On the Harish-Chandra Embedding

The purpose of this note is to link the Cartan and the root decompositions. In addition, it explains how we can view a Hermitian symmetric domain as $G_{\mathbf{C}}/P$ where P is a certain parabolic subgroup (more precisely, $P = K_{\mathbf{C}} \cdot P_-$ in the notation of the previous document).

1. Basic Notions from Lie Theory

1.1 Simple and semi-simple Lie algebras

A *simple Lie algebra* is a Lie algebra whose only ideals are 0 and itself. A semi-simple Lie algebra is a Lie algebra that is a direct sum of simple Lie algebras.

1.2 The killing form

Given a real Lie algebra $\mathfrak{g}_{\mathbf{R}}$ the Killing form on $\mathfrak{g}_{\mathbf{R}} \times \mathfrak{g}_{\mathbf{R}}$ is defined by

$$B(X, Y) = -\operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y) \in \mathbf{R}$$

Dimitar : **Define the killing form.**

Dimitar : **Maybe give some examples with $\mathbf{SL}_2(\mathbf{R})$.**

1.3 Cartan subalgebras

A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent Lie subalgebra that is equal to its centralizer, i.e., such that $\{X \in \mathfrak{g} \colon [X, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}$. For semi-simple Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ being Cartan is equivalent to $\mathfrak{h}$ being a maximal abelian subalgebra.

Example 1. We will keep in mind the example of $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$ and $\mathfrak{h} = \left\{ \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \colon t \in \mathbf{R} \right\}$. Note that $\mathfrak{h}$ is also the compact Lie algebra $\mathfrak{so}_2(\mathbf{R})$. **Dimitar** : **Describe the killing form, etc.**

1.4 Root decomposition

If $\mathfrak{g}_{\mathbf{C}}$ is an arbitrary semi-simple Lie algebra and $\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ is a Cartan subalgebra (one can show that every semi-simple Lie algebra contains a Cartan subalgebra [Hel78, Thm.III.4.1]), consider the linear subspace $\mathfrak{g}^{\alpha} \subset \mathfrak{g}_{\mathbf{C}}$ associated to a functional $\alpha \in \mathfrak{h}_{\mathbf{C}}^*$ and defined by

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} \colon [H, X] = \alpha(H)X, \ \forall H \in \mathfrak{h}_{\mathbf{C}}\}.$$

The functional $\alpha \in \mathfrak{h}_{\mathbf{C}}^*$ is called a root if $\mathfrak{g}^{\alpha} \neq 0$. If this is the case then $\mathfrak{g}^{\alpha}$ is called a root subspace. Let Δ be the set of roots. Associated to the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, there is a root decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$$

satisfying the following properties:

i) $\dim_{\mathbf{C}} \mathfrak{g}^{\alpha} = 1$,

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ii) If $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$ then $B(\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}) = 0$,

iii) The restriction of B to $\mathfrak{h}_{\mathbf{C}} \times \mathfrak{h}_{\mathbf{C}}$ is non-degenerate, i.e., for each root α, there is a unique element $H_\alpha \in \mathfrak{h}_{\mathbf{C}}$ such that $\alpha(H) = B(H, H_\alpha)$,

iv) For each $\alpha \in \Delta$, $-\alpha \in \Delta$ and $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}] = \mathbf{C}H_\alpha$.

Example 2. Consider the Cartan subalgebra $\mathfrak{h} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbf{R} \right\} \subset \mathfrak{sl}_2(\mathbf{R})$. Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The two roots for this subalgebra are $2, -2$ and the root spaces are generated by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In fact, we have $[H, X] = 2X$ and $[H, Y] = -2Y$. As we will see later, this will give us an identification of $\mathbf{SL}_2(\mathbf{R})/A \cong \mathbf{SL}_2(\mathbf{C})/P$ where $P \subset \mathbf{SL}_2(\mathbf{C})$ which is the Borel–Weil theorem.

Example 3. We can compute the root decomposition for the pair $(\mathfrak{sl}_2(\mathbf{R}), \mathfrak{so}_2(\mathbf{R}))$ arising from identifying the Poincaré upper-half plane with $\mathbf{SL}_2(\mathbf{R})/\mathbf{SO}_2(\mathbf{R})$ (by looking at the stabilizer of $i \in \mathcal{H}_1$). In this case the roots are $\alpha_\pm = \pm 2i$. As we will see later, this Cartan decomposition will be useful to identify $\mathbf{SL}_2(\mathbf{R})/\mathbf{SO}_2(\mathbf{R})$ with $\mathbf{SL}_2(\mathbf{C})/P$ (to give us the Harish-Chandra embedding).

2. Harish–Chandra Embedding

Dimitar : Assume that the domain is irreducible, i.e., $\mathfrak{g}_{\mathbf{C}}$ is simple (the latter means that it has non-trivial ideals). All of our analysis should go under this assumption.

We assume that the Hermitian symmetric domain D is irreducible. On the level of Lie algebras (or, the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$), this means that following:

- $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$ different from $\{0\}$,
- $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

Lemma 2.1. *Condition (ii) is equivalent to $\mathfrak{k}$ being a maximal proper subalgebra of $\mathfrak{g}$.*

Proof. If $\mathfrak{k}$ is not a maximal proper subalgebra then there exists a proper subalgebra $\mathfrak{k} \subsetneq \mathfrak{k}^* \subsetneq \mathfrak{g}$. But we can show that if $\mathfrak{p}^* = \mathfrak{k}^* \cap \mathfrak{p}$ then $\mathfrak{p}^*$ is an invariant subspace for the action of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ and hence, $\mathfrak{p}^* = \{0\}$ or $\mathfrak{p}^* = \mathfrak{p}$. The case $\mathfrak{p}^* = \{0\}$ yields a contradiction whereas if $\mathfrak{p}^* = \mathfrak{p}$ then $\mathfrak{k}^* = \mathfrak{g}$, a contradiction as well. The converse is easy: if $\mathfrak{p}^* \subset \mathfrak{p}$ is a proper $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$-invariant subspace then $\mathfrak{k} + \mathfrak{p}^* \subset \mathfrak{g}$ is a proper subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$. □

2.1 Maximal abelian subalgebra of $\mathfrak{k}$

Let $\mathfrak{h} \subset \mathfrak{k}$ be a maximal abelian subalgebra and let $\mathfrak{h}_{\mathbf{C}}$ be its complexification. Let $\mathfrak{c} \subset \mathfrak{k}$ be the center of $\mathfrak{k}$. Since $\mathfrak{c}$ is the center of $\mathfrak{k}$ and $\mathfrak{h}$ is maximal abelian subalgebra of $\mathfrak{k}$ then $\mathfrak{c} \subset \mathfrak{h}$. Consider the centralizer $C_{\mathfrak{g}}(\mathfrak{c})$. Since $\mathfrak{k} \subset C_{\mathfrak{g}}(\mathfrak{c})$ **Dimitar : Why?** and $C_{\mathfrak{g}}(\mathfrak{c})$ is a proper subalgebra of $\mathfrak{g}$ then it follows that $C_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. We will use this to show that $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$.

Lemma 2.2. *$\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ is a Cartan subalgebra.*

Proof. Consider the centralizer $C_{\mathfrak{g}_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}}) = \{X \in \mathfrak{g}_{\mathbf{C}} \colon [X, \mathfrak{g}_{\mathbf{C}}] \subset \mathfrak{h}_{\mathbf{C}}\}$. Since $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbf{C}}$, we have $C_{\mathfrak{k}_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$. If $C_{\mathfrak{k}_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}}) \subsetneq C_{\mathfrak{g}_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}})$ then there exists an element $X \in \mathfrak{p}$ such that X normalizes $\mathfrak{h}_{\mathbf{C}}$. But the latter is impossible since $[X, \mathfrak{h}_{\mathbf{C}}] \subset \mathfrak{p}_{\mathbf{C}}$. Hence, $C_{\mathfrak{k}_{\mathbf{C}}}(\mathfrak{h}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ is a Cartan subalgebra. □

2.2 Roots of compact and non-compact type and centralizers

Given the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, let $\mathfrak{h} \subset \mathfrak{k}$ be the maximal abelian subalgebra. We call a root α compact if $\mathfrak{g}^\alpha \subset \mathfrak{k}$ and non-compact if $\mathfrak{g}^\alpha \subset \mathfrak{p}_{\mathbf{C}}$. Let Q_+ be the set of positive roots that do not vanish identically on $\mathfrak{c}_{\mathbf{C}}$, i.e., for which $\mathrm{ad}_{\mathfrak{c}_{\mathbf{C}}}(\mathfrak{g}^\alpha) = [\mathfrak{c}_{\mathbf{C}}, \mathfrak{g}^\alpha] \neq 0$. We now consider two subspaces of $\mathfrak{p}_{\mathbf{C}}$:

$$\mathfrak{p}_+ = \sum_{\alpha \in Q_+} \mathfrak{g}^\alpha, \qquad \mathfrak{p}_- = \sum_{-\alpha \in Q_+} \mathfrak{g}^\alpha.$$

These subspaces decompose $\mathfrak{p}_{\mathbf{C}}$ as follows

Lemma 2.3. *We have* $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$, $[\mathfrak{k}, \mathfrak{p}_-] \subset \mathfrak{p}_-$ *and* $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

Proof. Clearly, if α is compact then α vanishes on $\mathfrak{c}$ so we get $\mathfrak{p}^+ + \mathfrak{p}^- \subset \mathfrak{p}_{\mathbf{C}}$. To see that $\mathfrak{p}_+$ is abelian, note that if $\alpha, \beta \in Q_+$ then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ and if $\alpha + \beta$ is a root then $\alpha + \beta \in Q_+$. But $[\mathfrak{p}_+, \mathfrak{p}_+] \in \mathfrak{k}$, so it follows that $[\mathfrak{p}_+, \mathfrak{p}_+] = 0$.

What is tricky to show is that $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{p}_+ + \mathfrak{p}_-$ in $\mathfrak{p}_{\mathbf{C}}$ with respect to the killing form. Define

$$\mathfrak{g}_+ = \mathfrak{p}_+ + \mathfrak{p}_- + [\mathfrak{p}_+, \mathfrak{p}_-].$$

One can check that $\mathfrak{g}_+ \subset \mathfrak{g}_{\mathbf{C}}$ is an ideal **Dimitar : Do the computation!** and since $\mathfrak{g}_{\mathbf{C}}$ is simple, it follows that $\mathfrak{g}_+ = \{0\}$ or $\mathfrak{g}_{\mathbf{C}}$. The first case is impossible (as all the roots will be compact), so we are in the second case and $\mathfrak{p}_+ + \mathfrak{p}_- = \mathfrak{p}_{\mathbf{C}}$. □

2.3 More on semi-simple Lie algebras

Consider a semi-simple Lie algebra $\mathfrak{g}_{\mathbf{C}}$ over $\mathbf{C}$. Consider also

- $\mathfrak{g}^{\mathbf{R}}$ - the Lie algebra $\mathbf{C}$ viewed as a real Lie algebra,
- G_c - any connected real Lie group with Lie algebra $\mathfrak{g}_{\mathbf{R}}$,
- $\mathfrak{u} \subset \mathfrak{g}_{\mathbf{C}}$ - a compact real form of $\mathfrak{g}_{\mathbf{C}}$,
- $\mathfrak{a} \subset \mathfrak{u}$ - a maximal abelian subalgebra,
- $\mathfrak{h}_{\mathbf{C}} = \mathfrak{a} + i\mathfrak{a}$ - a (one has to prove this) Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$.
- $\mathfrak{n}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ - the positive nilpotent (with respect to some ordering of the roots) considered as a real Lie algebra.
- J - a complex structure on $\mathfrak{g}^{\mathbf{R}}$ that coincides with i on $\mathfrak{g}_{\mathbf{C}}$.

Then (one has to prove this - see [Hel78, Thm.VI.6.3]) there is a direct sum decomposition

$$\mathfrak{g}^{\mathbf{R}} = \mathfrak{u} \oplus J\mathfrak{a} \oplus \mathfrak{n}_+.$$

Furthermore, if U, A^*, N_+ are the analytic subgroups of G_c with Lie algebras $\mathfrak{u}$, $\mathfrak{a}^* = J\mathfrak{a}$ and $\mathfrak{n}_+$, respectively, then the multiplication map gives an analytic diffeomorphism

$$U \times A^* \times N_+ \to G_c.$$

2.4 The parabolic subgroups

$\mathbf{G}(\mathbf{R})/K$ embeds into P_- via the following sequence of steps

- One needs to show that the multiplication map $P_- \times K_{\mathbf{C}} \times P_+ \to G_{\mathbf{C}}$ induces a diffeomorphism between $P_- \times K_{\mathbf{C}} \times P_+$ and an open submanifold of $G_{\mathbf{C}} = \mathbf{G}(\mathbf{C})$ that contains $G = \mathbf{G}(\mathbf{R})$.
- One identifies $\mathbf{G}(\mathbf{R})/K$ diffeomorphically with $\mathbf{G}(\mathbf{R})K_{\mathbf{C}}P_+/K_{\mathbf{C}}P_+$. For this, we need to know that $K_{\mathbf{C}}P_+ \cap \mathbf{G}(\mathbf{R}) = K$. Once this is proved, consider the map $gK_{\mathbf{C}}P_+ \mapsto gK$ for any

$g \in \mathbf{G}(\mathbf{R})$. It follows that this map is well-defined (if $g' K_{\mathbf{C}} P_+ = g'' K_{\mathbf{C}} P_+$ then $g' k p_+ = g''$ and hence, $k p_+ = (g')^{-1} g'' \in \mathbf{G}(\mathbf{R})$, hence $K_{\mathbf{C}} P_+ \cap \mathbf{G}(R) = K$) and a bijection.

LEMMA 2.4. *The multiplication map $P_- \times K_{\mathbf{C}} \times P_+ \to G_{\mathbf{C}}$ induces a diffeomorphism between $P_- \times K_{\mathbf{C}} \times P_+$ and an open submanifold of $G_{\mathbf{C}} = \mathbf{G}(\mathbf{C})$ that contains $G = \mathbf{G}(\mathbf{R})$.*

Proof. To show that the map is an injection, suppose that $q_1 k_1 p_1 = q_2 k_2 p_2$, i.e.,

$$q_2^{-1} q_1 = k_2 p_2 p_1^{-1} k_1^{-1}.$$

The right-hand side is $k_2 k_1^{-1} (k_1 p_2 p_1^{-1} k_1^{-1})$ and $k_1 p_2 p_1^{-1} k_1^{-1} \in P_+$ since $K_{\mathbf{C}}$ normalizes P_+. Hence, it suffices to show that $P_- K_{\mathbf{C}} \cap P_+ = \{e\}$ to get injectivity. The latter can be seen on the level of the Lie algebra as follows: suppose that $y \in P_- K_{\mathbf{C}} \cap P_+$ and let $Y \in \mathfrak{p}_+$ such that $\exp(Y) = y$. One one hand, we can write $Y = \sum_{\alpha \in Q_+} c_\alpha X_\alpha$. Since $[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{p}_-$, it follow that $\mathrm{ad}(y) \colon \mathfrak{p}_- \to \mathfrak{p}_-$. Let $\mathfrak{n}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ and $\mathfrak{n}_- = \sum_{-\alpha \in \Delta^+} \mathfrak{g}^\alpha$ (these are nilpotent Lie algebras whose Lie groups are unipotent groups **Dimitar** : **May need to be more precise about the latter.**). Let $\beta \in Q_+$ be the lowest root for which $c_\beta \neq 0$. After calculating

$$[Y, X_{-\beta}] \equiv c_\beta [X_\beta, X_{-\beta}] \bmod \mathfrak{n}_+,$$

we observe that it cannot be $0 \bmod \mathfrak{n}_+ + \mathfrak{n}_-$ and hence, $\mathrm{ad}(Y)(X_{-\beta}) \notin \mathfrak{p}_-$, a contradiction. Next, a dimension count shows that the image of the map is an open submanifold of $G_{\mathbf{C}}$. Finally, we want to show that any $p \in P = \exp(\mathfrak{p})$ is in the image (which, together with $G = KP = PK$ will imply that $G = \mathbf{G}(\mathbf{R})$ is contained in the image). □

Example 4. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{R})$, one can compute explicitly P_+ and P_- using the two non-compact root spaces. Since $\alpha = 2i$ is the positive root then the Lie algebra $\mathfrak{p}_-$ is simply the root space $\mathfrak{g}^{-\alpha}$. The embedding that we get is then the Cayley transform. **Dimitar** : **Complete the example.**

2.5 The Lie algebra of $\mathbf{SL}(2, \mathbf{C})$

The Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ of $\mathbf{SL}_2(\mathbf{C})$ consists of all trace-zero matrices in $M_2(\mathbf{C})$. A basis is given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They satisfy the relations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. The key point is that it is possible, given any $Z \in \mathbf{C}(X + Y)$, to decompose $\exp(Z)$ in the form $P_- K_{\mathbf{C}} P_+$ via the following lemma:

LEMMA 2.5. *Given any $Z = t(X + Y)$ then we have*

$$\exp t(X + Y) = \exp(\tanh tY) \exp(\log(\cosh t) H) \exp(\tanh X).$$

Proof. □

2.6 Boundedness

- Define a maximal abelian subalgebra $\mathfrak{a}_{\mathbf{C}} \subset \mathfrak{p}_{\mathbf{C}}$ as

$$\mathfrak{a}_{\mathbf{C}} = \sum_{i=1}^{s} \mathbf{C}(X_{\gamma_i} + X_{-\gamma_i}) \subset \mathfrak{p}_{\mathbf{C}}.$$

  Here, the vectors $X_\alpha \in \mathfrak{g}^\alpha$ are chosen such that

$$X_\alpha - X_{-\alpha}, \qquad i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}_{\mathbf{C}} := \mathfrak{k} \oplus i\mathfrak{p}, \qquad [X_\alpha, X_{-\alpha}] = 2/\alpha(H_\alpha) H_\alpha.$$

the latter being the compact form of $\mathfrak{g}_{\mathbf{C}}$.

- If the X_α's are chosen as above then
$$\mathfrak{a} = \sum_{i=1}^{s} \mathbf{R}(X_{\gamma_i} + X_{-\gamma_i}) \subset \mathfrak{p}.$$
- Suppose that $Z \in \mathfrak{a}$ and write
$$Z = \sum_{i=1}^{s} t_i(X_{\gamma_i} + X_{-\gamma_i}).$$
Using some nested commutator relations (à la Campbell–Hausdorff formula), one can show that
$$\exp(Z) = \exp(Y)\exp(H)\exp(X), \tag{1}$$
where $Y = \sum_{i=1}^{s}(\tanh t_i)X_{-\gamma_i}$, $H = \sum_{i=1}^{s}\log(\cosh t_i)[X_{\gamma_i}, X_{-\gamma_i}]$ and $X = \sum_{i=1}^{s}(\tanh t_i)X_{\gamma_i}$. Since $\exp(X) \in P_+$, $\exp(H) \in K_{\mathbf{C}}$ (since $H \in \mathfrak{h}_{\mathbf{C}} \subset \mathfrak{k}_{\mathbf{C}}$), so we compute that
$$\log \xi(a) = \sum_{i=1}^{s}(\tanh t_i)X_{-\gamma_i} \in \mathfrak{p}.$$

Remark 1. What helped in explicitly determining $\xi(a)$ was the fact that the Lie subalgebra generated by $X_{\gamma_i}, X_{-\gamma_i}, H_{\gamma_i}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ and for the latter, one can compute the decomposition (1) explicitly.

To understand why the image is bounded once we know that $P_+ \cdot K_{\mathbf{C}} \cdot P_- \to \mathbf{G}_{\mathbf{C}}$ is injective and contains $\mathbf{G}(\mathbf{R})$ in its image, we use the following

Lemma 2.6. *Given $x \in \mathbf{G}(\mathbf{R})$, let $\xi(x)$ be the unique element of P_- such that $x \in P_+K_{\mathbf{C}}\xi(x)$. Then $\|\log|\xi(x)|\|$ is bounded as x varies through $\mathbf{G}(\mathbf{R})$.*

Proof. We prove it using the following steps:

- Consider a Cartan decomposition $\mathbf{G}(\mathbf{R}) = KAK$ for some analytic subgroup A with Lie algebra $\mathfrak{a}$.
- Take $x \in \mathbf{G}(\mathbf{R})$ and write it as $x = kak'$. Then express $\xi(x)$ in terms of $\xi(a)$.

Dimitar : Define and discuss $\mathfrak{a}$.

□

3. Complex Structures on Homogeneous Spaces and the Borel–Weil Theorem

References

Hel78 S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press Inc., New York, 1978.