Hodge Structures and Shimura Data

It is interesting to understand when the example of $\mathbf{GL}_2(\mathbf{R})$ acting on the Hermitian symmetric space $\mathbf{C}-\mathbf{R}$ or $\mathbf{Sp}_{2g}(\mathbf{R})$ acting on $\mathcal{H}_g$, or $U(p,q)$ acting on X (from the last lecture) can be extended to more general reductive algebraic groups $\mathbf{G}$. In order to do this, one needs to extract the essential features of the domains. So far, the only common description of these domains was via $\mathbf{G}(\mathbf{R})^+$-conjugacy classes of morphisms $\mathbf{S}\to\mathbf{G}$ from the circle group $\mathbf{S}$ (considered as a real algebraic group) to the real algebraic group $\mathbf{G}$.

If $\mathbf{G}$ is an arbitrary reductive group and $h\colon\ \mathbf{S}\to\mathbf{G}_{\mathbf{R}}$ is an arbitrary morphism of real algebraic groups then the orbit space $X_h=\mathbf{G}(\mathbf{R})^+h$ for the conjugation action of $\mathbf{G}(\mathbf{R})^+$ will always have the structure of a real manifold, but will rarely have a Hermitian structure (i.e., structure of a Hermitian symmetric domain). We will thus attempt to understand how to axiomatize the properties that will not only make this orbit space a Hermitian symmetric space, but it will also give it a certain moduli interpretation.

The main idea is that X_h can be viewed as a moduli space of *Hodge structures* for a certain algebraic representation of $\mathbf{G}(\mathbf{R})$. Indeed, if $\rho\colon\ \mathbf{G}(\mathbf{R})\to\mathbf{GL}(V)$ is any algebraic representation of of the group of real points on $\mathbf{G}$ then for any $h'\in X_h$, one can considers the representation $\rho\circ h'$ as a representation of the circle group. This yields a family of Hodge structures as $h'\in X_h$ varies. It will turn out that for an algebraic representation ρ, a lot of the properties of this family of Hodge structures is determined by the properties of the same family when ρ is the adjoint representation on the Lie algebra $\mathrm{Lie}(\mathbf{G})_{\mathbf{C}}$ and by the corresponding weight space decomposition for this action. This allows us to axiomatize the properties of X_h in a way that is more intrinsic to the group $\mathbf{G}$.

This is the original point of view of Deligne that led to the formulation of the precise axioms for a Shimura datum. We will thus look for specific conditions on $\mathbf{G}$ and h that give an intrinsic description of the problem of the existence of a complex structure on X_h.

1. More on $\mathbf{GL}_2(\mathbf{R})$

- For any $h'\in X_h$, $h'(\mathbf{C}^\times)\subset\mathbf{GL}_2(\mathbf{R})$ is a non-split Cartan subgroup.
- Show that $Z_{\mathbf{GL}_2(\mathbf{R})}(h'(\mathbf{C}^\times))=h'(\mathbf{C}^\times)$ for any $h'\in X_h$.
- Calculating the normalizer $N_{\mathbf{GL}_2(\mathbf{R})}(h'(\mathbf{C}^\times))$. It is $\mathbf{R}^\times K_\infty^0$ where K_∞ is the unique maximal compact subgroup of $\mathbf{GL}_2(\mathbf{R})$ whose identity component K_∞ is the circle in $h'(\mathbf{C}^\times)$.

1.1 The case of $\mathbf{GL}_2$

1.1.1 $\mathbf{C}-\mathbf{R}$ *as a homogeneous space for* $\mathbf{GL}_2(\mathbf{R})$. $\mathbf{GL}_2(\mathbf{R})$ acts on $\mathbf{C}-\mathbf{R}$ by linear fractional transformations. The following exercise shows that one can identify (non-canonically, by choosing a base point from $\mathbf{C}-\mathbf{R}$) $\mathbf{C}-\mathbf{R}$ with $\mathbf{GL}_2(\mathbf{R})/Z(\mathbf{R})K_\infty^0$ where K_∞^0 is the component at the identity of some maximal compact subgroup $K_\infty\subset\mathbf{GL}_2(\mathbf{R})$.

Consider $\mathrm{Hom}_{\mathrm{alg}}(\mathbf{C}^\times,\mathbf{GL}_2(\mathbf{R}))$. One can show that any two such maps are conjugated under $\mathbf{GL}_2(\mathbf{R})$ (this follows from, e.g., the Skolem-Noether theorem). Here, $G=\mathbf{GL}_2(\mathbf{R})$.

2000 Mathematics Subject Classification
Keywords:

1.1.2 *Weight space decomposition.* Suppose that the circle group $S = \mathbf{S}(\mathbf{R})$ acts continuously on a finite dimensional $\mathbf{C}$-vector space V. We then get a weight decomposition of V as follows:

$$V = \bigoplus_n V_n, \qquad V_n = \{v \in V : z \cdot v = z^n v, \ \forall z \in S\}. \tag{1}$$

It is instructive to compute the weight decomposition in several examples.

1.1.3 *Computing the weight spaces for the action of $S = \mathbf{S}(\mathbf{R})$ on $\mathfrak{g}_\mathbf{C}$.* To compute the weight spaces for the action of S on the 4-dimensional Lie algebra $\mathfrak{g}_\mathbf{C} = \mathfrak{gl}_{2,\mathbf{C}}$, consider the closed subgroup $h(\mathbf{C}^\times) \subset \mathbf{GL}_2(\mathbf{R})$. It gives us a 2-dimensional Lie subalgebra of $\mathfrak{gl}_{2,\mathbf{C}}$. Since $h(\mathbf{C}^\times)$ commutes with $h(S)$, this Lie algebra is contained in V_0. Next, we discuss the following exercise:

Exercise 1. The action of $h(\mathbf{C}^\times)$ on $V = \mathfrak{gl}_{2,\mathbf{C}}$ is non-trivial, i.e., there are non-trivial weight spaces in the weight decomposition (1).

The exercise implies that there exists some $r > 0$ such that $V = V_0 \oplus V_r \oplus V_{-r}$. One can go further and compute the r by observing that for any $z \in \mathbf{S}(\mathbf{R})$, the matrix $h(z)$ is diagonalizable with eignevalues z and z^{-1}, hence, in the basis of eigenvalues, $\mathrm{ad}(h(z))$ is conjugation by $\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}$, i.e.,

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}^{-1} = \begin{pmatrix} a & z^2 b \\ z^{-2} c & d \end{pmatrix}.$$

From here, we see the two eigenspaces (with respect to the above eigenbasis): $\begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix}$ has eigenvalue z^{-2} and $\begin{pmatrix} 0 & 0 \\ \star & 0 \end{pmatrix}$ has eigenvalue z^2, i.e., $r = 2$.

1.1.4 *For any h, $Z_{\mathbf{GL}_2(\mathbf{R})}(h) \cap \mathbf{SL}_2(\mathbf{R})$ is a maximal compact subgroup.*

2. Hodge Structures and Representations of $\mathbf{C}^\times$ on Real Vector Spaces

2.1 Hodge structures

A Hodge structure can be thought of as a generalization of a complex structure.

Definition 2.1 (Hodge structure). Let V be an $\mathbf{R}$-vector space and let $V_\mathbf{C} = V \otimes_\mathbf{R} \mathbf{C}$. A *Hodge structure* on V is a decomposition

$$V_\mathbf{C} = \bigoplus_{p,q} V^{p,q},$$

such that $V^{p,q}$ are complex vector spaces for which $\overline{V^{q,p}} = V^{p,q}$ where for a complex vector space W, $\overline{W}$ indicates the complex vector space with the conjugated linear action, i.e., the map $W \to \overline{W}$, given by $\alpha \otimes w \mapsto \overline{\alpha} \otimes w$ for $\alpha \in \mathbf{C}$ and $w \in W$.

Given a Hodge structure on V, one can consider the associated *Hodge filtration*:

$$F^p = \bigoplus_{p' \geqslant p, \ q \in \mathbf{Z}} V^{p,q}.$$

2.2 Examples

2.2.1 *Hodge structures as generalizations of complex structures.* One can think of a complex structure as a Hodge structure as follows: suppose that V is an $\mathbf{R}$-vector space such that $\dim_\mathbf{R} V$ is even and suppose that V has a $\mathbf{C}$-linear structure. Let $\overline{V}$ denote complex vector space having the same underlying real vector space, but the conjugate $\mathbf{C}$-linear structure.

Exercise 2. Show that the map

$$V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} V \to V \oplus \overline{V}, \qquad \alpha \otimes v \mapsto (\alpha v, \overline{\alpha} v).$$

is an isomorphism (use that that $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \mapsto \mathbf{C} \times \mathbf{C},\ x \otimes y \mapsto (xy, \overline{x} y)$ is an isomorphism of complex vector spaces where the complex structure on the target is given by $\alpha(x, y) = (\alpha x, \overline{\alpha} y)$).

We thus have a decomposition $V_{\mathbf{C}} = V \oplus \overline{V}$ where

$$V = \{v \in V_{\mathbf{C}} \colon z \cdot v = z^{-(-1)} v, \forall z \in \mathbf{C}^\times\}, \qquad \overline{V} = \{v \in V_{\mathbf{C}} \colon z \cdot v = \overline{z}^{-(-1)} v, \forall z \in \mathbf{C}^\times\},$$

i.e., we have a Hodge structure of type $\{(-1, 0), (0, -1)\}$ (in other words, $V^{-1,0} = V$ and $V^{0,-1} = \overline{V}$).

Exercise 3. Show the converse, i.e., V is a real vector space then a Hodge structure $V_{\mathbf{C}} = V^{-1,0} \oplus V^{0,-1}$ of type $\{(-1, 0), (0, -1)\}$ yields a complex structure on V.

2.2.2 *Hodge structure on deRham cohomology.* Consider a smooth, proper variety Y over $\mathbf{C}$ and consider the deRham cohomology $V = \mathrm{H}^n(Y(\mathbf{C}), \mathbf{R})$. There is a natural Hodge filtration on $V_{\mathbf{C}}$ coming from the complex structure on $Y(\mathbf{C})$. Intuitively, F^p is the space generated by $\mathbf{C}$-valued differential forms that locally are wedge products of n differential 1-forms dz at least p of which must be holomorphic (i.e., dz_i). This gives us a Hodge filtration $\{F^p\}$ for which $F^0 = \mathrm{H}^n(Y(\mathbf{C}), \mathbf{C})$, F^n containing the space $\mathrm{H}^0(Y(\mathbf{C}), \Omega^n_{Y(\mathbf{C})})$ and $F^{n+1} = 0$.

In particular, if E is an elliptic curve then $F^0 = V_{\mathbf{C}} = \mathrm{H}^n(Y(\mathbf{C}), \mathbf{C})$, $F^1 = \mathrm{H}^0(E(\mathbf{C}), \Omega^1_{E(\mathbf{C})})$ is the space of regular (holomorphic) differential 1-forms on $E(\mathbf{C})$ and there is an exact sequence

$$0 \to F^1 = \mathrm{H}^1(E(\mathbf{C}), \Omega^1_{E(\mathbf{C})}) \to F^0 = \mathrm{H}^1(E(\mathbf{C}), \mathbf{C}) \to H^1(E(\mathbf{C}), \mathcal{O}_{E(\mathbf{C})}) \to 0.$$

The latter cohomology group is also $\mathrm{H}^1(E, \mathcal{O}_E)$.

Remark 1. The fact that $V^{p,q} = F^p \cap \overline{F}^q$ is a Hodge structure that is pure of weight n is not obvious. One needs to either use harmonic forms, or find an algebraic proof (a theorem of Deligne–Illusie). Yet, $V^{p,q}$ is canonically isomorphic to $\mathrm{H}^q(Y(\mathbf{C}), \Omega^p_{Y(\mathbf{C})})$.

2.2.3 *Tate twists.* Consider the Tate twist $\mathbf{Z}(1) = \ker\{\exp \colon \mathbf{C} \to \mathbf{C}^\times\}$. We make this pure of weight -2 by declaring $\mathbf{C}(1)^{-1,-1} = \mathbf{C}(1)$. With this convention, the Hodge filtration is very simple:

$$F^p = \begin{cases} V & \text{if } p < 0 \\ 0 & \text{if } p \geqslant 0. \end{cases}$$

Similarly, we get a Hodge structure for $\mathbf{Z}(n) := \mathbf{Z}(1)^{\otimes n}$ that is pure of weight $-2n$ by declaring that $\mathbf{C}(n)^{-n,-n} = \mathbf{C}(n)$. Here, we have $F^p = V$ for $p < -n+1$ and $F^p = 0$ for $p \geqslant -n+1$. Now, if $Y/\mathbf{C}$ is a smooth, proper and connected algebraic variety then we have an isomorphism of integral pure Hodge structures of weight $2n$, $\mathrm{H}^{2n}(Y(\mathbf{C}), \mathbf{Z}) \cong \mathbf{Z}(-n)$ (just think of top $\mathcal{C}^\infty$-differential form on an n-dimensional complex manifold as having bidegree (n, n) the form $dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$). Finally, if V is an $\mathbf{R}$-vector space equipped with a Hodge structure then $V(n) = V \otimes_{\mathbf{C}} \mathbf{C}(n)$ has a Hodge structure satisfying $V(n)^{p,q} = V^{p+n,q+n}$.

3. Parametrizing Hodge Structures

Let X denote (as usual), a $\mathbf{G}(\mathbf{R})$-conjugacy class of embeddings $h \colon \mathbf{C}^\times \to \mathbf{G}_{\mathbf{R}}$. We would like that the points on the space X encode information about Hodge structures on V in the following sense of Deligne: if $\rho\colon \mathbf{G}(\mathbf{R}) \to \mathbf{GL}(V)$ is any algebraic representation and $h \in X$ is any point then we would like that $\rho \circ h\colon \mathbf{C}^\times \to \mathrm{Aut}(V)$ is a Hodge structure in the following sense (normalized following the convention of [Del79]):

$$V_{\mathbf{C}} = \bigoplus_{p,q} V^{p,q}, \qquad V^{p,q} = \{v \in V_{\mathbf{C}}\colon (1 \otimes \rho \circ h(z))v = z^{-p}\overline{z}^{-q}v\}.$$

Under this identification, the Hodge structure determined by h and ρ is pure of weight n if and only if $\rho|_{\mathbf{R}^\times}(x)v = x^{-n}v$ for all $x \in \mathbf{R}^\times$.

In general, there is a weight decomposition coming just from the (algebraic) action of $\mathbf{R}^\times$ on V:

$$V = \bigoplus_{n \in \mathbf{Z}} V_{n,h}, \qquad V_{n,h} = \{v \in V\colon \rho_{\mathbf{R}} \circ h|_{\mathbf{R}^\times}(x)(v) = x^n v \ \forall x \in \mathbf{R}^\times\}.$$

Deligne's philosophy was to express properties for these Hodge structures arising from any fixed representation by just looking at the Hodge structures corresponding to the adjoint representation. One cannot expect that this is possible for an arbitrary X, so we now start (using the notion of variation of Hodge structure) to study what extra conditions are necessary.

3.1 Towards a family of Hodge structures

3.1.1 *The weight homomorphism.* We first would like to know when the weight spaces $V_{h,n}$ are independent of h for any algebraic representation $\rho\colon \mathbf{G}(\mathbf{R}) \to \mathrm{Aut}(V)$. In order for this to occur, we need the restriction $h|_{\mathbf{R}^\times}\colon \mathbf{R}^\times \to \mathbf{G}(\mathbf{R})$ to be independent of h. We next provide a group-theoretic description for the latter to occur:

Proposition 3.1. *The subspaces $V_{h,n}$ are independent of $h \in X_h$ and $\rho\colon \mathbf{G}(\mathbf{R}) \to \mathrm{Aut}(V)$ if and only if the algebraic morphism $w_h\colon \mathbf{GL}_{1,\mathbf{R}} \to \mathbf{G}_{\mathbf{R}}$ is independent of h. The latter is equivalent to the image of w_h being contained in the center $\mathbf{Z}_{\mathbf{G},\mathbf{R}}$ of $\mathbf{G}_{\mathbf{R}}$ for some (equivalently, all) h.*

The proof will rely on the following exercise:

Exercise 4. For any morphism of real algebraic groups $\mathbf{GL}_{1,\mathbf{R}} \to \mathbf{GL}(V)_{\mathbf{R}}$, there is a unique $\mathbf{GL}_1(\mathbf{R})$-stable decomposition $V = \bigoplus V_n$, where $\mathbf{GL}_1(\mathbf{R})$ acts on V_n by the nth power, i.e., $x \cdot v = x^n v$.

Proof of Prop. 3.1. Let w_h be as above and let for $g \in \mathbf{G}(\mathbf{R})$, $w_{g,h} := g w_h g^{-1}$. Using the above exercise, we note that if $V_{h,n}$'s are independent of h then for all $g \in \mathbf{G}(\mathbf{R})$, $\rho(w_{g,h}) = \rho(g)\rho(w_h)\rho(g)^{-1}$. Taking a faithful representation ρ, we obtain that $\mathbf{G}(\mathbf{R})$ centralizes $w_h(x)$ for all $x \in \mathbf{R}^\times$, i.e., $g w_h(x) g^{-1} = w_h(x)$. Now, $\mathbf{G}(\mathbf{R})$ is Zariski dense in $\mathbf{G}_{\mathbf{R}}$, hence the centralizer of $\mathbf{G}(\mathbf{R})$ is $\mathbf{Z}_G$ in the sense of algebraic group. Now, since $\mathbf{GL}_1(\mathbf{R})$ is Zariski dense in $\mathbf{GL}_{1,\mathbf{R}}$, w_h yields a morphism of algebraic groups $w_h\colon \mathbf{GL}_{1,\mathbf{R}} \to \mathbf{Z}_{\mathbf{G},\mathbf{R}}$. □

Using the lemma, we obtain a homomoprhism $w\colon \mathbf{GL}_1 \to \mathbf{Z}_{\mathbf{G}}$ called the *weight homomorphism*, independently of the choice of $h \in X$.

3.1.2 *Interpretation in terms of the adjoint representation.* Choosing the adjoint representation $\rho = \mathrm{ad}\colon \mathbf{G}(\mathbf{R}) \to \mathrm{Aut}(\mathfrak{g})$, a different way of stating the above proposition is that $\mathbf{R}^\times$ acts trivially on the "real" Lie algebra $\mathfrak{g}$. Another way of saying this is that the algebraic action $\mathbf{C}^\times \to \mathrm{Aut}(\mathfrak{g})$ yields an action of $\mathbf{R}^\times$ that is pure of weight 0. **Dimitar : This is exactly what we mean when we say that Deligne's philosophy is to deduce properties of Hodge structures for a general ρ from those for the adjoint representation.** This naturally gives us the first desired condition (axiom) for the pair $(\mathbf{G}, X)$:

Property 0: For $(\mathbf{G}, X)$, the induced Hodge structure on $\mathfrak{g}$ is pure of weight 0.

Remark 2. The above weight 0 should not be confused with the weight of the Hodge structures corresponding to the elements $h \in X$. Indeed, consider the case $V = \mathbf{R}^2$, $V_{\mathbf{C}} = \mathbf{C}^2$, $\mathbf{G} = \mathbf{GL}_2$ and X being the $\mathbf{GL}_2(\mathbf{R})$-conjugacy class of the embedding $h\colon \mathbf{C}^\times \to \mathbf{GL}_2(\mathbf{R})$, $z = a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. In this case, the Hodge structure corresponding to h is pure of weight -1 according to Deligne's convention. Indeed,

$$V^{p,q} = \{v \in V_{\mathbf{C}} \colon z \cdot v = z^{-p}\overline{z}^{-q} v,\ \forall z \in \mathbf{C}^\times\},$$

so $x \in \mathbf{R}^\times$ acts on $V^{p,q}$ by multiplication by $x^{-(p+q)}$. On the other hand, $h(x) = mtwoxx$, so $h(x)$ acts on $V_{\mathbf{C}}$ by multiplication by x, i.e., $x \cdot v = x^{-(-1)} v$. This means that the weight of the Hodge structure given by h is -1.

3.1.3 *The Hodge structure for the adjoint representation in the case of* $\mathbf{G} = \mathbf{GL}_2$. For the $G = \mathbf{GL}_2(\mathbf{R})$, the above property is satisfied. In fact, $\mathrm{ad}_{\mathbf{GL}_2(\mathbf{R})} \circ h$ yields a Hodge structure on $\mathfrak{g}$ given by

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0}, \qquad \dim \mathfrak{g}^{1,-1} = \dim \mathfrak{g}^{-1,1} = 1,\ \dim \mathfrak{g}^{0,0} = 2.$$

We say that the Hodge structure is of type $\{(-1,1),(0,0),(1,-1)\}$.

Note that so far, we have only used the action of $\mathbf{R}^\times$ (related to the weight decomposition) and have not done anything analysis with the actual Hodge structures or the complexification $V_{\mathbf{C}}$.

4. Motivation via Kähler manifolds

In order to better understand and define a suitable notion of a "family" of Hodge structures (and subsequently motivate Deligne's axioms of a Shimura datum), we need to look into how Hodge structures arise from the cohomology of the fibers of certain families of complex manifolds. These important manifolds in differential geometry carry the three structures (Riemannian, symplectic and hermitian) in a mutually compatible way. It is the cohomology of these manifolds that will naturally give rise to the properties that we will abstract in the next section to define variation of Hodge structures and Shimura data.

Our working example will be the case of K3 surfaces and as we will check, where we will get a moduli space of polarized Hodge structures for the orthogonal group $\mathbf{O}(2, 19)$.

4.1 Kähler manifolds and cohomology

A Kähler manifold is a manifold X that is simultaneously symplectic, Riemannian, complex and hermitian in a mutually compatible way. That is, for each $p \in X$, there is a symplectic form $\omega\colon \bigwedge^2 T_pX \to \mathbf{R}$ such that $\omega(Ju, Jv) = \omega(u, v)$ where $J\colon T_pX \to T_pX$ is the complex structure. In addition, ω is required to be closed (by the definition of a symplectic manifold) which means that ω gives us a class $[\omega] \in \mathrm{H}^2_{\mathrm{dR}}(X, \mathbf{C})$.

Definition 4.1 (Kähler manifold). A symplectic manifold (X, ω) is called Kähler if it has an integrable almost-complex structure $J_p\colon T_pX \times T_pX \to \mathbf{C}$ that is compatible with the symplectic form $\omega\colon \bigwedge^2 T_pX \to \mathbf{C}$ in the sense that $\omega(Ju, Jv) = \omega(u, v)$.

The associated Hermitian form is called a Kähler metric. In differential geometry, examples of Kähler manifolds are K3 surfaces (it is a theorem of Siu that they are Kähler) and Calabi–Yau manifolds.

Remark 3. It is not very often the case that a compact Kähler manifold admits the structure of a projective variety. In fact, if this is the case then we can see that $[\omega] \in \mathrm{H}^2(X, \mathbf{Z})$ since $[\omega]$ is the class of any hyperplane section in $\mathbf{P}^n$. There is a converse to this statement known as Kodaira's embedding theorem [Voi02, Thm.7.11]. For complex tori $A = V/L$ (V is a g-dimensional complex vector space), $\mathrm{H}^2(A, \mathbf{Z}) = \mathrm{Hom}(\bigwedge^2 V, \mathbf{Z})$ and hence, $\mathrm{H}^2(A, \mathbf{R}) = \mathrm{Hom}(\bigwedge^2 V, \mathbf{R})$. The Kähler class then corresponds to a form $\omega \colon \bigwedge^2 V \to \mathbf{R}$ and this, Kodaira embedding theorem says precisely that A is algebraic if $\omega|_{\Lambda \times \Lambda}$ takes values in $\mathbf{Z}$ (this is a restatement of the fact of the classical "algebraicity of a complex torus" statement that A is an abelian variety if and only if there exists a Riemann form on A).

4.1.4 *Hodge decomposition on the cohomology of a Kähler manifold.* The theory of harmonic forms allows us to show that the cohomology of Kähler manifolds comes equipped with a Hodge decomposition, namely

$$\mathrm{H}^k(X, \mathbf{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}.$$

This is a consequence of the fact that harmonic forms decompose into forms of type (p, q) under the natural action of complex conjugation on $\mathrm{H}^k(X, \mathbf{R}) \otimes \mathbf{C}$. As a consequence, if k is odd then the Betti number b_k must be even.

4.1.5 *A non-Kähler surface.* Consider the Hopf surface X obtained by quotienting $\mathbf{C}^2 - \{(0,0)\}$ by the action of $\Gamma = \mathbf{Z}$ via $(z_1, z_2) \mapsto \left(\frac{1}{2}z_1, \frac{1}{2}z_2\right)$. We can compute $\mathrm{H}^1(X, \mathbf{Z})$ by using that $\mathbf{C}^2 - \{0,0\}$ is simply connected and $\pi_1(X) = \mathbf{Z}$. To compute the Betti numbers, one could either use Hurewitz theorem or note that X is diffeomorphic to $S^3 \times S^1$ as follows:

$$\mathbf{C}^2 - \{0,0\} \to S^3 \times \mathbf{R}_{>0} \to S^3 \times \mathbf{R},$$

where the first map is $z \mapsto (z/\|z\|, \|z\|)$ and the second map is $(z, r) \mapsto (z, \log r)$. By taking the quotients, we obtain

$$X = (\mathbf{C}^2 - \{(0,0)\})/\Gamma = S^3 \times \mathbf{R}/\mathbf{Z}\log 2 \cong S^3 \times S^1.$$

Künneth's formula then allows us to compute the Betti numbers $b_0 = b_1 = b_3 = b_4 = 1$ and $b_2 = 0$. Since b_1 is odd, X cannot be a Kähler manifold.

4.1.6 *Lefschetz operator and Poincaré duality.* This is given by cup product with the Kähler class $[\omega] \in \mathrm{H}^2(X, \mathbf{R})$:

$$L \colon \mathrm{H}^k(X, \mathbf{R}) \to X^{k+2}(X, \mathbf{R}), \qquad \alpha \mapsto \alpha \cup [\omega],\ \alpha \in \mathrm{H}^k(X, \mathbf{R}).$$

The Poincaré duality gives us a perfect pairing

$$\langle\, , \rangle \colon \mathrm{H}^{2n-k}(X, \mathbf{R}) \otimes \mathrm{H}^k(X, \mathbf{R}) \to \mathbf{R}, \tag{2}$$

and the Hard Lefschetz theorem says that $L^{n-k} \colon \mathrm{H}^k(X, \mathbf{R}) \to \mathrm{H}^{2n-k}(X, \mathbf{R})$ is an isomorphism whenever $k \leqslant n$.

4.1.7 *Primitive cohomology.* The primitive cohomology in degree k is defined as

$$\mathrm{H}^k_{\mathrm{prim}}(X, \mathbf{R}) = \ker\{L^{n-k+1} \colon \mathrm{H}^k(X, \mathbf{R}) \to \mathrm{H}^{2n-k+2}(X, \mathbf{R})\}.$$

For instance, if X is a K3 surface and if $k = 2$ then $\mathrm{H}^2_{\mathrm{prim}}(X, \mathbf{R}) \subset \mathrm{H}^2(X, \mathbf{R})$ is the orthogonal complement of the Kähler class $[\omega] \in \mathrm{H}^2(X, \mathbf{R})$ under the pairing (2).

4.1.8 *Lefschetz decomposition and Hodge index theorem.* If $k \leqslant n$ then there is an intersection form Ψ on $\mathrm{H}^k(X, \mathbf{R})$ given by

$$\Psi\colon \mathrm{H}^k(X, \mathbf{R}) \times \mathrm{H}^k(X, \mathbf{R}) \to \mathbf{R}, \qquad \Psi(\alpha, \beta) = \int \omega^{n-k} \wedge \alpha \wedge \beta.$$

It is easily checked to be symplectic bilinear when k is odd and symmetric bilinear when k is even. Another way of writing it is as $\Psi(\alpha, \beta) = \langle L^{n-k}\alpha, \beta\rangle$. There is an associated Hermitian form on $\mathrm{H}^k(X, \mathbf{C})$ given by $H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta})$. Another decomposition of the cohomology in degree k is the Lefschetz decomposition:

$$\mathrm{H}^k(X, \mathbf{C}) = \bigoplus_{2r \leqslant k} L^r \, \mathrm{H}^{k-2r}_{\mathrm{prim}}(X, \mathbf{C}).$$

Finally, the subspaces $\mathrm{H}^{p,q}(X) \subset \mathrm{H}^k(X, \mathbf{C})$ are orthogonal for the Hermitian form H_k and the Hermitian form $(-1)^{k(k-1)/2} i^{p-q-k} H_k$ is positive definite on the subspace

$$\mathrm{H}^{p,q}_{\mathrm{prim}}(X) = \mathrm{H}^k_{\mathrm{prim}}(X) \cap \mathrm{H}^{p,q}(X).$$

The latter is a consequence of what is known as the Hodge index theorem [Voi02, §6].

4.2 Intersection forms, polarizations and integral polarized Hodge structures

The computation of the signature of H via the Hodge index theorem motivates the following general definition of an integral polarized Hodge structure:

Definition 4.2 (Integral polarized Hodge structure). An integral polarized Hodge structure that is pure of weight k is a $\mathbf{Z}$-module $V_{\mathbf{Z}}$ equipped with a symmetric (resp. alternating) bilinear pairing

$$\Psi\colon V_{\mathbf{Z}} \times V_{\mathbf{Z}} \to \mathbf{Z}$$

if k is even (resp., odd), together with a Hodge decomposition $V_{\mathbf{C}} = \bigoplus V^{p,q}$ that is pure of weight k such that if

$$H\colon V_{\mathbf{C}} \times V_{\mathbf{C}} \to \mathbf{C}, \qquad H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta})$$

is the associated Hermitian form then the $V^{p,q}$ are orthogonal with respect to H.

Remark 4. The definition is apparently motivated by the case of a Kähler manifold X where $V_{\mathbf{Z}} = \mathrm{H}^k(X, \mathbf{Z})$ and $\Psi(\alpha, \beta) = \int \omega^{n-k} \wedge \alpha \wedge \beta$ begin the intersection pairing which is 1) symmetric if k is even; 2) symplectic if k is odd.

4.2.9 *Period maps and period domains.* Let $f\colon \mathfrak{X} \to S$ be a morphism of complex manifolds and suppose that for every $s \in S$, $\mathfrak{X}_s$ is a Kähler manifold. The cohomology $\mathrm{H}^i(\mathfrak{X}_s, \mathbf{Z})$ gives us a local system $\mathcal{L} = R^i f_*(\mathbf{Z})$ on S (a locally constant sheaf of abelian groups). If S is connected, we get a representation $\rho\colon \pi_1(S, s) \to \mathrm{Aut}(\mathcal{L}_s)$, where $\mathcal{L}_s$ is that stalk of $\mathcal{F}$ at s.

For a given $s \in S$ let $V = \mathrm{H}^k(\mathfrak{X}_s, \mathbf{R})$ and let $r\colon s \mapsto s'$ be a path (or more precisely, a homothety class of paths) in S connecting s and s'. We can get different Hodge structures on the base vector space V by taking paths (up to homothety) and transporting the corresponding Hodge structure on the Känler manifold $\mathfrak{X}_{s'}$ to V via the map $\rho(r)\colon \mathcal{L}_{s'} \to \mathcal{L}_s$. That way, for each homothety class of paths, we are getting a Hodge structure, so we get a (period) map $\pi\colon \widetilde{S} \to \mathcal{D}$ where $\widetilde{S}$ is the universal covering space of S (i.e., the space of homothety classes of paths) and $\mathcal{D}$ is a domain whose points correspond to Hodge structures on V of the same type as the Hodge structure coming from $\mathfrak{X}_s$. This map measures the behavior of the period integrals $\displaystyle\int_{\gamma_s} \omega_s$ where the cycle γ_s and the k-form ω_s of type (p, q) vary continuously with s.

Example 1. Consider the famous parameter space for elliptic curves $S = \mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$ known back to Legendre (the Legendre parametrization of elliptic curves), namely, for each $\lambda \in S$, we look at the elliptic curve $E_\lambda \colon y^2 = x(x-1)(x-\lambda)$. The homology $\mathrm{H}_1(E_\lambda, \mathbf{Z})$ is generated by two loops: α around $\{0, 1\}$ and β around $\{\lambda, \infty\}$. This gives identification with $\mathbf{Z}^2$ of the polarized Hodge structure coming from the lattice $L_\lambda = \mathrm{H}^1(E_\lambda, \mathbf{Z})$. We have the Hodge decomposition $V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$ into holomorphic and anti-holomorphic differentials. The 1-dimensional $\mathbf{C}$-vector space $V^{1,0}$ is generated by $\omega = dx/y$. The universal covering space of S is known to be the Poincare upper-half plane $\mathcal{H}_1$. Indeed, the covering map $\mathcal{H}_1 \to S$ is given by the modular λ-function and the symmetry group is $\pi_1(S) = \Gamma(2) \cong F_2$. The period map is thus

$$\mathcal{H}_1 \to \mathcal{D}, \qquad \lambda \mapsto \left(\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \int_\lambda^\infty \frac{dx}{x(x-1)(x-\lambda)} \right) \in \mathbf{C}^2.$$

The ratio of the two periods turns out to be precisely $\lambda \in \mathcal{H}_1$, so the period map

$$\pi \colon \widetilde{S} = \mathcal{H}_1 \to \mathcal{H}_1 = \mathcal{D}$$

is the trivial map.

4.2.10 *Variation of Hodge structures.* The more general definition of variation of Hodge structures from last time was not sufficiently well-motivated. The reason we do the digression on Kähler manifolds is to motivate it naturally. Indeed, starting from a morpshim $f \colon \mathfrak{X} \to S$ as above (each fiber $\mathfrak{X}_s$ is a compact Kähler manifold), we get a "family" of Hodge structures on S coming from the standard Hodge structures $h_s = \{\mathrm{H}^{p,q}(\mathfrak{X}_s)\}$. In addition, if we restrict to the primitive cohomology $\mathrm{H}^k_{\mathrm{prim}}(\mathfrak{X}_s, \mathbf{R}) \subset \mathrm{H}^k(\mathfrak{X}_s, \mathbf{R})$, we also get an integral polarized Hodge structure. If we start with the period maps that we saw last time, namely,

$$\varphi \colon S \to \mathbf{Gr}_{d_1,\ldots,d_r}(V_{\mathbf{C}}), \qquad s \mapsto F^\bullet(h_s).$$

then one can show that these satisfy the following two properties:

- (Holomorphicity): φ is holomorphic [Voi02, Thm.10.9],
- (Griffiths' transversality): The image of
$$d\varphi \colon T_s S \mapsto T_{F^\bullet(h_s)}\, \mathbf{Gr}_{d_1,\ldots,d_r}(V_{\mathbf{C}})$$
is contained in $\bigoplus_p \mathrm{Hom}(F^p(h_s), F^{p-1}(h_s)/F^p(h_s))$ [Voi02, Prop.10.12].

Thus, variations of Hodge structures arise very naturally as generalizations of families of Kähler manifolds and their associated Hodge structures. We will not give a proof of Griffiths transversality here, but will try to spell out some of the details in the example below (which is relevant when we talk about Shimura varieties of orthogonal type).

4.2.11 *Example: domains of orthogonal type.* The following exercise will be very relevant for the later discussion of Shimura varieties for orthogonal groups:

Exercise 5. Let V be an $\mathbf{R}$-vector space of dimension n and let $\psi \colon V \times V \to \mathbf{R}$ be a symmetric bilinear form.

(i) Show that if V has a polarized Hodge structure $\{V^{p,q}\}$ of weight 2 then the signature of the quadratic space (V, ψ) is $(2h^{2,0}, n - 2h^{2,0})$ where $h^{2,0} = \dim V^{2,0}$. Thus, if $h^{2,0} = 1$ (this occurs for, e.g., Hodge structures for K3 surfaces), the signature is $(2, n-2)$.

(ii) Show that in the case $h^{2,0} = 1$, the polarized Hodge structures are parametrized by the domain

$$\mathcal{D}_n = \{u \in \mathbf{P}(V_{\mathbf{C}}) \colon \psi(u, u) = 0,\ \psi(u, \overline{u}) > 0\}.$$

This particular description of D helps us easily write down the Hodge structure corresponding to $u \in \mathbf{P}(V_{\mathbf{C}})$: namely, $V^{2,0} = u$, $V^{1,1} = (u \oplus \overline{u})^{\perp}$ and $V^{0,2} = \overline{u}$, so one easily checks that the Hodge filtration is $F^0 = V_{\mathbf{C}}$, $F^1 = u^{\perp}$, $F^2 = u$. I will not do it in detail, but rather leave it as a homework.

Exercise 6. Using the fact that a section for the line bundle determined by F_u^2 is a choice of a point $u(t) = (u_1(t), \ldots, u_n(t))$ on the line u for each fiber, show (using the fact that $\Psi(u, u) = 0$) that the derivative of $u(t)$ lands in F_u^1/F_u^2.

We will now try to compute the period map. Consider smooth quartic surfaces in $\mathbf{P}^3$, i.e., quartics given by a homogeneous polynomial f with $\deg f = 4$. The projective space of quartic surfaces in $\mathbf{P}^3$ has dimension $\binom{4+4-1}{4-1} - 1 = 34$. The space of non-singular projective quartic surfaces is thus an open subset $U \subset \mathbf{P}^{34}$ (and is also a subspace of the moduli space of K3 surfaces). The group $\mathbf{PGL}_4(\mathbf{C})$ (having dimension 15) acts on U and hence, the moduli space of smooth quartics has dimension $\dim U/\mathbf{PGL}_4(\mathbf{C}) = 34 - 15 = 19$. The first computation (which we omit) is showing that if X is any smooth quartic surface in $\mathbf{P}^3$ (in particular, a K3 surface), then $\mathrm{H}^1(X, \mathbf{Z}) = 0$, $\widetilde{L} = \mathrm{H}^2(X, \mathbf{Z})$ is torsion-free and $\operatorname{rank} \mathrm{H}^2(X, \mathbf{Z}) = 22$. We get a Hodge decomposition:

$$\mathrm{H}^2(X, \mathbf{C}) = \mathrm{H}^{2,0}(X, \mathbf{C}) \oplus \mathrm{H}^{0,2}(X, \mathbf{C}) \oplus \mathrm{H}^{1,1}(X, \mathbf{C}),$$

where $\dim \mathrm{H}^{2,0}(X, \mathbf{C}) = \dim \mathrm{H}^{0,2}(X, \mathbf{C}) = 1$ and $\dim \mathrm{H}^{1,1}(X, \mathbf{C}) = 20$. So far, this Hodge structure is not polarized. Yet, by considering the primitive cohomology $\mathrm{H}^2_{\text{prim}}(X, \mathbf{C})$, we get an integral (by taking $\mathbf{Z}$-coefficients) polarized Hodge structure (21-dimensional lattice) $L \subset \widetilde{L}$ of type $h^{2,0} = 1 = h^{0,2}$ and $h^{1,1} = 19$. We thus get the period map

$$\{\text{smooth quartics in } \mathbf{P}^3\} \cong U/\mathbf{PGL}_4(\mathbf{C}) \to \mathcal{D}_{21} = \{\text{polarized Hodge structures of type (1,1,19)}\}.$$

By Torelli's theorem for K3 surfaces proved by Piatetskii-Shapiro and Shafarevich [Voi02, Thm.7.21], this map is an open immersion (this is similar to the case of abelian varieties where the isomorphism class of an abelian variety over $\mathbf{C}$ is determined by the period matrix; here, we integrate 2-forms).

5. Axiomatic Description of Shimura Data

Let $\mathbf{G}$ be a reductive group over $\mathbf{Q}$. We start with an algebraic morphism $h\colon \mathbf{G}_{m,\mathbf{R}} \to \mathbf{G}_{\mathbf{R}}$ and as before, let X denote the $\mathbf{G}(\mathbf{R})$-conjugacy class of this morphism. Let $K_h \subset \mathbf{G}(\mathbf{R})$ be the centralizer of h. Then the orbit map identifies $X \cong \mathbf{G}(\mathbf{R})/K_h$. We would like to study when one can put a complex structure on X.

Example 2. Recall that for the case of unitary groups, we had $X_h \cong U(p,q)/U(p) \times U(q)$ where $h(a) \in \mathbf{GU}(p,q)$ was defined as the endomorphism $a \cdot (v_+ + v_-) = av_+ + \overline{a}v_-$.

Now, suppose that $\rho\colon\ \mathbf{G}(\mathbf{R}) \to \mathbf{GL}(V)$ is any representation and $h\colon\ \mathbf{S} \to \mathbf{G}_{\mathbf{R}}$ is any algebraic homomorphism. Then the composition $\rho \circ h$ is a representation of the circle group, hence, we can use the description from the previous handout to get a Hodge structure corresponding to h.

5.1 The Hodge structure corresponding to h

5.2 Variation of Hodge structures

What we would like to do is the following: suppose that we fix the weight n and the Hodge numbers $h^{p,q} = \dim V^{p,q}$ for all (p,q) with $p + q = n$. Let $\mathcal{S}$ be the moduli space of all polarized Hodge structures **Dimitar** : **Define polarized** with these discrete invariants. We would like to address the following question:

QUESTION 5.1. *Is there a family $\mathcal{X} \to \mathcal{S}$ such that for any $s \in M$ the Hodge structure corresponding to s is the middle cohomology of the variety $\mathcal{X}_s$?*

5.3 Hermitian symmetric domains as parameter spaces for certain Hodge structures

We would like to fix an $\mathbf{R}$-vector space V and consider Hodge structures on V parametrized by the points on some base $\mathcal{S}$ that is a complex manifold. This means that for any $s \in \mathcal{S}$, we have a Hodge structure of weight n that we denote by $h_s = \{V_s^{p,q}\}_{p+q=n}$. We would like these Hodge structures to vary continuously in the following sense:

- $\dim V_s^{p,q} = d(p,q)$ is constant,
- $d^{p,q} \colon \mathcal{S} \to \mathbf{Gr}_{d(p,q)}(V),\ s \mapsto V_s^{p,q}$ is continuous.

Moreover, we would like some holomorphicity condition. **Dimitar : State it precisely.** With these in mind, we can define what it means for a family of Hodge structures $\{h_s\}_{s\in\mathcal{S}}$ to be a *variation of Hodge structure.*

5.4 Griffiths transversality

Assume that the datum $(\mathbf{G}, X)$ satisfies Property 0. Take any algebraic representation $\rho\colon \mathbf{G}(R) \to \mathrm{Aut}(V)$ on a real vector space V. There is a weight decomposition $V = \bigoplus V_n$ such that $V_{n,\mathbf{C}}$ is the weight-n part of the Hodge decomposition $V_{\mathbf{C}} = \bigoplus V^{p,q}$ corresponding to $\rho \circ h$ for any h. For each n, for each $h \in X$ and each ρ, we consider the Hodge filtration $\{F_n^p(h,\rho)\}$. We would like to study the variation of this Hodge filtration as we vary $h \in X$.

Dimitar : Assume the existence of a holomorphic structure on X for the moment.

Consider the holomorphic vector bundle $\mathcal{H}_n(\rho) = X \times V_{n,\mathbf{C}}$ and the subbundle $\mathcal{F}_n^p(\rho) \subset \mathcal{H}_n(\rho)$ whose fibers are the spaces $F_n^p(h,\rho) \subset V_{n,\mathbf{C}}$. It turns out that there is a very special condition making this decreasing chain of subbundles holomorphic (recall that an n-dimensional holomorphic bundle over a complex manifold X is a vector bundle $\pi\colon E \to X$ such that the total space E is a complex manifold and π is holomorphic; equivalently, the trivialization maps $\pi^{-1}(U) \cong U \times \mathbf{C}^n$ are biholomorphic and the transition maps $t_{U,V}\colon U \cap V \to \mathbf{GL}_n(\mathbf{C})$ are holomorphic).

5.4.12 *Intuitive idea (motivation for Griffiths transversality).*

- If $f\colon Y \to S$ is a morphism of smooth algebraic varieties then one can consider the cohomology of the fibers $\mathrm{H}^n(Y_s, \mathbf{C})$ and the associated Hodge structure. By homological methods, the Hodge filtration yields a filtration on relative deRham cohomology
$$\mathrm{H}^n_{\mathrm{dR}}(Y/S) = R^n f_*(f^{-1}\mathcal{O}_S) \cong R^n f_*(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_S,$$
by $\mathcal{O}_S$-submodules $\mathcal{F}^p$.
- **Ehresmann's theorem:** $\mathrm{H}^n(Y_s, \mathbf{C})$ are locally constant.
- Yet, the complex structures on the fibers may vary. This can be detected by studying the Hodge structures on the fibers $\mathrm{H}^n(Y_s, \mathbf{C})$. Mapping a point to the corresponding Hodge filtration gives us a map from S to the flag variety corresponding to the Hodge filtration $\{\mathcal{F}^p\}$.
- **Griffiths transversality:** a property satisfied by these filtrations with respect to the first derivative
$$\nabla = 1 \otimes d\colon\ \mathrm{H}^n_{\mathrm{dR}}(X/S) \to \mathrm{H}^n_{\mathrm{dR}}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_S.$$

5.4.13 *Formal statement.* Here, we will state the theorem and will prove it and explain in in more detail in the next section.

Theorem 5.2 (Griffiths transversality). *Setting as above. Assume that the vector subbundles* $\{\mathcal{F}_n^p(\rho)\}$ *are holomorphic. If* $\nabla_\rho = 1 \otimes d\colon V_{n,\mathbf{C}} \otimes_{\mathbf{C}} \mathcal{O}_X \to V_{n,\mathbf{C}} \otimes_{\mathbf{C}} \Omega_X^1 = \mathcal{H}_n(\rho) \otimes_{\mathcal{O}_X} \Omega_X^1$ *is the derivation map then*

$$\nabla_\rho(\mathcal{F}_n^p(\rho)) \subset \mathcal{F}_n^{p-1}(\rho) \qquad \textit{(Griffiths transversality)}$$

if and only if for every h*, the weight zero Hodge structure* $\{V_h^{p,-p}\}_{p\in\mathbf{Z}}$ *on* $\mathfrak{g}$ *associated to the adjoint representation* $Ad_{\mathbf{G}(\mathbf{R})} \circ h$ *is of type* $\{(1,-1),(0,0),(-1,1)\}$*. In addition, the latter is equivalent to the transversality condition holding for a single* ρ *that is faithful.*

6. Variations of Hodge structures and Hermitian symmetric domains

6.1 Flag varieties

Here, $V = V_{\mathbf{C}}$ will be a complex vector space. Fix a sequence of integers $n = d_0 > d_1 > \cdots > d_r > 0$ and consider $\mathbf{Gr}_{d_0,d_1,\ldots,d_r}(V)$ to be the set of flags $F\colon V \supset V^1 \supset V^2 \supset \cdots \supset V^r \supset 0$.

6.1.14 $\mathbf{Gr}_{d_1,\ldots,d_r}(V)$ *as an algebraic variety.* Recall that map

$$W \mapsto \bigwedge^d W\colon\ \mathbf{Gr}_d(V) \to \mathbf{P}(\bigwedge^d V)$$

embeds the Grassmannian as a closed subset of $\mathbf{P}(\bigwedge^d V)$, hence $\mathbf{Gr}_d(V)$ is a projective variety. For general flags, consider

$$F \mapsto (V_i)\colon\ \mathbf{Gr}_{d_1,\ldots,d_n}(V) \hookrightarrow \prod_i \mathbf{Gr}_{d_i}(V) \subset \prod_i \mathbf{P}(\bigwedge^{d_i} V).$$

6.1.15 *The tangent space at a flag.* Given a flag $V \supset V^1 \supset V^2 \supset \cdots \supset V^r \supset \{0\}$, the tangent space at this flag is defined as all sequences $\{\phi_i \in\colon V^i \to V/V^i\colon i = 1,\ldots,r\}$ such that

$$\phi_i|_{V^{i+1}} \equiv \phi_{i+1} \bmod V_i, \qquad i = 1,\ldots,r-1.$$

This can be easily seen if we think of the flag variety $\mathbf{Gr}_{d_1,\ldots,d_r}(V)$ as a homogeneous space: indeed, choosing a flag $F^\bullet$ gives us an identification $\mathbf{GL}(V)/\mathbf{P}_{F^\bullet}(V)$, where $\mathbf{P}_{F^\bullet}(V) \subset \mathbf{GL}(V)$ is the parabolic subgroup fixing the flag $F^\bullet$. The tangent space $T_{F^\bullet}\mathbf{Gr}_{d_1,\ldots,d_r}(V)$ is then the quotient of the two Lie algebras, i.e., $\mathfrak{gl}(V)/\mathfrak{p}_{F^\bullet}(V)$, where $\mathfrak{p}_{\mathbf{F}^\bullet}(V) = \mathrm{End}_{F^\bullet}(V) = \{\phi\colon V \to V\colon \phi(V^i) \subset V^i\}$ is the Lie algebra of the Lie group $\mathbf{P}_{F^\bullet}(V)$ (considered as a Lie group over $\mathbf{C}$).

6.2 Variation of Hodge structures

If S is a connected complex manifold whose points correspond to Hodge structures that vary continuously, i.e., we have $\{h_s\}_{s\in S}$, where $h_s = \{V_s^{p,q}\}$ is a Hodge structure on V that is pure of a fixed weight n. In particular, suppose that the dimension of the corresponding Hodge filtrations are constant, i.e., $\dim F_s^p = d_p$ for $1 \leqslant p \leqslant r$. This gives us a holomorphic map

$$\varphi\colon S \to \mathbf{Gr}_{d_1,\ldots,d_r}(V), \qquad \varphi\colon s \mapsto F_s^\bullet$$

and so, an induced map on tangent spaces

$$d\varphi_s\colon T_sS \to T_{F_s^\bullet} \subset \bigoplus_{p=1}^r \mathrm{Hom}(F_s^p, V/F_s^p).$$

A quick observation is that φ is injective (this follows from the purity condition since $V^{p,q} = F^p \cap \overline{F^q}$, i.e., the Hodge filtration recovers uniquely the Hodge structure whenever the Hodge structure is pure).

Definition 6.1 (Variation of Hodge structures). We say that $\{h_s\}_{s\in S}$ is a variation of Hodge structures if the image of the above map $d\varphi_s$ lands in $\bigoplus_{p=1}^{r} \mathrm{Hom}(F_s^p, F_s^{p-1}/F_s^p)$, i.e., if any first-order direction in S makes the vector subbundle $\mathcal{F}^p = \{F_s^p\}$ of the trivial bundle $\mathcal{V} \cong V \times S$ vary at most within the subbundle $\mathcal{F}^{p-1} \subset \mathcal{V}$.

6.3 Proof of Theorem 5.2

Suppose that $\rho_{\mathbf{R}}\colon \mathbf{G}(\mathbf{R}) \to \mathbf{GL}(V)$ is an algebraic representation. Recall that

6.3.16 *Hodge structure on duals and tensor products.* Let V be a real vector space with a Hodge structure on its complexification. Then V^* has a natural Hodge structure defined by $(V^*)^{p,q} = (V^{-p,-q})^{\perp}$ so that $F^p(V^*) = (F^{1-p})^{\perp}$. In addition, if V' and V'' be two real vector spaces with Hodge structures on their complexifications, then we get a Hodge structure on the complexification of $V' \otimes_{\mathbf{R}} V''$ defined by

$$(V' \otimes V'')^{p,q} = \bigoplus_{a'+a''=p, b'+b''=q} (V')^{a',b'} \otimes_{\mathbf{C}} (V'')^{a'',b''}.$$

The corresponding Hodge filtration satisfies $F^p(V' \otimes_{\mathbf{C}} V'') = \sum_{a'+a''=p} F^{a'}(V') \otimes F^{a''}(V'')$.

Dimitar : Double check the details.

6.3.17 *Transversality condition in terms of the Lie algebra of* $\mathfrak{gl}(V)$. The Lie algebra $\mathfrak{gl}(V)$ comes equipped with a Hodge structure arising from the Hodge structure on V coming from the representation $\rho_{\mathbf{R}} \circ h$.

- Prove that $\phi(h(z)gh(z)^{-1}) = \rho(h(z))_{\mathbf{C}}(F^{\bullet}_{g\cdot h})$.
- What does this mean? Well, $\mathbf{G}(\mathbf{R})$ has an action of $\mathbf{C}^\times$ given by h and $\mathbf{Gr}_{d_1,\dots,d_r}(V)$ has an action of $\mathbf{C}^\times$ given by $\rho_{\mathbf{R}} \circ h$ and what φ does is it intertwines the two actions.
- Since $X = \mathbf{G}(\mathbf{R})/Z_{\mathbf{G}(\mathbf{R})}(h)$ is a homogeneous space, then $T_h(X)$ can be identified with $\mathfrak{g}/\mathfrak{g}^{0,0}$ where $\mathfrak{g}^{0,0}$ is the $\mathbf{R}$-descent of the $(0,0)$-piece $\mathfrak{g}_{\mathbf{C}}^{0,0} \subset \mathfrak{g}_{\mathbf{C}}$ of the Hodge structure determined by $Ad_{\mathbf{G}(\mathbf{R})} \circ h$ (just as in the $\mathbf{G} = \mathbf{GL}_2$-case).
- The transversality condition is equivalent to $d\varphi_s(\mathfrak{g}/\mathfrak{g}^{0,0}) \subset F^{-1}(\mathfrak{gl}(V))/F^0(\mathfrak{gl}(V))$.

7. Examples

7.1 The Siegel upper-half space $\mathcal{H}_g$

Dimitar : Describe the Siegel upper-half space as the set of polarized Hodge structures on $\mathbf{R}^{2g}$.

7.2 Six points in $\mathbf{P}^1$

Fixing $\alpha = (Q_1, \dots, Q_6) \in (\mathbf{P}^1)^6$, we consider the genus 4 curve in $\mathbf{P}^2$

$$C_\alpha\colon Y^3Z^3 = \prod_{i=1}^{6}(X - X(Q_i)Z).$$

There is an obvious action of $R = \mathbf{Z}[\mu_3] \subset \mathbf{Q}(\mu_3)$ on C_α and hence, on $L = \mathrm{H}^1(C_\alpha, \mathbf{Z})$ and this $\mathcal{O}$-lattice is free of rank 4 **Dimitar : Explain why?** It lives in the complex vector space $V_{\mathbf{C}} = \mathrm{H}^1(C_\alpha, \mathbf{C})$.

There is a complex conjugation coming from the action of μ_3 on this space yielding an eigenspace decomposition $V_{\mathbf{C}} = V^+ \oplus V^-$. Now, $L \otimes_R \mathbf{C} = V^+$ **Dimitar : Mention why?** . There is a Hermitian form on V^+ coming from the intersection form: $H(x) = \langle x, \overline{x} \rangle$ whose signature can be shown to be $(3,1)$. But $\mathbf{C}^\times$ acts on V^+ and yields a Witt decomposition $V^+ = V^{1,0} \oplus V^{0,1}$ where $H|_{V^{1,0}}$ is positive-definite and $H|_{V^{0,1}}$ is negative-definite. Let $D_\alpha = V^{0,1}$. Then we get a map

$$\phi\colon \{(Q_1, \ldots, Q_6)\colon Q_i \in \mathbf{P}^1\} \to \mathcal{S} = \Gamma \backslash \mathbf{SU}(3,1)/\mathbf{SU}(3), \qquad \phi(\alpha) = D_\alpha, \tag{3}$$

where $\Gamma = \mathbf{SU}_{3,1}(\mathcal{O})$ is the stabilizer of L in $\mathbf{SU}(3,1)(E)$ for $E = \mathbf{Q}(\mu_3)$.

Dimitar : Need to explain why the signature of H is $(3,1)$ - in fact, a basis for $\mathrm{H}^0(C_\alpha, \Omega^1)$ is given by $\left\{ \dfrac{dx}{y}, \dfrac{dx}{y^2}, \dfrac{xdx}{y^2}, \dfrac{x^2dx}{y^2} \right\}$ and the negative-definite part is spanned by the first differential. Show the details of the computation.

References

Del79 P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289.

Voi02 C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002, Translated from the French original by Leila Schneps.