de Rham complex, D-modules)
- covering spaces ($\pi_1 = \mathrm{Aut}(\widetilde{X}/X)$, local systems).

We will entirely focus on the second picture, since it's easier to generalize to arbitrary characteristic (the first picture becoming slightly obscure in positive characteristic, rendering it less intuitive for a complex geometer). Nonetheless, the importance of the first approach to the historical development of the subject cannot be underestimated. Grothendieck's proof of the algebraic de Rham theorem should be seen as the first result allowing the calculation of singular cohomology groups in a purely algebraic manner.

The second part of the talk is entirely settled in the world of finite fields, where the constructions obtained earlier are still reasonable but give rise to entirely new phenomena. By definition, all invariants constructed in the first chapter, will naturally be equipped with actions by Galois groups, giving the theory a new, arithmetic flavour. In the particularly important case of finite fields, the Galois element given by the Frobenius automorphism allows us introduce Deligne's theory of weights and Grothendieck's function-sheaf dictionary.

1.1 Fundamental groups

In the following we denote by X, Y, etc., nice topological spaces, e.g. those obtained as the underlying spaces of (complex) manifolds or more generally of analytic varieties.[1]

1.1.1 Motivation: manifolds

We define the category $Cov(X)$ whose objects are connected covering spaces $\pi : Y \to X$. Morphisms $Y \to Y'$ are given by a commutative diagram of coverings

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow & \downarrow \\ & & X, \end{array}$$

and denote by

$$\mathbf{Fib}_x : Cov(X) \to Set$$

the functor sending Y to the set $\pi^{-1}(x)$ and refer to it as the fibre functor at x. The group of natural self-transformations of the fibre functor $Aut(\mathbf{Fib}_x)$ is given by the collection of compatible automorphisms of $\pi^{-1}(Y)$; i.e., for every $Y \in Cov(X)$ a permutation σ_Y of the set $\mathbf{Fib}_x(Y) = \pi^{-1}(x)$, s.t. for every morphism of coverings $\phi : Y \to Y'$ we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Fib}_x(Y) & \xrightarrow{\phi} & \mathbf{Fib}_x(Y') \\ \downarrow{\scriptstyle \sigma_Y} & & \downarrow{\scriptstyle \sigma_{Y'}} \\ \mathbf{Fib}_x(Y) & \xrightarrow{\phi} & \mathbf{Fib}_x(Y'). \end{array}$$

Theorem 1.1. *There is a natural automorphism* $\pi_1(X, x) \cong Aut(\mathbf{Fib}_x)$.

Proof. Every element of $\pi_1(X, x)$ can be pictured as a closed path in X based at x. Every such path can be lifted to a non-necessarily closed path in a covering space

[1] From a purely topological viewpoint we need our spaces to be connected, locally path-connected, and locally simply-connected.

Y, depending only on the choice of a starting point given an element in $\pi^{-1}(x) = \mathbf{Fib}_x(Y)$. This construction obiously yields a compatible system of permutations of the set $\pi^{-1}(x)$. We have therefore obtained a natural morphism $\pi_1(X, x) \to Aut(\mathbf{Fib}_x)$ and to conclude the proof we have to verify that it is an isomorphism.

Let $\widetilde{X}$ denote a universal covering space of X. We recall that up to the choice of a base point $\tilde{x} \in \pi^{-1}(x)$ there exists an identification of $\mathbf{Fib}_x(\widetilde{X})$ with $\pi_1(X)$, by means of the above construction. Moreover there exists an identification of $\pi_1(X)$ with the group of deck transformations $Aut(\widetilde{X}/X)$.

Let now σ be the permutation of $\mathbf{Fib}_x(\widetilde{X})$ otained by restricting an arbitrary element of $Aut(\mathbf{Fib}_x)$ to $\widetilde{X}$. By the discussion in the paragraph above, we have to show that $\sigma(\tilde{x})$ determines σ uniquely. Every other element in $\mathbf{Fib}_x(\widetilde{X})$ can be uniquely written as $\gamma\tilde{x}$, where $\gamma \in \pi_1(X, x)$. Moreover, γ can be also viewed as a deck transformation of the universal covering space $\widetilde{X}$. By naturality of the permutation σ (definition of natural self-transformation of a functor), we obtain

$$\sigma(\gamma\tilde{x}) = \sigma(\gamma(\tilde{x})) = \gamma\sigma(\tilde{x}),$$

which allows us to conclude the proof. □

If the topological spaces X and Y can be endowed with the structure of differentiable manifold, the notion of covering can be expressed in terms of these extra structure.

Definition 1.2. *A map $f : Y \to X$ between two differentiable manifolds is called* étale *or* local diffeomorphism *if for every $x \in X$ and every $y \in f^{-1}(x)$, the differential $df_y : T_yY \to T_xX$ is an isomorphism.*

The proof of the proposition below is left to the reader. It will turn out to be the key ingredient in algebraizing the topological invariant π_1. We hope that the reader shares our belief of the given assertions being natural and geometrically evident.

Proposition 1.3. *Let $\pi : Y \to X$ be an étale morphism between two differentiable manifolds, which is additionally proper (i.e. preimages of compact subsets are compact), then π is a covering morphism with finite fibres. Moreover, all finite coverings of a differentiable manifold X arise in this way: i.e. every covering Y inherits the structure of a differentiable manifold, rendering the map π to be étale, and a covering map between manifolds has finite fibres if and only if it is proper.*

Proposition 1.3 gives a geometric characterization of finite covering maps, it is therefore an interesting question how far we can go by only using finite covering spaces. A more precise question being: let $Cov^{fin}(X) \subset Cov(X)$ be the full subcategory of finite connected covering spaces, and

$$\mathbf{Fib}_x^{fin} : Cov^{fin}(X) \to Set^{fin}$$

the restriction of the fibre functor. How does

$$\pi_1^{fin}(X, x) := Aut(\mathbf{Fib}_x^{fin})$$

relate to the fundamental group $\pi_1(X, x) = Aut(\mathbf{Fib}_x)$? The next definition contains a construction from abstract group theory, which allows us to formulate the answer.

Definition 1.4. *Let G be an abstract group, we denote by $F(G)$ the set of normal, finite-index subgroups N of G, i.e. G/N being a finite group. The set $F(G)$ is inductively ordered and the inverse limit of the finite quotients G/N, i.e.*

$$\widehat{G} := \{([g_N]_N)_{N \in F(G)} | [g_N]_N \in G/N, \text{ and } [g_{N'}]_N = [g_N]_N \text{ for } N' \subset N\},$$

is called the pro-finite completion *of G.*

It is important to know that pro-finite groups are more than just groups. The inverse limit construction endows them naturally with a topology (the subset topology of the product topology).[2] Moreover, by Tychonov's theorem, pro-finite groups are actually compact.

The relevance of this abstract notion to the determination of π_1^{fin} is due to a simple observation in the theory of covering spaces. Every finite-index subgroup N of $\pi_1(X, x)$ corresponds to a finite covering space $Y \to X$ by virtue of the fundamental theorem of covering theory. If N is moreover assumed to be a normal subgroup, it corresponds to finite regular covering spaces.[3] We hope that these remarks are already convincing enough to believe the statement of the following theorem, for the sake of clarity we have included a proof below.

Theorem 1.5. *The canonical morphism $\pi_1(X, x) \to \pi_1^{fin}(X, x)$, obtained by restricting an element of $Aut(Fib_x)$ to the subcategory $Cov^{fin}(X)$, induces an isomorphism*

$$\widehat{\pi_1(X, x)} \cong \pi_1^{fin}(X, x).$$

[2]Finite groups are viewed as topological groups with the trivial topology.

[3]Regularity is equivalent to the natural action of $\pi_1(X, x)$ on $\pi^{-1}(x)$ being transitive.

Proof. Let $Y_N \to X$ be the regular finite covering corresponding to a normal, finite-index subgroup $N \subset \pi_1(X, x)$, a compatible choice of these a collection $(Y_N)_{N \in F(\pi_1(X,x))}$ can be constructed by quotienting a universal covering space $\widetilde{X}$ by N. The group of deck transformations of Y_N is canonically given by $\pi_1(X, x)/N$. The choice of $\tilde{x} \in \mathbf{Fib}_x(\widetilde{X})$ gives rise to a base point x_N in every Y_N, which allows us to identify $\mathbf{Fib}_x(Y_N)$ with $\pi_1(X, x)/N$. Similarly to the proof of Theorem 1.1 we let (σ_N) be a compatible system of permuations of $\mathbf{Fib}_x(Y_N)$. For $y \in \mathbf{Fib}_x(Y_N)$ there exists a $\gamma \in \pi_1(X, x)$, whose class $[\gamma]_N$ is welldefined, s.t. $y = \gamma(y_N)$. As before we see by naturality of (σ_N) that σ_N is given by right multiplication with $\sigma_N(y_N) \in \pi_1(X, x)/N$. This construction associates to (σ_N) the compatible system $(\sigma_N(y_N))_N \in \pi_1(X, x)/N$, which can be seen to give an inverse

$$\widehat{\pi_1(X, x)} \to \pi_1^{fin}(X, x).$$

□

1.1.2 Étale coverings and fundamental groups

Proposition 1.3 contained a characterization of finite covering maps of manifolds as proper étale morphisms (i.e. proper local diffeomorphisms). Since tangent spaces can be defined in algebraic terms for varieties[4], this motivates the following definition.

Definition 1.6. *Let $f : Y \to X$ be a map between two smooth algebraic varieties. Then f is said to be* étale *if for every closed point $x \in X$ and every $y \in f^{-1}(x)$ the induced map of tangent spaces $df_x : T_yY \to T_xX$ is an isomorphism of vector spaces.*

It is important to note that the above definition is merely a characterization of étale maps for smooth varieties. A detailed discussion of general étale morphisms can be found in chapter 1.3 of [Mil80].

Also the notion of properness of a morphism is wonderfully captured by Grothendieck's approach to algebraic geometry (see chapter II.4 in [Har77]). Nonetheless it can be shown that for étale maps, properness is equivalent to the simpler notion of being *finite* (this is essentially exercise III.11.2 in [Har77]).

Definition 1.7. *A map between two affine varieties $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is called* finite*, if the induced map of rings $A \to B$ endows B with the structure of a finitely generated A-module. A map between two varieties $f : Y \to X$ is called* finite*, if*

[4]We define varieties to be separated schemes of finite type over a fixed base field. For algebraically closed fields it is possible to substitute the theory of schemes by one of the classical approaches to algebraic geometry.

there exists a covering $X = \bigcup_{i \in I} U_i$, s.t. each $f^{-1}(U_i)$ is affine, and the restriction $f : f^{-1}(U_i) \to U_i$ is finite.

Motivated by Theorem 1.5 we define the category $Cov^{et}(X)$ to be the category of (connected) finite étale covering spaces $\pi : Y \to X$ with morphisms being given by a finite étale map $Y \to Y'$ sitting in a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow & \downarrow \\ & & X. \end{array}$$

For every geometric point x of X, i.e. for every map $\operatorname{Spec} F^{sep} \to X$, where F^{sep} is a separably closed field, there is a fibre functor

$$\mathbf{Fib}_x^{et} : Cov^{et}(X) \to Set,$$

sending Y to the fibre $Hom_X(\operatorname{Spec} \bar{F}, Y)$.

Definition 1.8. *The étale fundamental group of a variety X at a geometric point x, is defined to be the group of natural self-transformations of the fibre functor $\mathbf{Fib}_x^{et}$, i.e.*

$$\pi_1^{et}(X, x) := Aut(\mathbf{Fib}_x^{et}).$$

For later use we record the following lemma, which will be useful in constructing representations of the étale fundamental group.

Lemma 1.9. *Let X be a variety and $\pi : Y \to X$ a finite étale covering. We say that π is regular (or Galois), if the action of $\pi_1^{et}(X, x)$ on $\mathbf{Fib}_x^{et}(Y)$ is transitive. Under these circumstances, the group of deck transformations $Aut(Y/X)$, i.e. the automorphism group of Y in the category $Cov^{et}(X)$, is a surjective image of $\pi_1^{et}(X, x)$.*

In case that X is a complex variety, and $x \in X(\mathbb{C})$ we would like to state a comparison theorem relating $\pi_1^{et}(X, x)$ with $\pi_1^{fin}(X, x)$. In order to acchieve this it suffices to construct a natural equivalence of categories

$$Cov^{fin}(X^a n) \cong Cov^{et}(X),$$

respecting fibre functors.

Theorem 1.10 (Riemann Existence Theorem)**.** *Let X be a complex variety, then there exists a canonical equivalence of finite étale coverings of X and finite coverings of X^{an}.*

A proof for non-projective X, avoiding resolution of singularities, is given in [GR58]. We restrict ourselves to some handwaving in the projective case, and invite the reader to find an alternative argument using Serre's GAGA theorem [Ser56].

Sketch for X projective. We have to show that a finite covering $\pi : Y^{an} \to X^{an}$ of a projective complex variety X^{an} is a projective complex variety itself. Let $\mathcal{L}$ be an ample line bundle on X^{an}, we would like to show that $\pi^* \mathcal{L}$ on Y^{an} is ample in the sense of complex analytic varieties. Using the cohomological criterion for ampleness, it suffices to show that $H^i(Y^{an}, \mathcal{F} \otimes \pi^* \mathcal{L}^m)$ vanishes for $m >> 0$, where $\mathcal{F}$ is an arbitrary coherent complex analytic sheaf on Y^{an}. Since π is a finite covering map, i.e. the fibres have no higher cohomology, this cohomology group is equivalent to $H^i(X^{an}, \pi_*(\mathcal{F} \otimes \pi^* \mathcal{L}^m))$. According to the projection formula, this agrees with $H^i(X^{an}, \pi_* \mathcal{F} \otimes \mathcal{L})$, and ampleness of $\mathcal{L}$ implies the requested vanishing for m big enough. □

Corollary 1.11 (Comparison theorem for π_1^{et})**.** *Let X be a complex variety and $x \in X(\mathbb{C})$ a $\mathbb{C}$-point, then there is a canonical equivalence*

$$\pi_1^{et}(X, x) \cong \pi_1 \widehat{(X^{an}, x)}.$$

Proof. The Riemann Existence Theorem 1.10 shows that there is an equivalence of categories $Cov^{et}(X, x) \cong Cov^{fin}(X^{an}, x)$, respecting fibre functors. In particular we obtain an equivalence of the groups of natural self-transformations

$$\pi_1^{et}(X, x) = Aut(\mathbf{Fib}_x^{et}) \cong Aut(\mathbf{Fib}_x^{fin}) = \pi_1^{fin}(X^{an}, x).$$

Since we have seen in Theorem 1.5 that $\pi_1^{fin}(X^{an}, x) \cong \pi_1 \widehat{(X^{an}, x)}$, finishing the proof of the theorem. □

1.1.3 The projective line

In this subsection we give a purely algebraic proof of the fact that $\pi_1^{et}(\mathbb{P}^1) = 1$ over an algebraically closed field k. This is equivalent to the statement below.

Proposition 1.12. *Let X be a smooth genus zero curve over an algebraically closed field k. Then every connected finite étale map $\pi : Y \to X$ is an isomorphism.*

The main step of the argument will be to show that Y has to be of genus zero. Topologically one would argue with the Euler characteristic of Y being $2 - 2g = 2n$, where n denotes the degree of the covering map. This implies $n = 1$ and $g = 0$.

Proof. For an arbitrary smooth curve C, the Euler characteristic of the sheaf T_C of tangent vector fields is given by $1-g$, where g denotes the genus. Definition 1.6 can be read as stating that the natural map $T_Y \to \pi^* T_X$ is an isomorphism. Therefore we obtain

$$\chi_Y(T_Y) = \chi(\pi_* T_Y) = \chi(T_X \otimes \pi_* \mathcal{O}_Y),$$

by virtue of the projection formula. The Riemann-Roch theorem implies

$$\chi(T_X \otimes \pi_* \mathcal{O}_Y) = n\chi(T_X) + \deg \pi_* \mathcal{O}_Y .$$

But since the same argument also works with the dual of the sheaves T_X and T_Y, we see that $\deg \pi_* \mathcal{O}_Y = 0$ with the help of Serre duality. As above we obtain $n = 1$ and $g(Y) = 0$.

There is a simple direct argument that such an étale morphism is an isomorphism. As we have seen above, π^* induces an isomorphism $\mathcal{P}\mathrm{ic}(X) \to \mathcal{P}\mathrm{ic}(Y)$. Picking a very ample line bundle (e.g. T_X) $\mathcal{L}$ on X, its pullback will again be very ample (T_Y in the example). Hence, $\pi : Y \to X$ is given by a grading preserving morphisms of graded rings

$$k[x,y] \to k[x',y'],$$

and thus is either constant or an isomorphism. □

1.1.4 Galois theory

While over algebraically closed fields $\bar{k}$ of arbitrary characteristic, the étale fundamental group behaves away from bad primes in analogy with what is known from complex geometry, a slightly different situation arises for non-algebraically closed fields k.

Lemma 1.13. *Let k be a field with separable closure k^{sep}, then*

$$\pi_1^{et}(\operatorname{Spec} k) \cong Gal(k^{sep}/k).$$

Proof. Proposition I.3.1 in [Mil80] implies that a connected finite étale covering of the spectrum of a field k is given by the spectrum of a finite separable field extension k'. The choice of a geometric base point for k is induced by fixing a separable closure k^{sep} of k. The fibre of k'/k at k^{sep} is given by the set of different embedding of k' into k^{sep}. The étale covering given by k'/k is regular if and only if it is Galois. Using the above insights it is easy to verify that $\pi_1^{et}(\operatorname{Spec} k)$ is given by $Gal(k^{sep}/k)$ endowed with the classical structure of a profinite group. □

In the second part of this talk the absolute Galois group of a finite field will play a special role.

Lemma 1.14. *Let $k = \mathbb{F}_q$ be a finite field, with algebraic closure $\bar{k}$. The Frobenius automorphism $x \mapsto x^q$ gives rise to an identification*

$$Gal(\bar{k}/k) \cong \widehat{\mathbb{Z}}.$$

Proof. Every finite field extension of k is Galois, and can be realized within $\bar{k}$ as the fixed-point set $\mathbb{F}_{q^d}$ of F^d. The Galois group $Gal(\mathbb{F}_{q^d}/\mathbb{F})$ is a cyclic group of order d, generated by F. Since $\mathbb{F}_{q^d} \subset \mathbb{F}_{q^{d'}}$, if and only if $d|d'$, we obtain the same pro-system of finite cyclic groups calculating the pro-finite completion of $\mathbb{Z}$. □

For a general scheme X over a field k one has a short exact sequence of groups, relating π_1^{et} with the absolute Galois group of k:

$$1 \to \pi_{et}^1(X \times_{\operatorname{Spec} k} \operatorname{Spec} k^{sep}) \to \pi_1^{et}(X) \to Gal(k^{sep}/k) \to 1.$$

In particular we conclude that the projective line over a finite field $\mathbb{P}^1_k$ has étale fundamental group isomorphic to $\widehat{\mathbb{Z}}$.

1.2 Sheaves

As we have just seen it is possible to recover a substantial part of the topological information contained in the fundamental group of an algebraic variety, by studying finite étale coverings. One might therefore hope for the existence of an *étale topology* on an algebraic variety X, e.g. given in form of a functorial assignment of a topological space X^{et} to X, s.t. $\pi_1(X^{et}) = \pi_1^{et}(X)$. If this was possible, we would also at once solve the problem of defining singular cohomology groups for X, since we could just calculate the cohomology of the constant sheaf $\underline{A}$ on X^{et} to compute singular cohomology with coefficients in A.

Nonetheless it seems unlikely that a topological space X^{et} in the traditional sense will be able to accomplish all of these tasks. Instead of generalizing the theory of topological spaces, we will try to guess how to modify the definition of Zariski sheaves by studying examples of étale sheaves provided by the theory of fundamental groups.

1.2.1 Local systems

Definition 1.15. *Let X be a topological space (sufficiently connected, as in the first subsection), a* local system *L on X is a locally constant sheaf of abelian groups, i.e. there exists an open covering (U_i) of X, s.t. each $L|_{U_i}$ is equivalent to a constant sheaf.*

There is an interesting link between locally constant sheaves L and covering spaces. For every sheaf in sets $\mathcal{F}$ on X there exists a universal local homeomorphism $\pi : Y_{\mathcal{F}} \to X$, s.t. $\mathcal{F}$ can be identified with the sheaf of sections of $Y_{\mathcal{F}}$. This construction is referred to as the étale space of the sheaf $\mathcal{F}$, and is given by appropriately topologizing the disjoint union of stalks $\coprod_{x \in X} \mathcal{F}_x$.

In case of a local system L the étale space Y_L can be seen to be a covering. The condition of L being locally constant translates directly into $Y_L \to X$ being locally trivial, since the étale space of the constant sheaf $\underline{A}$ is given by $X \times A$. This discussion reveals in particular that local systems on a simply connected space a trivial, which yields the following even more general description of locally constant sheaves.

Proposition 1.16. *Let L be a local system on X and $x \in X$ a base point. If $L_x \cong A$, A being a fixed abelian group, we call L a local system with fibre A, or an $Aut(A)$-local system. There is a natural equivalence of categories of $Aut(A)$-local systems and representations of the fundamental group*

$$\pi_1(X, x) \to Aut(A).$$

Proof. Let $\pi : \widetilde{X} \to X$ be a universal covering space for X, endowed with the canonical $\pi_1(X, x)$-action given by deck tranformations. Every local system L on X pulls back to a trivial local system π^*L on $\widetilde{X}$, since the universal covering space is simply connected. The pullback π^*L is endowed with a $\pi_1(X, x)$-equivariant structure, which is equivalent to giving a representation of $\pi_1(X, x) \to Aut(A)$, since π^*L is trivial. □

The representation of $\pi_1(X, x)$ on $Aut(A)$ can also be described in terms of monodromy. *Parallel transport* along closed paths representing elements in the fundamental group yields an automorphism of the stalk L_x.

The theory above motivates the following analogue of étale local systems.

Definition 1.17. *Let G be a topological group and X a variety with geometric base point x. An étale G-local system is a continuous representation*

$$\pi_1^{et}(X, x) \to G.$$

Of particular importance to us will be ℓ-adic local systems, which correspond to continuous representations taking values in $GL_n(\bar{\mathbb{Q}}_\ell)$. Nonetheless we will first study étale local systems with *discrete* structure groups to get a feeling for the theory. In this case it is easy to guess what the analogue of the étale space construction should be. Let therefore G be a finite discrete group, arising as the symmetry group of a mathematical object A (e.g. an abelian group). A continuous representation

$$\pi_1^{et}(X,x) \to G$$

then gives rise to an open kernel N (since $\{0\}$ is open in G), which induces a regular finite étale covering Y_N of X with deck transformation group

$$Aut(Y_N/X) \cong \pi_1^{et}(X,x)/N.$$

In the particularly interesting case where $G \cong Aut(Y_N/X)$, it seems plausible to call Y_N the étale space of the local system under consideration. In general, the definition as *twisted product*

$$Y_N \times^{Aut(Y_N/X)} A := (Y_N \times A)/Aut(Y_N/X)$$

seems justified.

1.2.2 Étale Sheaves

It is important to emphasize the analogy with the construction of topological local systems coming from representations of the fundamental group. There we noticed that every local system on X trivializes on the universal covering space and can therefore we described in terms of an equivariant structure on the trivial local system. The above story for étale local systems should thus be read as stating that every étale local system with discrete structure group trivializes along a finite étale covering. In light of Definition 1.15 this motivates the following notion of étale presheaves.

Definition 1.18. *An* étale presheaf $\mathcal{F}$ *(in abelian groups) is a rule that assigns to every étale map* $U \to X$ *an abelian group* $\mathcal{F}(U \to X)$ *(usually called* $\mathcal{F}(U)$ *to simplify notation), and to every commutative diagram of étale maps*

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X \end{array}$$

restriction morphisms $|_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$*, satisfying the rules* $|_U^U = \mathrm{id}_{\mathcal{F}(U)}$ *and* $|_V^W |_U^V = |_U^W$ *for commutative diagrams of étale maps*

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ & \searrow & & \searrow & \downarrow \\ & & & & X. \end{array}$$

We have certainly realized above that it is necessary to associate $\mathcal{F}(U)$ to every finite étale map, to cover the case of étale local system; and to every Zariski open subset, to include the traditional theory of sheaves in algebraic geometry. Allowing arbitrary étale maps $U \to X$ is a suitable compromise between the two, since the inclusion of a Zariski open subset is also étale.

Analogous to open covering in topology, an *étale covering* is a collection of étale morphisms $\{U_i \to X\}$, which are jointly surjective. We can now define étale sheaves in complete analogy with sheaves on topological spaces, replacing the intersection of two open subsets by the fibre product $U_{ij} := U_i \times_X U_j$.

Definition 1.19. *An* étale sheaf *is an étale presheaf, s.t. for every étale covering $\{U_i \to U\}$ of an étale map $U \to X$ the following is an equalizer diagram*

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{i,j}),$$

i.e. for every collection of $s_i \in \mathcal{F}(U_i)$ satisfying $s_i|^{U_i}_{U_{ij}} = s_j|^{U_j}_{U_{ij}}$ there exists a unique $s \in \mathcal{F}(U)$, s.t. $s|^U_{U_i} = s_i$.

In the theory of étale fundamental groups we have already seen that the role of base points is taken by geometric points $x : \operatorname{Spec} F^{sep} \to X$ in algebraic geometry. We will further emphasize this viewpoint by introducing the notion of stalks of an étale (pre)sheaf at a geometric point x. Recall that for classical sheaves $\mathcal{F}$ the stalk at a point $x \in X$ is defined to be the direct limit of $\mathcal{F}(U)$, where U is an open subset containing x.

Definition 1.20. *Let $\mathcal{F}$ be an étale presheaf on X and x a geometric point of X. The stalk $\mathcal{F}_x$ is defined to be the direct limit of $\mathcal{F}(U \to X)$ taken over the system of étale maps $U \to X$ sitting in a commutative diagram*

$$\begin{array}{ccc} x & \longrightarrow & U \\ & \searrow & \downarrow \\ & & X. \end{array}$$

1.3 Cohomology

The category of étale sheaves in abelian groups $Sh^{et}(X)$ can be shown to be abelian and having enough injectives (Proposition III.1.1 in [Mil80]). This allows us to apply the theory of (right) derived functors to the left exact *global section functor*, sending an étale sheaf $\mathcal{F}$ to the abelian group $\mathcal{F}(X)$. The obtained functors will be denoted by $H^i_{et}(X, \mathcal{F})$ and referred to as *étale cohomology groups* of $\mathcal{F}$.

Alternatively we can also apply a Cech-cohomology construction for étale covering $\{U_i\}$, yielding a slightly more restrictive theory (chapter III.2 in [Mil80]).

1.3.1 Singular cohomology with finite coefficients

The inability of algebraic geometry to account for possibly infinite covering spaces of a complex algebraic variety, already indicates that some care with the étale cohomology groups $H^i(X, \underline{A})$ for A being a general abelian group might be justified. Nonetheless, in the case of finite A we have the following remarkable comparison theorem.

Theorem 1.21. *Let X be a complex algebraic variety and A a finite abelian group. Then there is a natural equivalence $H^i(X^{an}, \underline{A}) \cong H^i_{et}(X, \underline{A})$ of cohomology groups.*

An outline of a proof is given in chapter III.3 of [Mil80]. As the comparison theorem for π_1^{et}, it is based on the Riemann Existence Theorem 1.10, but this time relying on its full strenght even for X projective. We invite the reader to try proving the theorem above for $i = 1$ using the comparison theorem for π_1^{et}.

1.3.2 ℓ-adic cohomology

Although the algebraic recovery of singular cohomology with torsion coefficients, as given by Theorem 1.21, is very exciting, our interest remains mainly algebraically retrieving the *Betti numbers* of X^{an}. For this reason we would like to find an étale analogue of the cohomology groups $H^i(X^{an}, K)$, where K is a field of characteristic zero. It turns out that this can be acchieved directly from the theory of torsion coefficients using a bit of prestidigitation.

Definition 1.22. *We formally define $H^i_{et}(X, \mathbb{Z}_\ell)$ to be inverse limit of the system of étale cohomology groups $H^i_{et}(X, \underline{\mathbb{Z}/\ell^n\mathbb{Z}})$. The ℓ-adic cohomology groups of X are defined to be*

$$H^i_{et}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Based on Theorem 1.21 we obtain

Theorem 1.23. *There is a canonical equivalence of ℓ-adic cohomology groups*

$$H^i(X^{an}, \underline{\mathbb{Q}_\ell}) \cong H^i_{et}(X, \mathbb{Q}_\ell).$$

Proof. The universal coefficent theorem implies that

$$H^i(X^{an}, \underline{A}) \cong H^i(X^{an}, \underline{\mathbb{Z}}) \otimes A \oplus Tor(H^{i+1}(X^{an}, \mathbb{Z}), A),$$

where the second part only depends on the torsion part of $H^{i+1}(X^{an}, \mathbb{Z})$, and can therefore be neglected from our analysis. Together with the comparison theorem for finite coefficient groups 1.21 we obtain that *up to torsion* we have

$$H^i_{et}(X, \mathbb{Z}_\ell) \cong H^i(X^{an}, \mathbb{Z}_\ell).$$

Tensoring by $\mathbb{Q}_\ell$ we conclude the proof of the assertion. □

1.3.3 Parental advisory disclaimer

It is important to know that many étale local systems actually don't give rise to étale sheaves as such, but rather to pro-systems of sheaves. Unless one is working with discrete finite structure groups, it is always necessary to complement the theory of étale sheaves with some abstract nonsense.

2 The function sheaf dictionary

In the section part of these notes we consider varieties X_0 over a *finite field*. It is not difficult to see that the purely algebraic concepts of étale fundamental groups, étale sheaves, and ℓ-adic cohomology are still well-defined in this geometrically exotic context.[5]

We denote by $\bar{k}$ a fixed algebraic closure of k, and by $\bar{X}$ the base change of X_0 to $\operatorname{Spec} \bar{k}$. Every element of $Gal(\bar{k}/k)$ induces a *scheme-theoretic automorphism* of X. This implies the existence of an interesting extra structure for the ℓ-adic cohomology, which is not present over the field of complex numbers: the action of the Galois group $\widehat{\mathbb{Z}} = Gal(\bar{k}/k)$ on $H^i_{et}(X, \mathbb{Q}_\ell)$. Since $\widehat{\mathbb{Z}}$ is topologically generated by the Frobenius automorphism $1 \in \widehat{\mathbb{Z}}$, it suffices to study the action of the Frobenius automorphism of an algebraic variety on ℓ-adic cohomology. If X is explicitly given by equations, F_x is the map of algebraic varieties given by raising the coordinates to the p^r-th power, where $p^r = |k|$.

Definition 2.1. *A Weil ℓ-adic local system on X_0 is an ℓ-adic local system L on X_0 together with an isomorphism*

$$F_X^* L \cong L.$$

[5]We emphasize that ℓ and $p := \operatorname{char} k$ are always assumed to be coprime.

2.1 Character sheaves on commutative algebraic groups

We refer the reader to Gaitsgory's chapter in [BCdS$^+$03]. Let $\mathcal{A}$ (resp. $\mathcal{A}_0$) be a commutative algebraic group scheme, defined over a finite field k, and let $A = \mathcal{A}(k)$ be the finite commutative group of k-points. The Frobenius morphism $F : \mathcal{A} \to \mathcal{A}$ can be shown to be an isomorphism of group schemes. The group A can be identified with the fixed-points of F, or alternatively with the zero fibre of the map

$$L : \mathcal{A} \to \mathcal{A},$$

which sends x to $Fx - x$. The map L is called the *Lang isogeny*, and can be shown to be a finite étale covering. In the special case of the additive group $\mathbb{G}_a$, the Lang isogeny is given by the Artin-Schreier map $\mathbb{A}^1 \to \mathbb{A}^1$, sending x to $x^p - x$.

Lemma 2.2. *The Lang isogeny L is a regular étale covering, with group of deck transformations canonically equivalent to A.*

Proof. Every non-trivial element of $\mathcal{A}$ induces a non-trivial action on $\mathcal{A}$ by translation. In particular we have a canonical action of A on $\mathcal{A}$, which by definition of L preserves the Lang isogeny. In particular we see that there is an injection

$$A \hookrightarrow Aut(L),$$

but since each fibre of L can be non-canonically identified with the kernel A of L, we conclude that A acts transitively on the fibres. This implies that L is a regular étale covering, and moreover that $A \cong Aut(L)$. □

This simple result, combined with Lemma 1.9, yields a construction of associating an Weil ℓ-adic local system on $\mathcal{A}$ (preserved by $F_{\mathcal{A}}$) to a character

$$\chi : A \to \bar{\mathbb{Q}}_\ell.$$

Lemma 2.3. *To every character $\chi : A \to \bar{\mathbb{Q}}_\ell$ we can naturally associate a Weil ℓ-adic local system L_χ on $\mathcal{A}$, satisfying $L_{\chi_1\chi_2} \cong L_{\chi_1} \otimes L_{\chi_2}$.*

Proof. Lemma 2.2 shows that the Lang isogeny $L : \mathcal{A} \to \mathcal{A}$ is a finite étale covering with group of deck transformations given by the finite commutative group A. In particular we have a surjection $\pi_1^{et}(\mathcal{A}, 0) \twoheadrightarrow A$ by Lemma 1.9. By composing with the character $\chi : A \to \bar{\mathbb{Q}}_\ell$ we obtain a continuous representation of the fundamental group, giving rise to an ℓ-adic local system L_χ. □

Applying this construction to the Artin-Schreier morphism, gives rise to interesting local systems on the affine line, usually referred to as *Artin-Schreier sheaves.*[6] Similarly, *Kummer sheaves* on the multiplicative group $\mathbb{G}_m$ and *Hecke eigensheaves* on Jacobians of curves, can be constructed.

2.1.1 Extracting a function out of a sheaf

Conversely to the process described in the proceeding subsection, we would like to associate a function on $X(k)$ to a Weil local system L on X. In order to do that we let $x : \operatorname{Spec} k \to X$ be a k-point of X. Pulling back L to x, we simply obtain a representation of $\pi_1^{et}(\operatorname{Spec} k) = Gal(\bar{k}/k) = \widehat{\mathbb{Z}}$, which is determined by the action of the Frobenius morphism. In analogy with the above character-theoretic construction, we therefore associate the trace of the Frobenius element on the stalk at x. The corresponding function will be denoted by

$$f_L : X(k) \to \bar{\mathbb{Q}}_\ell.$$

The lemma below follows directly from the definition of the Lang isogeny, and establishes a compatibility with the character sheaf construction of Lemma 2.3.

Lemma 2.4. *We have* $f_{L_\chi} = \chi$.

This is not the only convenient property of the function-sheaf correspondance.

Lemma 2.5. *The following properties hold for Weil local systems L_1, L_2 on X, and a map $\pi : Y \to X$:*

(a) $f_{L_1 \oplus L_2} = f_{L_1} + f_{L_2}$,

(b) $f_{L_1 \otimes L_2} = f_{L_1} \cdot f_{L_2}$,

(c) $f_{\pi^* L} = \pi^* f_L$.

The functor π^* has an adjoint given by push-forward π_*. At least for a proper morphism $Y \to X$ we would expect that $f_{\pi_* L}$ should be calculated by fibrewise (discrete) integration of the function f_L. In order for this to be true, it is necessary to generalize the definition of f to complexes of étale local systems and even more generally complexes of *constructible* ℓ-adic sheaves. There exists a triangulated category $D^b_{const}(X, \bar{\mathbb{Q}}_\ell)$, containing the categories of ℓ-adic local systems supported on an

[6]This statement is to be contrasted with the analogous situation over the complex numbers, where the affine line, due to its simply-connectedness, does not carry any interesting local systems. As we can see, $\mathbb{A}^1$ is not simply-connected in positive characteristic!

arbitrary closed subvarieties $Z \subset X$. Moreover, it contains complexes of ℓ-adic local systems and generalizations thereof. The functors π_*, π^*, $\otimes$ and $\oplus$ admit derived analogues, which in the particular important case of π_* compute fibrewise cohomology. An object $L^\bullet$ in $D^b_{const}(X, \bar{\mathbb{Q}}_\ell)$, fixed by F_X, gives now rise to a function on $X(k)$, by taking an alternating sum of traces

$$\sum_{i \in \mathbb{Z}} (-1)^i Tr(f^* : H^i(L^\bullet) \to H^i(L^\bullet)).$$

Theorem 2.6 (Grothendieck-Lefschetz)**.** *For a proper map $\pi : Y \to X$ and $L^\bullet \in D^b_{const}(X, \bar{\mathbb{Q}}_\ell)$ we have $f_{\pi_* L^\bullet} = \int_\pi f_{L^\bullet}$.*

The Grothendieck-Lefschetz theorem is a generalization of a classical theorem in algebraic topology, which is due to Lefschetz. In connection with its relevance to the Weil conjectures, it should be understood to be the main motivation to study étale analogues of topological invariants.

Theorem 2.7 (Lefschetz)**.** *Let X be a nice compact topological space, and $f : X \to X$ a map with only finitely many fixed-points. Then the number of fixed points equals the following alternativing sum of traces*

$$\sum_{i \in \mathbb{N}} (-1)^i Tr(f^* : H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})).$$

Naively applying an ℓ-adic analogue of the Lefschetz fixed-point Theorem to the Frobenius F_X of our variety X, we see that the number of k-points can be calculated as

$$\sum_{i \in \mathbb{N}} (-1)^i Tr(F_X^* : H^i(X, \bar{\mathbb{Q}}_\ell) \to H^i(X, \bar{\mathbb{Q}})_\ell),$$

which agrees with $f_{\pi_* \bar{\mathbb{Q}}_\ell}$, where $\pi : X \to \operatorname{Spec} k$. In general we should think of functions obtained by fibrewise integration as giving rise to a *twisted count of rational points.*

References

[BCdS+03] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla, *An introduction to the Langlands program*, Birkhäuser Boston Inc., Boston, MA, 2003, Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001, Edited by Joseph Bernstein and Stephen Gelbart. MR 1990371 (2004g:11037)

[GR58] Hans Grauert and Reinhold Remmert, *Komplexe Räume*, Math. Ann. **136** (1958), 245–318. MR 0103285 (21 #2063)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR 559531 (81j:14002)

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42. MR 0082175 (18,511a)