Connected Shimura Varieties and Shimura Varieties

As before, $\mathbf{S}$ and $\mathbf{S}^1$ will denote the real algebraic groups for which $\mathbf{S}(\mathbf{R}) = \mathbf{C}^\times$ ($\mathbf{S}$ is sometimes called the Deligne circle group) and $\mathbf{S}^1(\mathbf{R}) = \{z\colon |z| = 1\}$, respectively (the usual circle group).

1. Connected Shimura Data and Connected Shimura Varieties

1.1 Congruence subgroups

1.2 Connected Shimura datum

Definition 1.1 (Connected Shimura datum). A **connected Shimura datum** is a pair $(\mathbf{G}, X^+)$ where $\mathbf{G}$ is a semi-simple algebraic group over $\mathbf{Q}$ and X^+ is a $\mathbf{G}^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class of morphisms $h\colon\ \mathbf{S}^1_{\mathbf{R}} \to \mathbf{G}^{\mathrm{ad}}$ satisfying the following axioms:

SV1: $\forall h \in X^+$, only $\{z/\overline{z}, 1, \overline{z}/z\}$ occur in the representation of $\mathbf{S}^1(\mathbf{R})$ on $\mathrm{Lie}(\mathbf{G}^{\mathrm{ad}}(\mathbf{R}))_{\mathbf{C}}$.

SV2: The involution $\mathrm{Ad}h(i)$ is a Cartan involution.

SV3: $\mathbf{G}^{\mathrm{ad}}$ has no simple factor on which the projection of h is trivial.

As a baby case, we consider the classical modular curve:

Example 1.

1.3 Connected Shimura varieties

Definition 1.2 (Connected Shimura variety). A connected Shimura variety is the inverse system of locally symmetric varieties $(D(\Gamma))_\Gamma$ where Γ runs over the arithmetic subgroups of $\mathbf{G}^{\mathrm{ad}}(\mathbf{Q})^+$ whose inverse image in $\mathbf{G}(\mathbf{Q})^+$ is a congruence subgroup.

1.4 Shimura data of Hodge type

A Shimura datum $(\mathbf{G}, X)$ is of *Hodge type* if there exists a symplectic space (V, ω) and a closed embedding $\mathbf{G} \hookrightarrow \mathbf{GSp}(V)$ that carries $X \hookrightarrow X(V, \omega)$. Here, $X(V, \omega)$ is the space of Hodge structures of type $\{(-1, 0), (0, -1)\}$ on V that are $\pm$-polarized by ω. Below, we will see examples of Shimura data for unitary groups that are Shimura data of Hodge type.

1.5 Shimura data of abelian type

Dimitar : **Mention Deligne's classification of Shimura data of abelian type.**

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4. Abelian varieties and Hodge structures

Let $\mathbf{AV}$ be the category of abelian varieties over $\mathbf{C}$ and let $\mathbf{IPHS}_{\{(-1,0),(0,-1)\}}$ be the category of integral, polarizable Hodge structures of type $\{(-1,0),(0,-1)\}$. Moreover, let $\mathbf{AV}^0$ be the category whose objects are exactly the same as those of $\mathbf{AV}$, but whose morphisms are $\mathrm{Hom}_{\mathbf{AV}^0}(A,B) = \mathrm{Hom}_{\mathbf{AV}}(A,B)$ (i.e., $\mathbf{AV}^0$ is the *up-to-isogeny* category). Similarly, $\mathbf{RPHS}_{\{(-1,0),(0,-1)\}}$ will denote the category of rational polarized Hodge structures.

PROPOSITION 4.1. *(i) The functor $A \mapsto \mathrm{H}^1(A,\mathbf{Z})$ is an equivalence of categories*

$$\mathbf{AV} \xrightarrow{\sim} \mathbf{IPHS}_{\{(-1,0),(0,-1)\}}.$$

(ii) The functor $A \mapsto \mathrm{H}^1(A,\mathbf{Q})$ is an equivalence of categories

$$\mathbf{AV}^0 \xrightarrow{\sim} \mathbf{RPHS}_{\{(-1,0),(0,-1)\}}.$$

Proof. In (i), to give the inverse functor, let $V_{\mathbf{Z}}$ be an object of $\mathbf{IPHS}_{\{(-1,0),(0,-1)\}}$. Then $V_{\mathbf{Z}}$ comes equipped with a polarization $\psi\colon V_{\mathbf{Z}} \times V_{\mathbf{Z}} \to \mathbf{Z}(-1)$ giving rise to a Hermitian form $\langle v,w\rangle = \psi_{\mathbf{C}}(v,iw) + i\psi_{\mathbf{C}}(v,w)$ on $V_{\mathbf{C}}$. This gives a Riemann form on the complex torus $V^{-1,0}/V_{\mathbf{Z}}$ and hence, a complex abelian variety. Here, we use that if $A = \mathbf{C}^g/\Lambda$ then $\mathrm{H}_1(A,\mathbf{Z}) \cong \Lambda$ and the correspondence between complex structures on a real vector space V and Hodge structures on V of type $\{(-1,0),(0,-1)\}$ is given by $(V,J) \cong V(\mathbf{C})/\operatorname{Fil}^0 V(\mathbf{C})$. □

5. Examples

5.1 The Siegel modular variety

5.1.1 *The domain and the Siegel upper-half space* Here, let (V,ω) be a symplectic space over $\mathbf{R}$ (i.e., V is an even-dimensional vector space and $\omega\colon V \times V \to \mathbf{R}$ is a non-degenerate symplectic form). Let J be a complex structure on V such that $\omega(Jv,Jw) = \omega(v,w)$. Equivalently, we have chosen a Hodge structure on V of type $\{(-1,0),(0,-1)\}$. The structure J yields a morphism of real algebraic groups

$$h\colon \mathbf{S} \to \mathbf{GL}(V_{\mathbf{R}}), \qquad h(a+ib)\colon v \in V_{\mathbf{R}} \mapsto (a+bJ)v \in V_{\mathbf{R}}.$$

LEMMA 5.1. *The morphism h factors through $\mathbf{GSp}(V)$.*

Proof. It suffices to show that for every $z = a+ib \in \mathbf{C}^\times$, $h(z)\colon V_{\mathbf{R}} \to V_{\mathbf{R}}$ is a symplectic similitude. We calculate

$$\omega(h(z)v,h(z)w) = \omega(av,aw) + \omega(bJv,aw) + \omega(av,bJw) + \omega(bJv,bJw).$$

Since

$$\omega(bJv,aw) = ab\omega(Jv,w) = ab\omega(J^2v,Jw) = -ab\omega(v,Jw) = -\omega(av,bJw),$$

we get $\omega(h(z)v,h(z)w) = |z|^2\omega(v,w)$, i.e., the $h(z)$ is a symplectic similitude with factor $|z|^2$. □

Let $X(V,\omega)$ be the $\mathbf{GSp}(V)(\mathbf{R})$-conjugacy class of h. Then $X = \mathcal{H}_g^+ \sqcup \mathcal{H}_g^-$.

5.1.2 *What does this mean in terms of Hodge structures.*

5.1.3 *Verification of the axioms.*

5.1.4 *Moduli interpretation in terms of rational polarized Hodge structures.* Let $\mathbf{G} = \mathbf{GSp}(V)$. If $K \subset \mathbf{GSp}(V)(\mathbf{A}_f)$ then let $\mathcal{H}_K$ be the set of triples $\{(W, h), s, \eta K\}$ where

- (W, h) is a rational Hodge structure of type $\{(-1, 0), (0, -1)\}$,
- $\pm\lambda$ is a polarization for (W, h),
- ηK is a K-orbit of $\mathbf{A}_f$-isomorphisms $\eta\colon W \otimes \mathbf{A}_f \to V \otimes \mathbf{A}_f$ sending ω to an $\mathbf{A}_f^\times$ multiple of s.

Two triples $((W, h), \pm\lambda, \eta K)$ and $((W', h'), \pm\lambda', \eta' K)$ are said to be isomorphic if there exists an isomorphism of Hodge structures $f\colon (W, h) \to (W', h')$ such that $f(\lambda)$ is a $\mathbf{Q}^\times$-multiple of λ' and $f \circ \eta = \eta' \bmod K$.

There is a map $\mathcal{H}_K \to \mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}_f) \times X(V, \omega)/K$ defined as follows: suppose that (W, h) is a rational Hodge structure and λ is a polarization for (W, h). Since η is an isomorphism, $\dim W = \dim V$ and hence, there is an isomorphism $a\colon W \to V$ sending λ to ψ. Fix such an isomorphism for the particular rational polarized Hodge structure and notice that

$$g = a \circ \eta\colon V_{\mathbf{A}_f} \xrightarrow{\eta} W_{\mathbf{A}_f} \xrightarrow{a} V_{\mathbf{A}_f}$$

is a symplectic similitude for $V_{\mathbf{A}_f}$ and hence, an element of $\mathbf{G}(\mathbf{A}_f)$. Moreover ah yields a rational Hodge structure on V and hence, an element of $X = X(V, \omega)$. Then consider $[g, ah] \in \mathrm{Sh}_K(\mathbf{G}, X)$ and note that it is independent of the choice of a (a is defined up to an element of $\mathbf{G}(\mathbf{Q})$). It is not hard to check that this map yields a bijection

$$\mathcal{H}_{K/\cong} \to \mathrm{Sh}_K(\mathbf{G}, X) = \mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}_f) \times X/K.$$

5.1.5 *Moduli interpretation in terms of polarized abelian varieties.* Using the moduli interpretation in terms of rational polarized Hodge structures as well as the equivalence of the categories $\mathbf{AV}^0$ and $\mathbf{RPHS}_{\{(-1,0),(0,-1)\}}$, we can establish the following:

Theorem 5.2. *The Siegel modular variety* $\mathrm{Sh}_K(\mathbf{G}, X)$ *classifies isomorphism classes of triples of the form* $(A, \lambda, \eta K)$ *where*

- *A is an abelian variety over $\mathbf{C}$ up to isogeny (i.e., an object in $\mathbf{AV}^0$),*
- *$\pm\lambda$ is a polarization on A*
- *ηK is a K-orbit of $\mathbf{A}_f$-linear isomorphisms*

$$\eta\colon V \otimes \mathbf{A}_f \to V_f(A)$$

that carry the symplectic form ψ onto an $\mathbf{A}_f$-multiple of λ

Proof. □

Dimitar : Explain why working with rational polarized Hodge structures.

5.2 Unitary Shimura varieties

5.2.6 *The domains.* Let $\mathcal{K} = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field for some $D < 0$ and let $(V_0, \langle\,,\rangle)$ be a Hermitian $\mathcal{K}$-vector space. Let V denote the underlying $\mathbf{Q}$-vector space and let $\omega\colon V \times V \to \mathbf{Q}$ be the (alternating) bilinear form obtained by projecting $\langle\,,\rangle$ to the $\sqrt{-D}$-component of E, i.e., we write $\omega(v, w) = \mathrm{im}\langle v, w\rangle$.

Exercise 1. Check that ω defined as above is an alternating form.

The idea that unitary Shimura varieties are Shimura varieties of Hodge type is built upon the fact that one associated the symplectic space (V, ω) to the Hermitian space $(V_0, \langle\,,\rangle)$. We thus get an embedding of algebraic groups $\mathbf{GU}(V_0) \hookrightarrow \mathbf{GSp}(V)$. Indeed, if $g \in \mathbf{GU}(V_0)(k)$ for a $\mathbf{Q}$-algebra k then

$$\omega(gv, gw) = \mathrm{im}\langle gu, gv\rangle = \mathrm{im}\langle u, v\rangle = \omega(v, w).$$

5.2.7 *Moduli interpretation.* Consider quadruples $(A, \lambda, \iota, \eta K)$ where

- A is an abelian variety that is an object of $\mathbf{AV}^0$,
- $\lambda\colon A \to A^\vee$ is a polarization,
- $\iota\colon \mathcal{K} \to \mathrm{End}(A)$ is an action of $\mathcal{K}$ up to isogeny (explain what this means)
- ηK is a K-orbit of $\mathcal{K}$-linear isomorphisms

$$\eta\colon V \otimes \mathbf{A}_f \to \widehat{V}(A).$$

In addition, we require that

i) η carries ω to a $\mathbf{A}_f^\times$-multiple of λ.

ii) There exists a $\mathcal{K}$-linear isomorphism $a\colon V \to \mathrm{H}_1(A; \mathbf{Q})$ carrying ω to a $\mathbf{Q}^\times$-multiple of λ.

We decree that two quadruples $(A, \lambda, \iota, \eta K)$ and $(A', \lambda', \iota', \eta' K)$ are isomorphic if there exists a $\mathcal{K}$-linear isogeny $\varphi\colon A \to A'$ such that carries λ to a $\mathbf{Q}^\times$-multiple of λ' as well as ηK to $\eta' K$.

Proposition 5.3. *The Shimura variety* $\mathrm{Sh}_K(\mathbf{GU}(V_0), X)$ *classifies isomorphism classes of quadruples* $(A, \lambda, \iota, \eta)$ *as above.*