On the Harish-Chandra Embedding

The purpose of this note is to link the Cartan and the root decompositions. In addition, it explains how we can view a Hermitian symmetric domain as $G_C/P$ where $P$ is a certain parabolic subgroup (more precisely, $P = K_C \cdot P_-$ in the notation of the previous document).

1. Basic Notions from Lie Theory

1.1 Simple and semi-simple Lie algebras
A *simple Lie algebra* is a Lie algebra whose only ideals are 0 and itself. A semi-simple Lie algebra is a Lie algebra that is a direct sum of simple Lie algebras.

1.2 The killing form
Given a real Lie algebra $\mathfrak{g}_R$ the Killing form on $\mathfrak{g}_R \times \mathfrak{g}_R$ is defined by
$$B(X,Y) = -\text{Tr}(\text{ad} X \circ \text{ad} Y) \in \mathbb{R}$$

Dimitar: Define the killing form.

Dimitar: Maybe give some examples with $\text{SL}_2(\mathbb{R})$.

1.3 Cartan subalgebras
A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent Lie subalgebra that is equal to its centralizer, i.e., such that $\{X \in \mathfrak{g} : [X, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}$. For semi-simple Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ being Cartan is equivalent to $\mathfrak{h}$ being a maximal abelian subalgebra.

*Example 1.* We will keep in mind the example of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h} = \left\{ \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : t \in \mathbb{R} \right\}$. Note that $\mathfrak{h}$ is also the compact Lie algebra $\mathfrak{so}_2(\mathbb{R})$. Dimitar: Describe the killing form, etc.

1.4 Root decomposition
If $\mathfrak{g}_C$ is an arbitrary semi-simple Lie algebra and $\mathfrak{h}_C \subset \mathfrak{g}_C$ is a Cartan subalgebra (one can show that every semi-simple Lie algebra contains a Cartan subalgebra [Hel78 Thm.III.4.1]), consider the linear subspace $\mathfrak{g}^\alpha \subset \mathfrak{g}_C$ associated to a functional $\alpha \in \mathfrak{h}_C^*$ and defined by
$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X, \forall H \in \mathfrak{h}_C\}.$$The functional $\alpha \in \mathfrak{h}_C^*$ is called a root if $\mathfrak{g}^\alpha \neq 0$. If this is the case then $\mathfrak{g}^\alpha$ is called a root subspace. Let $\Delta$ be the set of roots. Associated to the pair $(\mathfrak{g}_C, \mathfrak{h}_C)$, there is a root decomposition
$$\mathfrak{g}_C = \mathfrak{h}_C \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$$satisfying the following properties:

i) $\dim_{\mathbb{C}} \mathfrak{g}^\alpha = 1$, 

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ii) If $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$ then $B(g^\alpha, g^\beta) = 0$,

iii) The restriction of $B$ to $\mathfrak{h}_C \times \mathfrak{h}_C$ is non-degenerate, i.e., for each root $\alpha$, there is a unique element $H_\alpha \in \mathfrak{h}_C$ such that $\alpha(H) = B(H, H_\alpha)$,

iv) For each $\alpha \in \Delta$, $-\alpha \in \Delta$ and $[g^\alpha, g^{-\alpha}] = CH_\alpha$.

**Example 2.** Consider the Cartan subalgebra $\mathfrak{h} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathfrak{sl}_2(\mathbb{R})$. Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The two roots for this subalgebra are $2, -2$ and the root spaces are generated by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In fact, we have $[H, X] = 2X$ and $[H, Y] = -2Y$. As we will see later, this will give us an identification of $\mathfrak{SL}_2(\mathbb{R})/A \cong \mathfrak{SL}_2(\mathbb{C})/P$ where $P \subseteq \mathfrak{SL}_2(\mathbb{C})$ which is the Borel–Weil theorem.

**Example 3.** We can compute the root decomposition for the pair $(\mathfrak{so}_2(\mathbb{R}), \mathfrak{so}_2(\mathbb{R}))$ arising from identifying the Poincaré upper-half plane with $\mathfrak{SL}_2(\mathbb{R})/\mathfrak{SO}_2(\mathbb{R})$ (by looking at the stabilizer of $i \in \mathcal{H}_1$). In this case the roots are $\alpha_{\pm} = \pm 2i$. As we will see later, this Cartan decomposition will be useful to identify $\mathfrak{SL}_2(\mathbb{R})/\mathfrak{SO}_2(\mathbb{R})$ with $\mathfrak{SL}_2(\mathbb{C})/P$ (to give us the Harish-Chandra embedding).

### 2. Harish–Chandra Embedding

Dimitar : Assume that the domain is irreducible, i.e., $g_C$ is simple (the latter means that it has non-trivial ideals). All of our analysis should go under this assumption.

We assume that the Hermitian symmetric domain $D$ is irreducible. On the level of Lie algebras (or, the decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$), this means that following:

- $\mathfrak{k}$ contains no ideal of $g$ different from $\{0\}$,
- $\text{ad}_g(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

**Lemma 2.1.** Condition (ii) is equivalent to $\mathfrak{k}$ being a maximal proper subalgebra of $g$.

**Proof.** If $\mathfrak{k}$ is not a maximal proper subalgebra then there exists a proper subalgebra $\mathfrak{k} \subsetneq \mathfrak{k}^* \subsetneq g$. But we can show that if $\mathfrak{p}^* = \mathfrak{k}^* \cap \mathfrak{p}$ then $\mathfrak{p}^*$ is an invariant subspace for the action of $\text{ad}_g(\mathfrak{k})$ and hence, $\mathfrak{p}^* = \{0\}$ or $\mathfrak{p}^* = \mathfrak{p}$. The case $\mathfrak{p}^* = \{0\}$ yields a contradiction whereas if $\mathfrak{p}^* = \mathfrak{p}$ then $\mathfrak{k}^* = g$, a contradiction as well. The converse is easy: if $\mathfrak{p}^* \subsetneq \mathfrak{p}$ is a proper $\text{ad}_g(\mathfrak{k})$-invariant subspace then $\mathfrak{k} + \mathfrak{p}^* \subsetneq g$ is a proper subalgebra of $g$ containing $\mathfrak{k}$. \hfill \Box

#### 2.1 Maximal abelian subalgebra of $\mathfrak{k}$

Let $\mathfrak{h} \subset \mathfrak{k}$ be a maximal abelian subalgebra and let $\mathfrak{h}_C$ be its complexification. Let $\mathfrak{c} \subset \mathfrak{k}$ be the center of $\mathfrak{k}$. Since $\mathfrak{c}$ is the center of $\mathfrak{k}$ and $\mathfrak{h}$ is maximal abelian subalgebra of $\mathfrak{k}$ then $\mathfrak{c} \subset \mathfrak{h}$. Consider the centralizer $C_{\mathfrak{g}}(\mathfrak{c})$. Since $\mathfrak{k} \subset C_{\mathfrak{g}}(\mathfrak{c})$ Dimitar : Why? and $C_{\mathfrak{g}}(\mathfrak{c})$ is a proper subalgebra of $g$ then it follows that $C_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. We will use this to show that $\mathfrak{h}_C$ is a Cartan subalgebra of $g_C$.

**Lemma 2.2.** $\mathfrak{h}_C \subset g_C$ is a Cartan subalgebra.

**Proof.** Consider the centralizer $C_{\mathfrak{g}_C}(\mathfrak{h}_C) = \{ X \in \mathfrak{g}_C : [X, \mathfrak{g}_C] \subset \mathfrak{h}_C \}$. Since $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{k}_C$, we have $C_{\mathfrak{g}_C}(\mathfrak{h}_C) = \mathfrak{h}_C$. If $C_{\mathfrak{k}_C}(\mathfrak{h}_C) \subseteq C_{\mathfrak{g}_C}(\mathfrak{h}_C)$ then there exists an element $X \in \mathfrak{p}$ such that $X$ normalizes $\mathfrak{h}_C$. But the latter is impossible since $[X, \mathfrak{h}_C] \subset \mathfrak{p}_C$. Hence, $C_{\mathfrak{k}_C}(\mathfrak{h}_C) = \mathfrak{h}_C$ and $\mathfrak{h}_C \subset g_C$ is a Cartan subalgebra. \hfill \Box
2.2 Roots of compact and non-compact type and centralizers

Given the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, let $\mathfrak{h} \subset \mathfrak{k}$ be the maximal abelian subalgebra. We call a root $\alpha$ compact if $\mathfrak{g}^{\alpha} \subset \mathfrak{k}$ and non-compact if $\mathfrak{g}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$. Let $Q_+$ be the set of positive roots that do not vanish identically on $\mathfrak{c}_{\mathbb{C}}$, i.e., for which $\text{ad}_{\mathfrak{c}_{\mathbb{C}}} (\mathfrak{g}^{\alpha}) = [\mathfrak{c}_{\mathbb{C}}, \mathfrak{g}^{\alpha}] \neq 0$. We now consider two subspaces of $\mathfrak{p}_{\mathbb{C}}$:

$$p_+ = \sum_{\alpha \in Q_+} \mathfrak{g}^{\alpha}, \quad p_- = \sum_{-\alpha \in Q_+} \mathfrak{g}^{\alpha}.$$ 

These subspaces decompose $\mathfrak{p}_{\mathbb{C}}$ as follows

**Lemma 2.3.** We have $[\mathfrak{k}, p_+] \subset p_+$, $[\mathfrak{k}, p_-] \subset p_-$ and $\mathfrak{p}_{\mathbb{C}} = p_+ \oplus p_-$. 

**Proof.** Clearly, if $\alpha$ is compact then $\alpha$ vanishes on $\mathfrak{c}$ so we get $p_+ + p_- \subset \mathfrak{p}_{\mathbb{C}}$. To see that $p_+$ is abelian, note that if $\alpha, \beta \in Q_+$ then $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha + \beta}$ and if $\alpha + \beta$ is a root then $\alpha + \beta \in Q_+$. But $[p_+, p_+] \subset \mathfrak{k}$, so it follows that $[p_+, p_+] = 0.$

What is tricky to show is that $\mathfrak{p}_{\mathbb{C}} = p_+ + p_-$. Let $\mathfrak{q}$ be the orthogonal complement of $p_+ + p_-$ in $\mathfrak{p}_{\mathbb{C}}$ with respect to the killing form. Define

$$\mathfrak{g}_+ = p_+ + p_- + [p_+, p_-].$$

One can check that $\mathfrak{g}_+ \subset \mathfrak{g}_{\mathbb{C}}$ is an ideal **Dimitar: Do the computation!** and since $\mathfrak{g}_{\mathbb{C}}$ is simple, it follows that $\mathfrak{g}_+ = \{0\}$ or $\mathfrak{g}_{\mathbb{C}}$. The first case is impossible (as all the roots will be compact), so we are in the second case and $p_+ + p_- = p_{\mathbb{C}}$. \hfill $\Box$

2.3 More on semi-simple Lie algebras

Consider a semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ over $\mathbb{C}$. Consider also

- $\mathfrak{g}^{\mathbb{R}}$ - the Lie algebra $\mathbb{C}$ viewed as a real Lie algebra,
- $G_c$ - any connected real Lie group with Lie algebra $\mathfrak{g}^{\mathbb{R}}$,
- $\mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$ - a compact real form of $\mathfrak{g}_{\mathbb{C}}$,
- $\mathfrak{a} \subset \mathfrak{u}$ - a maximal abelian subalgebra,
- $\mathfrak{h}_{\mathbb{C}} = \mathfrak{a} + i\mathfrak{a}$ - a (one has to prove this) Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.
- $\mathfrak{n}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}$ - the positive nilpotent (with respect to some ordering of the roots) considered as a real Lie algebra.
- $J$ - a complex structure on $\mathfrak{g}^{\mathbb{R}}$ that coincides with $i$ on $\mathfrak{g}_{\mathbb{C}}$.

Then (one has to prove this - see [Hel78 Thm.VI.6.3]) there is a direct sum decomposition

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{a} \oplus \mathfrak{n}_+.$$ 

Furthermore, if $U, A^*$, $N_+$ are the analytic subgroups of $G_c$ with Lie algebras $\mathfrak{u}$, $\mathfrak{a}^* = Ja^*$ and $\mathfrak{n}_+$, respectively, then the multiplication map gives an analytic diffeomorphism

$$U \times A^* \times N_+ \to G_c.$$ 

2.4 The parabolic subgroups

$G(\mathbb{R})/K$ embeds into $P_-$ via the following sequence of steps

- One needs to show that the multiplication map $P_- \times K_{\mathbb{C}} \times P_+ \to G_{\mathbb{C}}$ induces a diffeomorphism between $P_- \times K_{\mathbb{C}} \times P_+$ and an open submanifold of $G_{\mathbb{C}} = G(\mathbb{C})$ that contains $G = G(\mathbb{R})$.
- One identifies $G(\mathbb{R})/K$ diffeomorphically with $G(\mathbb{R})K_{\mathbb{C}}P_+/K_{\mathbb{C}}P_+$. For this, we need to know that $K_{\mathbb{C}}P_+ \cap G(\mathbb{R}) = K$. Once this is proved, consider the map $gK_{\mathbb{C}}P_+ \mapsto gK$ for any
\( g \in \mathbf{G}(\mathbb{R}) \). It follows that this map is well-defined (if \( g'K_+ = g''K_+P_+ \) then \( g'kp_+ = g'' \) and hence, \( kp_+ = (g')^{-1}g'' \in \mathbf{G}(\mathbb{R}) \), hence \( K_+P_+ \cap \mathbf{G}(\mathbb{R}) = K \) and a bijection.

**Lemma 2.4.** The multiplication map \( P_- \times K_+ \times P_+ \to G_C \) induces a diffeomorphism between \( P_- \times K_+ \times P_+ \) and an open submanifold of \( G_C = G(\mathbb{C}) \) that contains \( G = \mathbf{G}(\mathbb{R}) \).

**Proof.** To show that the map is an injection, suppose that \( q_1k_1p_1 = q_2k_2p_2 \), i.e.,

\[
 q_2^{-1}q_1 = k_2p_2p_1^{-1}k_1^{-1}.
\]

The right-hand side is \( k_2k_1^{-1}(k_1p_2p_1^{-1}k_1^{-1}) \) and \( k_1p_2p_1^{-1}k_1^{-1} \in P_+ \) since \( K_+ \) normalizes \( P_+ \). Hence, it suffices to show that \( P_-K_+ \cap P_+ = \{e\} \) to get injectivity. The latter can be seen on the level of the Lie algebra as follows: suppose that \( y \in P_-K_+ \cap P_+ \) and let \( Y \in \mathfrak{p}_+ \) such that \( \exp(Y) = y \). One one hand, we can write \( Y = \sum c_\alpha X_\alpha \). Since \( [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{p}_- \), it follow that \( \text{ad}(y) : \mathfrak{p}_- \to \mathfrak{p}_- \). Let \( n_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \) and \( n_- = \sum_{-\alpha \in \Delta^+} \mathfrak{g}^\alpha \) (these are nilpotent Lie algebras whose Lie groups are unipotent groups Dimitar: May need to be more precise about the latter). Let \( \beta \in \Delta^+ \) be the lowest root for which \( c_\beta \neq 0 \). After calculating

\[
 [Y, X_-] \equiv c_\beta [X_\beta, X_-] \mod n_+,
\]

we observe that it cannot be \( 0 \mod n_+ + n_- \) and hence, \( \text{ad}(Y)(X_-) \notin \mathfrak{p}_- \), a contradiction. Next, a dimension count shows that the image of the map is an open submanifold of \( G_C \). Finally, we want to show that any \( p \in P = \exp(\mathfrak{p}) \) is in the image (which, together with \( G = K_+P_+PK \) will imply that \( G = \mathbf{G}(\mathbb{R}) \) is contained in the image).

**Example 4.** For \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \), one can compute explicitly \( P_+ \) and \( P_- \) using the two non-compact root spaces. Since \( \alpha = 2i \) is the positive root then the Lie algebra \( \mathfrak{p}_- \) is simply the root space \( \mathfrak{g}^{-\alpha} \). The embedding that we get is then the Cayley transform. Dimitar: Complete the example.

### 2.5 The Lie algebra of \( \text{SL}(2, \mathbb{C}) \)

The Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) of \( \text{SL}_2(\mathbb{C}) \) consists of all trace-zero matrices in \( M_2(\mathbb{C}) \). A basis is given by the matrices

\[
 H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

They satisfy the relations \( [X, Y] = H, [H, X] = 2X \) and \( [H, Y] = -2Y \). The key point is that it is possible, given any \( Z \in \mathbb{C}(X + Y) \), to decompose \( \exp(Z) \) in the form \( P_-K_+P_+ \) via the following lemma:

**Lemma 2.5.** Given any \( Z = t(X + Y) \) then we have

\[
 \exp t(X + Y) = \exp(\tanh tY) \exp(\log(\cosh t)H) \exp(\tanh X).
\]

**Proof.**

### 2.6 Boundedness

- Define a maximal abelian subalgebra \( \mathfrak{a}_C \subset \mathfrak{p}_C \) as

\[
 \mathfrak{a}_C = \sum_{i=1}^s \mathbb{C}(X_\gamma_i + X_{-\gamma_i}) \subset \mathfrak{p}_C.
\]

Here, the vectors \( X_\alpha \in \mathfrak{g}^\alpha \) are chosen such that

\[
 X_\alpha - X_{-\alpha}, \quad i(X_\alpha + X_{-\alpha}) \in \mathfrak{u}_C := \mathfrak{t} \oplus i\mathfrak{p}, \quad [X_\alpha, X_{-\alpha}] = 2/\alpha(H_\alpha)H_\alpha.
\]
the latter being the compact form of $\mathfrak{g}_C$.

- If the $X_\alpha$’s are chosen as above then

$$a = \sum_{i=1}^{s} R(X_{\gamma_i} + X_{-\gamma_i}) \subset p.$$ 

- Suppose that $Z \in a$ and write

$$Z = \sum_{i=1}^{s} t_i (X_{\gamma_i} + X_{-\gamma_i}).$$

Using some nested commutator relations (à la Campbell–Hausdorff formula), one can show that

$$\exp(Z) = \exp(Y) \exp(H) \exp(X), \quad (1)$$

where $Y = \sum_{i=1}^{s} (\tanh t_i) X_{-\gamma_i}$, $H = \sum_{i=1}^{s} \log(\cosh t_i)[X_{\gamma_i}, X_{-\gamma_i}]$ and $X = \sum_{i=1}^{s} (\tanh t_i) X_{\gamma_i}$. Since $\exp(X) \in P_+$, $\exp(H) \in K_C$ (since $H \in \mathfrak{h}_C \subset \mathfrak{k}_C$), so we compute that

$$\log \xi(a) = \sum_{i=1}^{s} (\tanh t_i) X_{-\gamma_i} \in p.$$ 

Remark 1. What helped in explicitly determining $\xi(a)$ was the fact that the Lie subalgebra generated by $X_{\gamma_i}, X_{-\gamma_i}, H_{\gamma_i}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2(C)$ and for the latter, one can compute the decomposition $(1)$ explicitly.

To understand why the image is bounded once we know that $P_+ \cdot K_C \cdot P_- \to G_C$ is injective and contains $G(R)$ in its image, we use the following

**Lemma 2.6.** Given $x \in G(R)$, let $\xi(x)$ be the unique element of $P_-$ such that $x \in P_+ K_C \xi(x)$. Then $\|\log |\xi(x)||$ is bounded as $x$ varies through $G(R)$.

**Proof.** We prove it using the following steps:

- Consider a Cartan decomposition $G(R) = KAK$ for some analytic subgroup $A$ with Lie algebra $a$.
- Take $x \in G(R)$ and write it as $x = kak'$. Then express $\xi(x)$ in terms of $\xi(a)$.

Dimitar: Define and discuss $a$.

\[ \square \]

### 3. Complex Structures on Homogeneous Spaces and the Borel–Weil Theorem

**References**