Introduction to Shimura Varieties

1. Introduction

Shimura varieties play a crucial role for providing a link between two major aspects of modern number theory: the automorphic and arithmetic aspect. The interplay between these two dates back to the mid-19th century with the work of Kronecker and Kummer. It has later been taken over by Hilbert and some of his students in the attempt to describe explicitly abelian extensions of number fields in a similar way to the field of rational numbers (Kronecker–Weber's theorem) and quadratic imaginary fields (Kronecker Jugendtraum). With the remarkable discoveries and developments in the arithmetic properties due to Shimura in the 1950s and the later generalization to arbitrary reductive groups by Deligne, Shimura varieties took a central role in the Langlands program, in particular, in the construction of Galois representations associated to automorphic forms. The latter turned out to be key in the study of elliptic curves and the proof of Fermat's last theorem as well as in the major developments related to the Birch and Swinnerton-Dyer conjecture.

To explain in a little more detail the importance of Shimura's contribution, we take as an example the quotient of the Poincaré upper half plane $\mathcal{H}_1 = \{z \in \mathbf{C} \colon \operatorname{im}(z) > 0\}$ by the left action of congruence subgroups of $\mathbf{SL}_2(\mathbf{Z})$ via linear fractional transformation, i.e., $Y(N) = \Gamma(N)\backslash\mathcal{H}_1$ where $\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) \colon \begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \bmod N\right\}$. The latter inherits the complex structure on $\mathbf{C}$ and it is not hard to check that it has the structure of a complex analytic manifold. There are general methods (e.g., Serre's GAGA principle) that tell us that often[1] such complex analytic manifolds have algebraic structure (i.e., can be viewed as algebraic varieties over $\mathbf{C}$). Thus, it is not surprising that one can look at $Y(N)$ as a complex curve. The remarkable discovery of Shimura was that the latter are in fact defined over number fields.

Here is some historical account:

- **Kronecker Jugendtraum (1800s):** Arithmetic properties of elliptic modular functions and modular forms turned out to be important for the beautiful Kronecker Jugendtraum.
- **Hilbert's 12th problem (1900):** Aims at extending the Kronecker–Weber theorem on the abelian extensions of $\mathbf{Q}$. More precisely, find analogues of the roots of unity that can be used to completely describe the abelian extensions of the field.
- **Blumenthal (1903):** Studied modular forms for real quadratic fields. A student of Hilbert, this gave rise to Hilbert–Blumenthal varieties, the first examples of Shimura varieties.
- **Hecke (1912):** Using Hlibert modular forms, attempted to study abelian extensions of real quadratic fields.
- **Taniyama–Shimura (1960):** Developed the theory of abelian varieties with complex multiplication, subsequently leading to the connection between elliptic curves and modular forms (i.e., elliptic curves over $\mathbf{Q}$ are modular) that is now a theorem due to Wiles et al.
- **Langlands (1973):** The modern version of Shimura varieties should deal with the Hasse–Weil Zeta function of a Shimura variety. These should be considered as sources for Galois

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[1]Saying often because the original Serre's GAGA applies to compact manifolds and $Y(N)$ is not compact.

representations associated to automorphic forms.

- **Mazur (1977):** Modular curves and the Eisenstein ideal - congruences between modular forms turned out to be relevant to the proof of Fermat's last theorem.
- **Deligne (1979):** Recasts the theory of Shimura on the language of reductive groups; extension of results on canonical models.

2. The course

This course is a basic introduction to Shimura varieties and the work of Shimura, subsequently revisited and generalized by Deligne.

2.1 Course syllabus

Tentative topics that I am planning to cover are the following:

- Basic notions from differential geometry, complex manifolds, symmetric spaces, hermitian symmetric domains. The lecture will include important examples of hermitian symmetric domains for the symplectic groups and the unitary groups. We will follow [Mil05] with certain references to [Hel78]. We will look at examples such as the Poincaré upper-half plane, the Siegel upper-half space as well as domains associated to unitary groups.
- Introducing the notion of a Hodge structure. Here, I will follow some notes due to Brian Conrad.
- Defining Shimura data, connected Shimura varieties and Shimura varieties à la Deligne [Del71] and understanding the motivation behind these.
- Shimura varieties of PEL type (will recall the theory of abelian varieties over the complex numbers and will discuss certain moduli problems).
- CM theory and special points.
- Theory of canonical models.
- Galois action on special points and on connected components, reciprocity laws.

2.2 Grading

If you need a grade for the course, you have two options: 1) you do a final project (paper) and a presentation (in which case you have to discuss with me in advance); 2) You turn in homeworks (these appear in the lecture notes and are marked as **exercises**). If you choose the second option, you do not need to turn in all the exercises, yet, you are allowed to select exercises that you find relevant for a better understanding of the course.

3. Hermitian Symmetric Domains

Here, we will assume that (M, g) is a real Riemannian manifold with Riemannian metric g. Typical examples we will consider are

- $\mathbf{R}^n$ with the Euclidean metric $g = dx^2 + dy^2$.
- The Poincaré upper-half plane $\mathcal{H}_1 = \{z \in \mathbf{C} \colon \operatorname{im}(z) > 0\}$ together with the Poincaré metric $g = \dfrac{(dx)^2 + (dy)^2}{y^2}$. Note that $\mathbf{SL}_2(\mathbf{R})$ acts on $\mathcal{H}_1$ by linear fractional transformations, i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$.

By an *isometry* we will mean a morphism that preserves the metric. For instance

- Rotations or translations in $\mathbf{R}^n$ are isometries since they preserve the standard metric g.
- $\mathbf{SL}_2(\mathbf{R})$ acts on $\mathcal{H}_1$ by isometries (the Poincaré metric is preserved by $\mathbf{SL}_2(\mathbf{R})$).

Exercise 1. Prove that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{R})$, $z \mapsto \gamma \cdot z := \dfrac{az+b}{cz+d} \colon \mathcal{H}_1 \to \mathcal{H}_1$ is an isometry with respect to the Poincaré metric g defined above..

3.1 Complex structures and almost complex structures

3.1.1 *Complex manifolds and complex structures.* A **complex manifold** M is given by an atlas of coordinate charts to $\mathbf{C}^n$ such that the transition maps are holomorphic. If M is a smooth real manifold then a **complex structure** on M is an atlas of coordinate charts $\mathbf{C}^n$ such that the transition maps are holomorphic. Note that a real manifold M may or may not have a complex structure (e.g., $M = \mathbf{R}^{2n+1}$ has no complex structure, but $M = \mathbf{R}^{2n}$ has).

3.1.2 *Almost-complex structure.* To understand complex structures on real manifolds, we need the weaker notion of an **almost-complex structure**. An almost complex structure is a smoothly varying family of complex structures on the tangent spaces of the manifold, i.e., a smooth tensor field $(J_p)_{p\in M}$, $J_p \colon T_pM \to T_pM$, $J_p^2 = -1$, where T_pM denotes the tangent space at p. A complex structure on a manifold M induces an almost complex structure in the following way: if $z_1, \ldots, z_n$ are the coordinates and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are the real coordinates then

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}, \qquad \frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}, \tag{1}$$

gives an almost-complex structure on M.

3.1.3 *Integrability.* An almost-complex structure may or may not arise from a complex structure. The notion of **integrability** of an almost-complex structure allows us to understand when an almost-complex structure does arise from a complex structure. A almost-complex structure is integrable if there is an atlas of charts and local coordinates so that J is given via (1). If (M, J) is an almost complex manifold and if J is integrable then M is a complex manifold (the formal statement is known as the theorem of Newlander–Nierenberg).

3.1.4 *Hermitian manifold.* A Hermitian manifold is a Riemannian manifold (M, g) together with a complex structure J that acts by isometries, i.e., $g(JX, JY) = g(X, Y)$ for any two vector fields X, Y (such a metric g is called a Hermitian metric).

3.2 Symmetric spaces

A Riemannian manifold (M, g) is said to be **symmetric** if its automorphism group acts transitively and if there exists a point p for which there is an automorphism s_p of (M, g) such that $s_p^2 = 1$ and p is the unique fixed point of s_p in some neighborhood of p. A **Hermitian symmetric space** is a Hermitian manifold that is connected and symmetric. Before we provide the classification of Hermitian symmetric spaces, we discuss a bit more the group of isometries of a symmetric space.

We consider three main examples:

Example 1. The Poincaré upper half plane is a homogeneous space (i.e., the automorphism group acts transitively). Indeed, if $z = x + iy \in \mathcal{H}_1$ is any point then

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \cdot i = x + iy = z.$$

Here, we have used $\mathrm{im}(z) > 0$. In addition, the upper-half plane $\mathcal{H}_1$ has a symmetry $s_i \colon \mathcal{H}_1 \to \mathcal{H}_1$ at $z = i$ defined by $s_i(z) = -1/z$ for i is an isolated fixed point.

Example 2. The Riemann sphere $\mathbf{P}^1(\mathbf{C})$ is a Hermitian symmetric space when endowed with the metric coming from $\mathbf{R}^3$. Here, rotations act transitively and are isometries. In addition, take any point $p \in \mathbf{P}^1(\mathbf{C})$. For any point p', consider the geodesic connecting p and p' (the large circle) and define $s_p(p') = p''$, the reflected point across this geodesic. One can check that this is an isometry.

Example 3. Let $\Lambda \subset \mathbf{C}$ and consider $M = \mathbf{C}/\Lambda$ endowed with the standard metric. Clearly, the group of translations acts transitively (translations are isometries) and clearly $z \mapsto -z$ is a symmetry having $0 + \Lambda$ as the unique fixed point (here, we use discrete).

3.2.5 *Groups of isometries of symmetric spaces.* To get the link between symmetric spaces and topological groups, we need to look at groups of isometries. If (M, g) is a symmetric space then we look at the group $\mathbf{Is}(M, g)$ of diffeomorphisms $M \to M$ that are also isometries for the Riemann metric. Given any point $p \in M$, one can consider the stabilizer K_p of p for the action of $\mathbf{Is}(M, g)$ on M. One endows $\mathbf{Is}(M, g)$ with the compact-open topology to make it a topological group (we need to show that multiplication and inverse are continuous for this topology).

Exercise 2. Show that $m \colon \mathbf{Is}(M, g) \times \mathbf{Is}(M, g) \to \mathbf{Is}(M, g)$ given by $m(x, y) = xy$ and $\iota \colon \mathbf{Is}(M, g) \to \mathbf{Is}(M, g)$ given by $\iota(x) = x^{-1}$ are continuous maps when $\mathbf{Is}(M, g)$ is equipped with the compact-open topology.

Let $\mathbf{Is}(M, g)^+$ indicate the connected component at the identity. We first note that $\mathbf{Is}(M, g)$ is a locally compact topological groups and the stabilizer of a point $p \in M$ in $\mathbf{Is}(M, g)$ is compact.

In fact, you will show this in the following two exercise:

Exercise 3. Show that the compact-open topology turned $\mathbf{Is}(M, g)$ into a locally compact topological space via the following steps:

(i) Show that if $\{f_i \colon f_i \in \mathbf{Is}(M, g)\}$ is a sequence for which there exists a point $p \in M$ such that the sequence $\{f_i(p)\}$ is convergent then the sequence $\{f_i\}$ has an accumulation point in $\mathbf{Is}(M, g)$ with respect to the compact-open topology.

(ii) Given a point $p \in M$ and a open relatively compact neighborhood U of p (i.e., having compact closure), consider $W(\{p\}, U) = \{f \in \mathbf{Is}(M, g) \colon f(p) \in U\}$ (which is open by the definition of the compact-open topology). Using (i), show that it has a compact closure. Since the stabilizer of p is a closed subset of $W(\{p\}, U)$, it is necessarily compact.

Next, we distinguish three types of groups of automorphisms:

- The real-analytic isometries $\mathbf{Is}(M^\infty, g)$,
- The automorphisms $\mathrm{Hol}(M)$ of M as a complex manifold,
- The holomorphic isometries $\mathbf{Is}(M, g)$; clearly, $\mathbf{Is}(M, g) = \mathbf{Is}(M^\infty, g) \cap \mathrm{Hol}(M)$.

Remark 1. We will see that for the case of hermitian symmetric domains, the inclusions $\mathbf{Is}(M^\infty, g) \supset \mathbf{Is}(M, g) \subset \mathrm{Hol}(M)$ induce equalities of the connected components of the identity for the three groups.

Example 4. (upper-half plane $\mathcal{H}_1$) The group of isometries of $\mathcal{H}_1$ is generated by the holomorphic isometries together with the following anti-holomorphic isometry: $z \mapsto \overline{z}^{-1}$ (we need to check that this is an isometry).

Exercise 4. Show that $z \mapsto \overline{z}^{-1}$ is an isometry of $\mathcal{H}_1$ that is, in addition, anti-holomorphic.

3.3 Classification of Hermitian symmetric spaces

For the classification of Hermitian symmetric spaces, we first look at the group $\mathbf{Is}(M^\infty, g)$ of all isometries (as a real Riemannian manifold) and look at the subgroup $\mathbf{Is}(M, g)$ of holomorphic ones (a closed subgroup and hence, a Lie subgroup). Recall that a Lie group is called **adjoint** if it is semi-simple and has a trivial center. To give the classification, we distinguish three families of Hermitian symmetric spaces:

- **Non-compact type** (such as $\mathcal{H}_1$): negative curvature, non-compact and adjoint group $\mathbf{Is}(M, g)^+$, simply-connected.
- **Compact type** (such as $\mathbf{P}^1(\mathbf{C})$): positive curvature, compact and adjoint group $\mathbf{Is}(M, g)^+$, simply connected.
- **Euclidean type** (such as $\mathbf{C}/\Lambda$): zero curvature, not necessarily simply connected (e.g., torus). These are quotients of $\mathbf{C}^g$ by discrete additive subgroups.

A good reference for the classification is, e.g., [Hel78, VIII]. A Hermitian symmetric space (as a Hermitian manifold) can be decomposed as $M^- \times M^+ \times M^0$ where M^- is of non-compact type, M^+ is of compact type and M^0 is of Euclidean type. An **irreducible** Hermitian symmetric space is one which is not a product of two lower-dimensional ones. The spaces M^- and M^+ are products of Hermitian symmetric spaces for which the isometry group $\mathbf{Is}(M, g)^+$ is simple.

For the theory of shimura varieties, we restrict to Hermitian symmetric spaces of non-compact type - these are called **Hermitian symmetric domains**.

3.4 Cartan involutions

Let $\mathbf{G}$ be a connected real algebraic group. Let $g \mapsto \overline{g}$ denotes the complex conjugation on $\mathbf{G}(\mathbf{C})$.

Definition 3.1 (Cartan involution). An involution $\theta\colon \mathbf{G} \to \mathbf{G}$ (a morphism of real algebraic groups) is called **Cartan** if $\mathbf{G}^{(\theta)}(\mathbf{R}) = \{g \in \mathbf{G}(\mathbf{C})\colon g = \theta(\overline{g})\}$ is compact.

Example 5. Let $\mathbf{G} = \mathbf{SL}_2$ over $\mathbf{R}$ and consider the the involution θ on $\mathbf{G}(\mathbf{R})$ given by ad $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We compute

$$\mathbf{G}^{(\theta)}(\mathbf{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{C})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \right\}.$$

It is easy to check that the latter is $\mathbf{SU}_2(\mathbf{R})$ which is compact, i.e., θ is a Cartan involution.

For the algebraic groups $\mathbf{G}$ that we will be working with (i.e., **reductive groups** - I may recall some basics on algebraic groups at some point), Cartan involutions always exist and any two are conjugated by an element of $\mathbf{G}(\mathbf{R})$. Rather than diving into more generalities on algebraic groups, it is instructive to look at some examples:

Example 6. Let V be a finite-dimensional real vector space and let $\mathbf{G} = \mathbf{GL}(V)$. Fixing a basis of V gives us an involution $\theta\colon\ \mathbf{G}(\mathbf{R}) \to \mathbf{G}(\mathbf{R})$, $M \mapsto (M^t)^{-1}$. We claim that this is a Cartan involution. It is easy to calculate that $\mathbf{G}^{(\theta)}(\mathbf{R}) = \{M \in \mathbf{GL}_n(\mathbf{C})\colon M\overline{M}^t = I_n\}$ and the latter is the compact unitary group $\mathbf{U}(n)$.

3.5 Representations of the circle group

An algebraic group that we will often need and refer to is the circle group $\mathbf{S}$. It is a real algebraic that satisfies $\mathbf{S}(\mathbf{R}) = \{z \in \mathbf{C}^\times\colon z\overline{z} = 1\}$. Note that $\mathbf{S}$ is a non-split torus over $\mathbf{R}$ and it splits over $\mathbf{C}/\mathbf{R}$, i.e., $\mathbf{S}(\mathbf{C}) \cong \mathbf{C}^\times = \mathbf{G}_m(\mathbf{C})$.

Exercise 5. The group $\mathbf{S}$ is more precisely defined as follows: for any $\mathbf{R}$-algebra R, if $R_{\mathbf{C}} = R\otimes_{\mathbf{R}}\mathbf{C}$ then the R-points on $\mathbf{S}$ are

$$\mathbf{S}(R) = \{z \in \mathbf{S}(R_{\mathbf{C}})\colon z\overline{z} = 1\},$$

where $z \mapsto \overline{z}$ denotes the complex conjugation involution. Prove that $\mathbf{S}(R_{\mathbf{C}}) \cong \mathbf{G}_m(R_{\mathbf{C}})$. (Hint: consider the isomorphism $\mathbf{C}\otimes_{\mathbf{R}} R_{\mathbf{C}} \cong R_{\mathbf{C}} \times R_{\mathbf{C}}$ given by $\alpha\otimes x \mapsto (\overline{\alpha}x, \alpha x)$).

Now, to give a finite-dimensional representation V over $\mathbf{R}$ of $\mathbf{S}(\mathbf{R})$ amounts to giving a decomposition (grading according to the characters of $\mathbf{S}$)

$$V(\mathbf{C}) = \bigoplus_{\chi\colon\ \mathbf{S}\to\mathbf{G}_m} V_\chi,$$

such that $V_{c\cdot\chi} = \overline{V_\chi}$, where c denotes complex conjugation action on the character group $X^*(\mathbf{S})$. Since the characters $X^*(\mathbf{S})$ are given by $z \mapsto z^n$, $X^*(\mathbf{S}) \cong \mathbf{Z}$ and we look at gradings $V(\mathbf{C}) = \oplus V^n$ where $\mathbf{S}$ acts on V^n via the character $z \mapsto z^n$. We easily see that complex conjugation sends the character $z \mapsto z^n$ to $z \mapsto z^{-n}$, so the representations of $\mathbf{S}(\mathbf{R})$ are direct sums of the following two representations:

- $\mathbf{R}$ with the trivial action of $\mathbf{S}(\mathbf{R})$ (this corresponds to V^0).
- $\mathbf{R}^2$ where $x+iy$ acts by the matrix $\begin{pmatrix} x & y \\ -y & x\end{pmatrix}^n$ (this corresponds to $V^n\oplus V^{-n}$).

4. Examples

4.1 Hermitian symmetric domains for symplectic groups (the Siegel upper-half space)

The Siegel upper-half space is defined by

$$\mathcal{H}_g = \{Z \in M_g(\mathbf{C})\colon Z^t = Z,\ \mathrm{im}(Z) > 0\},$$

where $\mathrm{im}(Z) > 0$ means that the imaginary part of Z is positive definite. As a complex manifold, $\mathcal{H}_g$ locally looks like $\mathbf{C}^{\frac{g(g+1)}{2}}$. If $g = 1$ then we get the Poincaré upper-half plane. First of all, we will note that there is an action of the group $\mathbf{Sp}_{2g}(\mathbf{R})$ on $\mathcal{H}_g$, but before that, let us give a definition of $\mathbf{Sp}_{2g}(\mathbf{R})$ (I will define it using the canonical symplectic form).

4.1.1 *Symplectic spaces and symplectic groups.* Let $V = \mathbf{R}^{2n}$ where we think of the vectors as the coordinates as $\mathbf{x} = (x_{-1},\dots,x_{-n},x_1,\dots,x_n)$ and consider the symplectic form $\omega\colon V\times V \to \mathbf{R}$ given by

$$\omega(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{g}(x_i y_{-i} - x_{-i}y_i).$$

We define the group of symplectic isometries as the group

$$\mathbf{Sp}(V)(\mathbf{R}) = \mathbf{Sp}_{2g}(\mathbf{R}) = \{g \in \mathbf{GL}(V)(\mathbf{R})\colon \omega(gv,gw) = \omega(v,w),\ \forall v,w\in V\}.$$

Similarly, we define the group of unitary similitudes

$$\mathbf{GSp}_{2g}(V)(\mathbf{R}) = \{g \in \mathbf{GL}(V)(\mathbf{R})\colon \exists\nu(g)\in\mathbf{R}^\times,\ \omega(gv,gw) = \nu(g)\omega(v,w),\ \forall v,w\in V\}.$$

With respect to the standard symplectic basis, we get

$$\mathbf{Sp}_{2g}(\mathbf{R}) = \left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathbf{GL}_{2g}(\mathbf{R})\colon \begin{pmatrix} A^t & C^t\\ B^t & D^t\end{pmatrix}\begin{pmatrix} 0 & -I_g\\ I_g & 0\end{pmatrix}\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \begin{pmatrix} 0 & -I_g\\ I_g & 0\end{pmatrix}\right\}.$$

Here, $J_g = \begin{pmatrix} 0 & -I_g\\ I_g & 0\end{pmatrix}$ is the matrix corresponding to the standard symplectic form on $\mathbf{R}^{2g}$.

4.1.2 *The symmetry.* One can check that the group $\mathbf{Sp}_{2g}(\mathbf{R})$ acts on $\mathcal{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

Under this action, the point $Z = iI_g \in \mathcal{H}_g$ is the only fixed for the involution $Z \mapsto J_g \cdot Z$.

4.1.3 *Comparison with the g-dimensional complex ball $\mathcal{D}_g$.* The Poincaré upper half plane $\mathcal{H}_1$ is conformally equivalent to the unit disc $\mathcal{D}_1$ via the Möbius transform

$$\mathcal{H}_1 \to \mathcal{D}_1, \qquad z \mapsto \frac{z-i}{z+i}.$$

Similarly, define the bounded domain

$$\mathcal{D}_g = \{Z \in \mathbf{GL}_g(\mathbf{C})\colon Z = Z^t,\ I_g - \overline{Z}^t Z \text{ is positive definite}\}.$$

The higher-dimensional analogue of the Möbius transform is

$$\mathcal{H}_g \to \mathcal{D}_g, \qquad Z \mapsto (Z - I_g)(Z + I_g)^{-1}.$$

Since $\mathcal{D}_g$ is a bounded domain, it has a canonical hermitian metric that has negative curvature (Bergman metric) which makes it a hermitian symmetric domain.

Remark 2. It is known (we will not prove this) that every hermitian symmetric domain can be embedded into some $\mathbf{C}^n$ as a bounded symmetric domain. Hence, every hermitian symmetric domain $\mathcal{D}$ has a hermitian metric that corresponds to the canonical Bergman metric.

4.2 Hermitian symmetric domains for unitary groups

We will now define the real unitary groups $U(p,q)$ of signature (p,q) and associate to it a symmetric domain $X = X_{p,q}$. It will be important to consider three different descriptions of the domain X.

4.2.4 *Unitary groups over* $\mathbf{R}$. Let $(V, \langle\, ,\rangle)$ be a Hermitian $\mathbf{C}$-vector space of signature (p,q) (and dimension $n = p+q$). Let

$$\mathbf{U}(V)(\mathbf{R}) = \{g \in \mathbf{GL}(V)(\mathbf{C})\colon \langle gv, gw\rangle = \langle v, w\rangle,\ \forall v, w \in V\}$$

be the group of unitary isometries and let

$$\mathbf{GU}(V)(\mathbf{R}) = \{g \in \mathbf{GL}(V)(\mathbf{C})\colon \exists \nu(g) \in \mathbf{R}^\times,\ \langle gv, gw\rangle = \nu(g)\langle v, w\rangle,\ \forall v, w \in V\}$$

be the group of unitary similitudes. We can decompose $V = V_+ \perp V_-$ where $(V_+, \langle\, ,\rangle)$ is positive definite and $(V_-, \langle\, ,\rangle)$ is negative-definite. We will often use $U(p,q)$ for $\mathbf{U}(V)(\mathbf{R})$.

4.2.5 *A matrix representation of* $U(p,q)$. Choose a basis $\mathcal{B} = \{v_1^+, \dots, v_p^+, v_1^-, \dots, v_q^-\}$ for V that diagonalizes the Hermitian form, i.e., such that $\mathcal{B}^+ = \{v_1^+, \dots, v_p^+\}$ is a basis for V^+ and $\mathcal{B}^- = v_1^-, \dots, v_q^-\}$ is a basis for V_- and the matrix for $\mathcal{B}$ is $J_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$. With respect to this basis, $U(p,q)$ is described as follows:

$$U(p,q) = \{M \in \mathbf{GL}_{p+q}(\mathbf{C})\colon \overline{M}^t J_{p,q} M = J_{p,q}\}.$$

We will typically divide each matrix $M \in U(p,q)$ into 2×2 blocks $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is a $p \times p$ matrix, B is a $p \times q$ matrix, C is $q \times p$ and D is $q \times q$.

4.2.6 *First description (as a bounded domain via operators of bounded norm).* Consider $\mathrm{Hom}(V_-, V_+)$ as a complex vector space. It comes equipped with a norm $\|\cdot\|$ (the operator norm) defined by

$$\|\varphi\| = \sup_{v \in V, v \neq 0} \sqrt{\left| \frac{\langle \varphi(v), \varphi(v) \rangle}{\langle v, v \rangle} \right|}.$$

Define a set $X = \{\varphi \in \mathrm{Hom}(V_-, V_+) \colon \|\varphi\| < 1\}$. Before we understand why X is a hermitian symmetric domain, we will define an action of $U(p, q)$ on X and will show that this action is transitive. Indeed, consider any $M \in U(p, q)$ and write it in the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as above. With respect to the same choice of basis, we notice that $A\varphi + B \in \mathrm{Hom}(V_-, V_+)$ and $C\varphi + D \in \mathrm{End}(V_-)$. We would like to define the action of M on X in a way similar to the action of $\mathbf{SL}_2(\mathbf{R})$ on the complex upper-half plane $\mathcal{H}_1$ by linear fractional transformation. To do this, note that if $\varphi \in X \subset \mathrm{Hom}_{\mathbf{C}}(V_-, V_+)$ then $A\varphi + B \in \mathrm{Hom}_{\mathbf{C}}(V_-, V_+)$ and $C\varphi + D \in \mathrm{End}_{\mathbf{C}}(V_-)$. Suppose that we know that $C\phi + D \in \mathbf{GL}(V_-)$. Then we can consider $(A\varphi + B) \circ (C\varphi + D)^{-1} \in \mathrm{Hom}_{\mathbf{C}}(V_-, V_+)$ and try to prove that it belongs to X.

Lemma 4.1. *The endomorphism $C\varphi + D \in \mathrm{End}(V_-)$ is invertible for all $\varphi \in X$.*

Proof. Assume the contrary, i.e., for some $\varphi \in X$, $\exists v_- \in V_-$ such that $C\varphi v_- + Dv_- = 0$. Consider the vector $v = \begin{pmatrix} \varphi v_- \\ v_- \end{pmatrix} \in V$. On one hand, we know that $Q(Mv) = Q(v)$ where $Q(w) = \langle w, w \rangle$ since M preserves the hermitian form. Yet, if we calculate Mv, we get $(A\varphi v_- + B\varphi v_-, 0)$, so $Q(Mv) = \|A\varphi v_- + B\varphi_-\|_V > 0$. Yet, $Q(v) = \|\varphi v_-\|_V - \|v_-\|_V < 0$, hence a contradiction. □

Next, we need to know that if $\varphi \in X$ then $M \cdot \varphi \in X$.

Lemma 4.2. *If $\varphi \in X$ then $(A\varphi + B) \circ (C\varphi + D)^{-1} \in X$.*

Proof. Take any $v_- \in V_-$ and let $w_- = (C\varphi + D)^{-1} v_-$. We then observe that

$$M \begin{pmatrix} \varphi w_- \\ w_- \end{pmatrix} = \begin{pmatrix} (M\varphi) v_- \\ v_- \end{pmatrix}.$$

As before, if $w = \begin{pmatrix} \varphi w_- \\ w_- \end{pmatrix} \in V$ then by definition of φ, $Q(w) < 0$, hence (since M preserves the Hermitian form), it follows that for $v = \begin{pmatrix} \varphi v_- \\ v_- \end{pmatrix}$, we have $Q(v) = 0$ which means that $\|M\varphi\| < 1$ since $v_- \in V_-$ was arbitrary. □

Finally, we only need to know that $U(p, q)$ acts transitively on X in order to prove the following proposition:

Proposition 4.3. *If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as above then the map $\varphi \mapsto M\varphi = (AZ + B) \circ (CZ + D)^{-1}$ defines a transitive action of $U(p, q)$ on X.*

Exercise 6. Show that the action of $U(p, q)$ on X is transitive.

Remark 3. We also need to discuss the symmetry of X at a given point in order to view X as a symmetric manifold. For the moment, we will postpone this until the discussion of the second description where the symmetry will be seen in a very general way. Yet, just as a remark, since X looks like a complex ball, it is pretty easy to guess what the symmetry of X is - we simply reflect points across the center of the ball.

Remark 4. We also need to describe the metric on this domain (since it is a bounded symmetric domain, the Bergmann metric). Again, we postpone this until the second description where it can more easily be seen via a trace form.

Remark 5. The complex structure on X is the one induced from the complex structure on $\mathrm{Hom}_{\mathbf{C}}(V_-, V_+)$ considered as a complex vector space.

4.2.7 *Second description (subspaces of complex flag varieties).* One can define the domain in the following way: consider

$$X' = \{W \subset V \colon \dim_{\mathbf{C}} W = q,\ \langle\, , \rangle|_W \text{ is negative definite}\}.$$

Clearly, the action of $U(p, q)$ on V induces an action on X'. We first discuss explicitly the $U(p, q)$-equivariant isomorphism of complex manifolds between X and X':

Lemma 4.4. *The map $\phi\colon X \to X'$ that sends $\varphi \in X$ to the subspace $W_\varphi := \{\varphi(v_-) + v_- \colon v_- \in V_-\} \subset V$ is a $U(p, q)$-equivariant isomorphism of complex manifolds.*

Proof. First, the map is well-defined since for any $\varphi \in X$ and $v_- \in V_-$, if $v = \begin{pmatrix} \varphi v_- \\ v_- \end{pmatrix}$ then $Q(v) < 0$ for the same reason is above, i.e., $\langle\, , \rangle|_{W_\varphi}$ is negative-definite. Next, take any $W \in X'$ and consider $\pi_-(W) \subseteq V_-$ where $\pi_- \colon V \to V_-$ is the projection. We claim that π_- maps W isomorphically to V_-. This is easy to check by checking that $\ker(\pi_-) \cap W$ is trivial. Indeed, if $w \in \ker(\pi_-) \cap W$ is non-zero then $w \in V_+$ and hence, $\langle w, w\rangle > 0$ contradicting the negative-definiteness of W.

We can already use this to define ϕ^{-1} as follows: given $W \in X'$, let $\phi^{-1}(W)$ be the $\mathbf{C}$-linear map $\varphi_W \colon V_- \to V_+$ defined by $\varphi_W = \pi_+ \circ (\pi_-|_W)^{-1} \colon V_- \to V_+$. $\square$

Remark 6. The above proposition is a particular case of a very general phenomena that Hermitian symmetric domains can be embedded in complex flag varieties. In this case we have the Grassmannian consisting of negative definite q-planes in V (let us denote it by $\mathbf{Gr}_q^-(V)$ for the moment). To define the Riemannian metric, we think of each negative-definite q-plane $W \subset V$ as a matrix P_W giving the orthogonal projection $V \to W$. This gives us an embedding of the above Grassmannian into Euclidean space (more precisely, into $M_{p+q}(\mathbf{C})$) and hence, we get a Hermitian metric inherited from the Hermitian metric on the $\mathbf{C}$-vector space $M_{p+q}(\mathbf{C})$ defined by $\langle P, Q\rangle = \mathrm{tr}(\overline{P}Q)$.

4.2.8 *Third description (conjugacy classes of embeddings).* We start by defining an action of $\mathbf{C}^\times$ on $V = V_+ \perp V_-$ as follows: given $a \in \mathbf{C}^\times$, define

$$a \cdot (v_+ + v_-) = av_+ + \overline{a}v_-, \qquad v_+ \in V_+, v_- \in V_-.$$

It is easy to check that

$$\langle a \cdot (v_+ + v_-), a \cdot (w_+ + w_-)\rangle = a\overline{a}\langle v_+ + v_-, w_+ + w_-\rangle,$$

i.e., a acts as a unitary similitude. This gives us a homomorphism $h\colon \mathbf{C}^\times \to GU(p, q)$ and we denote by X_h the $GU(p, q)$-conjugacy class of h. We claim that X_h is isomorphic to X' (and hence, to X) via a $U(p, q)$-equivariant isomorphism.

Proposition 4.5. *The map*

$$\psi\colon X_h \to X', \qquad \psi(h') = W_{h'} = \{v \in V \colon h'(i)v = -iv\}.$$

is a $U(p, q)$-equivariant isomorphism.

Proof. It is not hard to check that $\psi(h) = V_- \subset V$. Note that $GU(p, q)$ acts transitively on X', so if we check $U(p, q)$-equivariance, we get subjectivity. For the injectivity, suppose that $h_1, h_2 \in X_h$ are such that $W_{h_1} = W_{h_2}$. To show that $h_1 = h_2$, it suffices to check that $h_1(i) = h_2(i)$. But h_1 and h_2 agree on W_{h_1} (by the definition of W_{h_1} and W_{h_2}) and by the equivalence of the Hermitian pairing, they also agree on $W_{h_1}^{\perp}$. Since $V = W_{h_1} \perp W_{h_1}^{\perp}$, h_1 and h_2 are the same on the whole V. □

Finally, we will calculate the stabilizer $\mathrm{Stab}_{U(p,q)}(h)$.

Exercise 7. Calculate the stabilizer of h inside $U(p,q)$ and show that the latter is a maximal compact subgroup. (Hint: you should get that the stabilizer is $U(p) \times U(q) \subset U(p,q)$.)

4.2.9 *The complex structure on X_h.* Although the latter description (as conjugacy class of embeddings) is quite general and is what we will use in the future for a general reductive group, it has the drawback that one does not see so explicitly the complex structure on X_h. We will do this by providing explicitly the almost complex structure, i.e., the complex structure on the tangent space $T_h X_h$. Indeed, if $K = U(p) \times U(q)$ is the compact and if $\mathfrak{g} = \mathrm{Lie}\, U(p,q)$ then we can identify (as real vector spaces), $T_h X_h \cong \mathfrak{g}/K$. The advantage of such a decomposition is that the action of $\mathrm{ad}(i)$ on $\mathfrak{g}$ yields a decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. We will be done if we manage to identify $T_h X_h$ with $\mathfrak{g}^-$. Indeed, considering the action of $J_h = \mathrm{ad}(e^{\pi i/4})$ on $\mathfrak{g}^-$ then we have a linear map $J_h \colon T_h X_h \to T_h X_h$ satisfying $J_h^2 = -1$, the definition of a complex structure). To show that $\mathfrak{g}/K$ is naturally isomorphic to $\mathfrak{g}^{-1}$, we simply note that $\mathrm{Lie}\, K = \mathfrak{g}^-$ since K is the centralizer of h in $U(p,q)$.

Exercise 8. Show that under the identification $X_h \cong X'$, the complex structure J_h defined above coincides with the (natural) complex structure on X'.

5. Classification of Hermitian Symmetric Domains in Terms of Real Groups

The starting point of this section is a hermitian symmetric domain (M, g). We first associate to (M, g) a connected, adjoint algebraic group $\mathbf{G}$. If $p \in M$ is a point, we will then consider a homorphism $u_p \colon \mathbf{S} \to \mathrm{Hol}(M)$, such that $u_p(z)$ fixes p and acts as multiplication by z on $T_p M$ (such a homomorphism will be unique).

5.1 The associated connected adjoint algebraic group

– One can show that $\mathbf{Is}(M, g)^+ = \mathbf{Is}(M^{\infty}, g)^+ = \mathrm{Hol}(M)^+$.

Since $\mathrm{Hol}(M)^+$ has the structure of a connected, adjoint real Lie group H then the adjoint representation $\mathrm{ad} \colon H \to \mathbf{GL}(\mathfrak{h})$ is faithful where $\mathfrak{h} = \mathrm{Lie}\, H$ **Dimitar** : **Explain why?** . One can show that there is a connected adjoint algebraic group $\mathbf{G} \subset \mathbf{GL}(\mathfrak{h})$ such that $H = \mathbf{G}(\mathbf{R})^+$ (since the adjoint representation of $\mathrm{Hol}(M)^+$ on the Lie algebra $\mathfrak{h}$ is faithful) **Dimitar** : **Explain why or give a reference!** . We illustrate this with an example:

Example 7. If $M = \mathcal{H}_g$ then $\mathbf{G} = \mathbf{PGL}_2$ (since H is adjoint). Yet, $\mathbf{PGL}_2(\mathbf{R})$ has two connected components and acts holomorphically on $X = \mathbf{C} - \mathbf{R}$. The stabilizer of $\mathcal{H}_1$ is then $\mathbf{PGL}_2(\mathbf{R})^+$.

5.2 Action of the circle group $\mathbf{S}(\mathbf{R})$ via holomorphic isometries

Let X be a Hermitian symmetric domain. In order to describe the points of X more "group-theoretically", we need the following fact:

PROPOSITION 5.1. *For every $p \in X$, there exists a unique homomorphism $u_p \colon \mathbf{S}(\mathbf{R}) \to \mathrm{Hol}(X)$ such that $u_p(z)$ fixes p and acts as multiplication by z on the tangent space $T_p X$.*

Example 8. Start from $h\colon \mathbf{S}(\mathbf{R}) \to \mathbf{PSL}_2(\mathbf{R})$ given by $a+ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ mod $\pm I$. We can calculate the action of $h(z)$ on the tangent space at $p = i$ as follows:

$$\frac{d}{dz}\left(\frac{az+b}{-bz+a}\right)|_{z=i} = \frac{a^2+b^2}{(a-ib)^2}, i.e,$$

it acts by multiplication by $z/\overline{z}$. To define the desired homomorphism $u\colon \mathbf{S}(\mathbf{R}) \to \mathbf{PSL}_2(\mathbf{R})$, we note that if we set $u(z^2) = h(z)$ then $u(z^2)$ acts on $T_i\mathcal{H}_1$ by multiplication by z^2. To extend u to $\mathbf{S}(\mathbf{R})$, given $z \in \mathbf{S}(\mathbf{R})$, choose $\sqrt{z} \in \mathbf{S}(\mathbf{R})$ and define $u(z) = h(\sqrt{z})$ mod $\pm I$. Note that $u(z) \in \mathbf{PSL}_2(\mathbf{R})$ is independent of the choice of $\sqrt{z}$ since $h(-1) = -I$.

Remark 7. We will not prove in gory detail Proposition 5.1, but will rather indicate what ingredients go into the proof: the latter uses a classical concept from Riemannian geometry that we have not yet discussed - the sectional curvature. This is a way of computing the curvature by picking a 2-dimensional subspace $E_p \subset T_pX$ of the tangent space of the manifold X at p and computing the Gauss curvature of the surface that has the plane E as the tangent space. We denote this curvature by $K(p, E_p)$.

- **Integrability property:** we need to know the following: if $a\colon T_pX \to T_{p'}X'$ is a linear isometry that preserves $K(p, E)$ for **every** 2-dimensional plane $E \subset T_pX$ then the exponential map $\exp_p(Y) \mapsto \exp_{p'}(aY)$ is an isometry of a neighborhood of p to a neighborhood of p'.
- **Uniqueness:** if X is complete, connected and simply-connected then there is a **unique** isometry $\alpha\colon M \to M'$ such that $\alpha(p) = p'$ and $d\alpha_p\colon T_pX \to T_{p'}X'$ coincides with a.
- We need to know that multiplication-by-z on the tangent space preserves the sectional curvature tensor (this should be a computation).
- Using the above uniqueness, we can show that for any $z, z' \in \mathbf{S}(\mathbf{R})$, $u_p(z) \circ u_p(z')$ acts by multiplication by zz' on the tangent space, i.e., (again by uniqueness) it coincides with $u_p(zz')$, and hence, get a unique homomorphism $u_p\colon \mathbf{S}(\mathbf{R}) \to \mathrm{Hol}(X)$ **Dimitar : Something needs to be said about $u_p(z)$ being a holomorphic isometry.**

Exercise 9. If you are very curious, try to prove the above claims. That will undoubtedly

5.2.10 *Essential properties of u_p.* We are interested in what properties of u_p classify p as being a point of a Hermitian symmetric domain. We state a very general theorem (often known as **Cartan's classification of Hermitian symmetric domains**) and discuss it in a much greater detail via Hodge theory leading to Deligne's notion of a Shimura datum.

Theorem 5.2. *Suppose that X is a Hermitian symmetric domain and let $\mathbf{G}$ be the associated adjoint real algebraic group (i.e., $\mathrm{Hol}(X)^+ = \mathbf{G}(\mathbf{R})^+$). Given a point $p \in X$, the associated morphism $u_p\colon \mathbf{S}_\mathbf{R} \to \mathbf{G}_\mathbf{R}$ of real algebraic groups has the following properties:*

i) *The only characters occurring in the representation $\mathrm{ad} \circ u_p$ of $\mathbf{S}(\mathbf{R})$ on $\mathrm{Lie}(\mathbf{G}(\mathbf{R}))_\mathbf{C}$ are $z, 1, z^{-1}$.*
ii) *$Ad(u_p(-1))$ is a Cartan involution.*
iii) *$u_p(-1)$ does not project to 1 on any simple factor of $\mathbf{G}$.*

Conversely, let $u\colon \mathbf{S}_\mathbf{R} \to \mathbf{G}_\mathbf{R}$ be a morphism of algebraic groups satisfying i), ii) and iii). Then the $\mathbf{G}(\mathbf{R})^+$-conjugacy class X_u of u is a Hermitian symmetric domain.

Remark 8. We will not prove the theorem immediately, but rather look at sufficiently many examples first to arrive naturally to its understanding and Deligne's notion of Shimura datum.

Remark 9. This is known as Cartan's classification of Hermitian symmetric domains.

Proof. If we start with $u\colon \mathbf{S}_{\mathbf{R}} \to \mathbf{G}_{\mathbf{R}}$ and the $\mathbf{G}(\mathbf{R})^+$-conjugacy class X_u of u, we have to do the following to show that X_u is a Hermitian symmetric domain:

- Show that the centralizer $K_u = Z_{\mathbf{G}(\mathbf{R})^+}(u)$ is a compact. To do this, consider the Cartan involution $\theta = Ad(u(-1))$ and note that
$$K_u = \{g \in \mathbf{G}(\mathbf{R})^+ \colon gu(-1) = u(-1)g\} \subset \{g \in \mathbf{G}(\mathbf{C})\colon gu(-1) = u(-1)\overline{g}\} = \mathbf{G}^\theta(\mathbf{R}).$$
Since $\mathbf{G}^\theta(\mathbf{R})$ is compact and K_u is closed then K_u is compact. It then follows **Dimitar : Is it possible to give a good reference for that?** that the orbit space $X_u = (\mathbf{G}(\mathbf{R})^+/K_u) \cdot u$ has the structure of a smooth real analytic manifold. The tangent space to X_u at u is then identified with
$$T_u X_u \cong \mathrm{Lie}(\mathbf{G}(\mathbf{R}))/K_u.$$
- (Almost complex structure) **Dimitar : There is an integrability question to get a complex structure out of the almost complex structure.** We get that by using condition (ii). Indeed, to equip T_uX_u with a complex structure, we observe that $u(z)$ acts on T_uX_u as z (implied by the second condition), so we can use $u(i)$ to define the complex structure (for that, $u(i)^2$ will act as -1). We need to show that this complex structure is integrable **Dimitar : The latter is not a priori obvious, give a reference** .
- (Hermitian metric): since K_u is compact, there is a K_u-invariant positive-definite form on $T_uX_u \cong \mathfrak{g}/\mathfrak{g}^+$. Indeed, taking any symmetric, positive-definite form $\langle\, , \rangle$ on the quotient $\mathfrak{g}/\mathfrak{g}^+$ and defining
$$\langle v, w\rangle_{K_u} = \int_{K_u} \langle kv, kw\rangle dk$$
where dk is the Haar measure on K_u. This gives us a K_u-invariant symmetric bilinear form on T_uX_u. To show that this is indeed a Hermitian metric, we simply note that $u(i) \in K_u$. We thus choose a Hermitian metric on T_uX_u and use the homogeneity of X_u to move it to the other tangent spaces by elements of $\mathbf{G}(\mathbf{R})^+$.
- Condition (iii) implies that X_u does not have an irreducible factor of compact type.

□

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