## Locally Symmetric Varieties

So far, we have only looked at Hermitian symmetric domains. These have no algebraic structure (e.g., think about the case of the unit disk $\mathcal{D}_{1}$ ). This part of the course will deal with studying quotients of the form $D(\Gamma)=\Gamma \backslash D$ where $\Gamma$ is a discrete subgroup of $\mathbf{G}(\mathbf{R})$ ( $\mathbf{G}$ being the reductive algebraic group associated to the domain $D$ ).

## 1. Quotients of Hermitian Symmetric Domains by Discrete Subgroups

- Discuss quotients $\Gamma \backslash D$ where $\Gamma$ is discrete and torsion-free
- Discuss that principal congruence subgroups for sufficiently large $N$ are torsion-free.
- Discuss *large* discrete subgroups
- Arithmetic subgroups, criteria for finite covolume, compactness/unipotent elements, etc.
- Add a remark on discrete co-compact vs. arithmetic (Margulis' theorem, $\mathbf{S L}_{2}$-exception, etc.).


### 1.1 Discrete subgroups acting on hermitian symmetric domains

Proposition 1.1. Let $D$ be a Hermitian symmetric domain and let $\Gamma \subset \operatorname{Hol}(D)^{+}$be a discrete subgroup of holomorphic isometries. If $\Gamma$ is torsion-free then there is a unique complex structure on the quotient $D(\Gamma)=\Gamma \backslash D$ making the quotient map $D \rightarrow \Gamma \backslash D$ a local isomorphism of complex manifolds.

Proof. Let $p \in D$ be a point. We already saw in one of the early exercises that $K_{p}=\operatorname{Stab}_{\mathrm{Hol}(D)^{+}}(p)$ is compact. The map $g \in \operatorname{Hol}(D)^{+} \mapsto g p \in D$ gives a homeomorphism $\operatorname{Hol}(D)^{+} / K_{p} \simeq D$. Note that $K_{p} \cap \Gamma$ is finite since $\Gamma$ is discrete and $K_{p}$ is compact. This already shows that if $\Gamma$ is torsion-free then $\Gamma$ acts freely on $D$.

Next, we need to show that $D(\Gamma)=\Gamma \backslash D$ equipped with the quotient topology for $\pi$ : $D \rightarrow \Gamma \backslash D$ is Hausdorff. Let $x, y \in D$ be two points not in the same $\Gamma$-orbit. Then there exist opens $U \ni x$ and $V \ni y$ such that $U \cap g V=\varnothing$ for any $g \in \Gamma$. This shows that $\pi(x)$ and $\pi(y)$ are separated by $\pi(U)$ and $\pi(V)$, hence, $D(\Gamma)$ is Hausdorff.

To show that $D(\Gamma)$ is a manifold, we note that $p$ has a neighborhood $U \ni p$ such that $g U \cap U=\varnothing$ for any $g \in \Gamma$ (which means that $U \rightarrow \pi(U)$ is a local homeomorphism at $p$ ).

To get the complex structure on $D(\Gamma)$, declare a function $f: D(\Gamma) \rightarrow C$ to be homeomorphic if $f \circ$ $\pi$ is homeomorphic. The holomorphic functions on $D(\Gamma)$ form a sheaf and $\pi$ is a local isomorphism of ringed spaces. Thus, the sheaf defines complex structure on $D(\Gamma)$ for which $\pi$ is a local isomorphism of complex manifolds.

### 1.2 Subgroups of finite co-volume

To get interesting quotients $D(\Gamma)$, we need the discrete subgroup to be sufficiently large. The latter means that we would like $\operatorname{vol}(\Gamma \backslash D):=\int_{\Gamma \backslash D} \Omega<\infty$ where $\Omega$ is a volume form (recall that $D$ is a Riemannian manifold). As usual, we will study this condition in terms of the real Lie group of holomorphic isometries as opposed to the domain directly.

Exercise 1. Show that when $D=\mathcal{H}_{1}$ and $\Gamma=\mathbf{P S L}_{2}(\mathbf{Z})$ then $\operatorname{vol}(\Gamma \backslash D)<\infty$. (Hint: use that the volume form for $\mathcal{H}_{1}$ is $\Omega=\frac{d x d y}{y^{2}}$ and the fundamental domain for the action of $\mathbf{S L}_{2}(\mathbf{Z})$ on $\left.\mathcal{H}_{1}\right)$.

### 1.3 Arithmetic subgroups

Recall that two groups $G_{1}$ and $G_{2}$ are called commensurable if both $\left[G_{1}: G_{1} \cap G_{2}\right]$ and $\left[G_{2}: G_{1} \cap G_{2}\right]$ are finite. Let $\mathbf{G}$ be an algebraic group over $\mathbf{Q}$. An subgroup $\Gamma \subset \mathbf{G}(\mathbf{Q})$ is called arithmetic if there exists an embedding $\mathbf{G} \hookrightarrow \mathbf{G L}_{n}$ such that $\Gamma$ is commensurable with $\mathbf{G}(\mathbf{Q}) \cap \mathbf{G} \mathbf{L}_{n}(\mathbf{Z})$. If this is the case then one can show that $\Gamma$ is commensurable with $\mathbf{G}(\mathbf{Q}) \cap \mathbf{G L}_{n^{\prime}}(\mathbf{Z})$ for any embedding $\mathbf{G} \hookrightarrow \mathbf{G L}_{n^{\prime}}$. Arithmetic subgroups are always discrete.

Example 1. It is not always the case that an arithmetic subgroup has finite co-volume. Take $\mathbf{G}=\mathbf{G}_{m}$ and $\Gamma=\{ \pm 1\} \subset \mathbf{G}_{m}(\mathbf{Q})$. Then $\Gamma$ is obviously an arithmetic subgroup since $\Gamma=\mathbf{G}(\mathbf{Q}) \cap \mathbf{G L}_{1}(\mathbf{Z})$, yet, $\Gamma \backslash \mathbf{G}(\mathbf{R}) \cong \mathbf{R}_{>0}$ has infinite volume.

Theorem 1.2 (Borel). Let $\mathbf{G} / \mathbf{Q}$ be a reductive group and let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be an arithmetic subgroup.
(i) The quotient $\Gamma \backslash \mathbf{G}(\mathbf{R})$ has finite volume if and only if $\operatorname{Hom}\left(\mathbf{G}, \mathbf{G}_{m}\right)=0$.
(ii) The quotient $\Gamma \backslash \mathbf{G}(\mathbf{R})$ is compact if and only if $\operatorname{Hom}\left(\mathbf{G}, \mathbf{G}_{m}\right)=0$ and $\mathbf{G}(\mathbf{Q})$ has no non-trivial unipotent elements.

For the next two examples, $B / \mathbf{Q}$ will denote a quaternion algebra and let $\mathbf{G}$ be the algebraic group over $\mathbf{Q}$ defined by $\mathbf{G}(k)=\left(B \otimes_{\mathbf{Q}} k\right)^{1}:=\left\{b \in B \otimes_{\mathbf{Q}} k: \operatorname{nr}(b)=1\right\}$.

Example 2. Let us explain what the second condition means in the case of $\Gamma \subset \mathbf{S L}_{2}(\mathbf{Z})$ acting on the upper-half plane $\mathcal{H}_{1}$. In this case, the rational unipotent matrices in $\mathbf{S L}_{2}(\mathbf{R})$ correspond to the cusps. You can see this explicitly by fixing the cusp at 0 and noticing that $\left(\begin{array}{ll}1 & r \\ 0 & 1\end{array}\right) \cdot 0=r$ and $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right) \cdot 0=i \infty$.

Example 3. If $B \otimes_{\mathbf{Q}} \mathbf{R}$ is a division algebra, then $\mathbf{G}(\mathbf{R})$ has no unipotent elements (otherwise, $B(\mathbf{R})^{\times}$will have nilpotents), so by the theorem $\Gamma \backslash \mathbf{G}(\mathbf{R})$ will be compact (this is the classical fact that Shimura curves associated to quaternion algebras of discriminant $D>1$ are always compact, i.e., have no cusps).
1.3.1 Neat subgroups. If $V$ is a finite-dimensional $\mathbf{Q}$-vector space then an element $\alpha \in \mathbf{G L}(V)(\mathbf{Q})$ is called neat if it eigenvalues generate a torsion-free subgroup of $\mathbf{C}^{\times}$. If $\alpha$ is neat then clearly $\alpha$ is not torsion. An element $g \in \mathbf{G}(\mathbf{Q})$ is neat if $\rho(g)$ is neat for some faithful representation $\rho: \mathbf{G} \rightarrow \mathbf{G L}(V)$ (in this case, it will be neat for any representation). A subgroup of $\mathbf{G}(\mathbf{Q})$ is neat if every element of that subgroup is neat. Neat arithmetic subgroups are important to make sure we are getting smooth quotients. We first show some basic results (due to Borel) on the existence of those (since the arguments are interesting on their own):

Proposition 1.3. For any $\mathbf{Q}$-group $\mathbf{G}$ and an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbf{Q})$, then there exists a neat subgroup $\Gamma^{\prime} \subset \Gamma$ of finite index.

Proof. Identify $\mathbf{G}(\mathbf{Q})$ with a $\mathbf{Q}$-subgroup of $\mathbf{G L}_{n}(\mathbf{Q})$ via a morphism that sends $\Gamma$ to a subgroup of $\mathbf{G} \mathbf{L}_{n}(\mathbf{Z})$. It suffices to prove the statement for $\Gamma=\mathbf{G} \mathbf{L}_{n}(\mathbf{Z})$. Let $S$ be the set of all cyclotomic polynomials different from $\Phi_{1}(X)=X-1$ of degree at most $n$. Choose a prime $p$ such that $p \nmid \prod_{f \in S} f(1)$. Consider the congruence subgroup $\Gamma^{\prime}=\left\{g \in \mathbf{G L}_{n}(\mathbf{Z}): g \equiv 1 \bmod p\right\}$. We claim that $\Gamma$ is neat. If not, there exists $g \in \Gamma^{\prime}$ and $s$ that is in the subgroup generated by the eigenvalues

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$\lambda_{1}, \ldots, \lambda_{n}$ of $g$ such that $s$ is a root of unity. We want to show that $s=1$. Assume the contrary. Let $K=\mathbf{Q}\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ (a number field of degree at most $n$ ) since $\lambda_{i}$ is a root of the characteristic polynomial of $g$. Since $s$ is a root of unity and the degree of $s$ is at most $n$, there exists $f \in S$ such that $f(s)=0$ (if $s \neq 0$ then the minimal polynomial of $s$ is necessarily one of the cyclotomic polynomials in $S$ ). On the other hand the eigenvalues $\lambda_{i}$ reduce to 1 modulo a prime ideal $\mathfrak{p} \mid p$ of $K$. This means that $f(1) \equiv 0 \bmod p$ which is a contradiction. Hence, $\Gamma^{\prime}$ is neat.

LEMMA 1.4. The principal congruence $\operatorname{subgroup} \Gamma(N) \subset \mathbf{G L}_{n}(\mathbf{Q})$ is torsion-free for $N \geqslant 3$.
Proof. Assume that $g \in \Gamma(N)$ is a torsion element and consider the eigenvalues of $g$ (as algebraic integers). Since $g$ is torsion, the eigenvalues are all roots of unity. It suffices to show that they are all equal to 1 . Indeed, a unipotent matrix over a field of characteristic 0 that has finite order is necessarily the identity (this can be seen using the Jordan canonical form). Let $\zeta \in \bar{Z}$ be an eigenvalue of $g$. Since $g \in \Gamma(N), \zeta \equiv 1 \bmod N \overline{\mathbf{Z}}$. If $N$ has an odd prime factor $p$ then $\zeta \equiv 1 \bmod p$ and so $\log _{p}(1+x)$ is convergent for $x=\zeta-1$, so it is in a domain where the function $\log _{p}(1+x)$ is invertible (has inverse $\exp _{p}$ ). But $\log _{p} \zeta=0$ since $\zeta$ is a root of unity, hence, $\zeta=1$.

### 1.4 Arithmeticity of discrete co-compact subgroups of non-compact Lie groups

Similarly to the notion of arithmetic subgroups of algebraic groups, one can define the notion of an arithmetic subgroup of a Lie group. Let $H$ be a real Lie group. A subgroup $\Gamma \subset H$ is called arithmetic if there exists an algebraic group $\mathbf{G}$ over $\mathbf{Q}$, a surjective homomorphism of Lie groups $\varphi: \mathbf{G}(\mathbf{R})^{+} \rightarrow H$ with compact kernel and an arithmetic subgroup $\Gamma_{0} \subset \mathbf{G}(\mathbf{Q})$ such that $\Gamma_{0} \cap \mathbf{G}(\mathbf{R})^{+}$ whose image (under $\varphi$ ) is $\Gamma$.

Arithmetic subgroups of Lie groups are discrete and have finite co-volume as is shown by the following proposition:

Proposition 1.5. If $H$ is a semi-simple real Lie group and $\Gamma \subset H$ is an arithmetic subgroup then $\Gamma$ is discrete and has finite co-volume. Moreover, $\Gamma$ contains a torsion-free subgroup of finite index.

The converse to that is Margulis' arithmeticity theorem:
THEOREM 1.6 (Margulis arithmeticity). Let $H$ be a non-compact simple Lie group and let $\Gamma \subset H$ be a discrete subgroup of finite co-volume. Then $\Gamma$ is arithmetic unless $H$ is isogenous to $S O(1, n)$ or $H=S U(1, n)$.

Remark 1. The theorem does not apply to $\mathbf{S L}_{2}(\mathbf{R})$ since the latter is isogenous $S O(1,2)$. In fact, $\mathbf{S L}_{2}(\mathbf{R})$ has uncountably many discrete subgroups of finite co-volume whereas there are only countably many arithmetic subgroups.

## 2. Complex Algebraic Varieties and Complex Manifolds

### 2.1 Complex manifolds associated to complex algebraic varieties

Given an algebraic variety $X$ over $\mathbf{C}$, then the set of complex points $X(\mathbf{C})$ has a natural structure of a complex manifold. More precisely, we have a functor $F:\left(X, \mathcal{O}_{X}\right) \longrightarrow\left(X^{\text {an }}, \mathcal{O}_{X^{\text {an }}}\right)$ from the category of smooth algebraic varieties to the category of smooth complex manifolds satisfying the following properties:

- Every Zariski open set is open for the complex topology and every regular map is holomorphic.
- If $X=\mathbf{A}^{n}$ then $X^{\text {an }}=\mathbf{C}^{n}$ with the usual complex topology.
- If $\varphi: X \rightarrow Y$ is an étale morphism then $\varphi^{\mathrm{an}}: X^{\mathrm{an}} \rightarrow Y^{\text {an }}$ is a local isomorphism.

Remark 2. Clearly, if $\varphi: X \rightarrow Y$ is a regular map then $\varphi$ is determined by the holomorphic map $\varphi^{\text {an }}: X^{\text {an }} \rightarrow Y^{\text {an }}$. This means that $F$ is faithful. Yet, not every holomorphic map determines a regular map, i.e., a holomorphic map need not be algebraic. For instance, exp: $\mathbf{C} \rightarrow \mathbf{C}$ is holomorphic, but not algebraic.

Remark 3. Not every complex manifold is a complex algebraic variety. For example, the open unit disk is a complex manifold, but it is not an algebraic variety. In addition, as we will see below, two non-isomorphic non-singular algebraic varieties might be isomorphic as complex manifolds. Thus, $F$ is a faithful functor that is neither full, nor essentially surjective.

### 2.2 Algebraicity criteria

2.2.2 Necessary criteria in terms of compactification and meromorphic functions. One can state two necessary conditions for a complex manifold $X$ to be algebraic.

- (Compactification): $X$ embeds as an open submanifold in a compact complex manifold $X^{*}$ such that $X^{*} \backslash X$ is a union of finitely many components of dimension at most $\operatorname{dim} X-1$.
- (Transcendence degree): If $X$ is compact then the field of meromorphic functions on $X$ has transcendence degree equal to $\operatorname{dim} X$.
What do these conditions mean? The first condition is a consequence of a celebrated theorem of Hironaka on the resolution of singularities:

Theorem 2.1 (Hironaka). Let $X$ be a non-singular variety over $\mathbf{C}$. Then $X$ can be embedded into a complete non-singular variety $X^{*}$ over $\mathbf{C}$ such that the boundary $X^{*} \backslash X$ is a divisor with normal crossings.

Remark 4. The divisor $X^{*} \backslash X$ being a divisor with normal crossings is a technical condition which is a slightly weaker condition than the condition of it being smooth. In fact, it is not always possible to require that the complement is smooth, but the divisor with normal crossings condition works as well. In terms of the complex topology, the latter means that locally, the inclusion $X \hookrightarrow X^{*}$ looks like $\left(\mathcal{D}_{1}^{\times}\right)^{r} \times \mathcal{D}_{1}^{s} \hookrightarrow \mathcal{D}_{1}^{r+s}$.

The second condition is a consequence of another classical result in complex algebraic geometry, namely, the fact that if $X$ is a smooth and projective complex algebraic variety then the field of meromorphic functions on $X$ coincides with the field of rational functions. We will elaborate a bit on this via a sequence of statements some of which are presented as spelled-out exercises.

Theorem 2.2. If $X$ is a compact complex manifold then the field of meromorphic functions on $X$ has transcendence degree at most $\operatorname{dim} X$ over $\mathbf{C}$.

Since the theorem is quite fundamental, we will prove it in two exercises
Exercise 2. Let $f\left(z_{1}, \ldots, z_{n}\right)$ be a holomorphic function defined on the polydisc $\left|z_{i}\right| \leqslant 1$. Let $M=\max _{\left|z_{i}\right| \leqslant 1}|f(z)|$. Let $\mathfrak{m}_{0}$ be the maximal ideal of the local ring of analytic functions at the origin. If $f \in \mathfrak{m}_{0}^{h}$ then prove that

$$
|f(z)| \leqslant M \max \left|z_{i}\right|^{h},
$$

for any $z$ in the polydisc $\left|z_{i}\right|<1$ for $i=1, \ldots, n$. (Hint: let $|z|=\max _{i}\left|z_{i}\right|$. Given $z=\left(z_{i}\right)$ with $|z| \leqslant 1$, define the function $g(t)=f(t z)$ for $t \in \mathbf{C}^{n}$ and use that $g(t)$ is holomorphic for $|t| \leqslant|z|^{-1}$. Using the maximal modulus principle, deduce the desired result.)

Exercise 3. Let $f_{1}, \ldots, f_{n+1}$ be meromorphic functions on an $n$-dimensional compact complex manifold $X$. In this exercise, you will show that there exists a polynomial $F\left(T_{1}, \ldots, T_{n+1}\right) \in \mathbf{C}\left[T_{1}, \ldots, T_{n}\right]$ such that $F\left(f_{1}, \ldots, f_{n+1}\right)=0$.

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(i) Show that each point $x \in X$ has open neighborhoods $U_{x} \ni x$ such that each $f_{i}$ is a ratio of two holomorphic functions on $x$. Show that there exists a neighborhood $V_{x} \subset U_{x}$ whose closure is contained in $U_{x}$ and $V_{x}$ has local coordinates $z=\left(z_{1}, \ldots, z_{n}\right)$ with $|z|<1$. Finally, Let $W_{x} \subset V_{x}$ be the open neighborhood for which $|z|<1 / 2$.
(ii) Let $h>0$ be an integer and let $\mathfrak{m}_{x}$ be the maximal ideal of the local ring of analytic functions at $x$. Show that there exists a polynomial $F \neq 0$ such that

$$
F\left(f_{1}, \ldots, f_{n+1}\right)=\frac{R_{x}}{Q_{x}^{k}}, \quad \text { in } V_{x},
$$

where $R_{x} \in \mathfrak{m}_{x}^{h}$. (Hint: using the compactness of $X$, choose a set of finitely many points $x \in X$ such that $X=\bigcup_{x \in S} W_{x}$. Define $Q_{x}$ as the product of the denominators for the functions $f_{i}$ on $W_{x}$; To define $R_{x} \in \mathfrak{m}_{x}^{k}$, use the conditions $D^{s} R_{x}(x)=0$ for all $\left.s<h\right)$.
(iii)

Theorem 2.3. If $X$ is a complete algebraic variety over $\mathbf{C}$ then a meromorphic function on the complex manifold $X^{\text {an }}$ is necessarily a rational function on $X$.

We do the proof in a series of exercises:
Exercise 4. The setting is as in the statement of Theorem 2.3. Let $f$ be a meromorphic function on $X^{\text {an }}$. We will show that $f$ is rational in the following sequence of exercises.
(i) Show that there exists an irreducible polynomial $F(T)=T^{m}+c_{1} T^{m-1}+\cdots+c_{m} \in \mathbf{C}(X)[T]$ such that $F(f)=0$. (Hint: use Theorem 2.2),
(ii) Let $F$ be as in (i) and consider

$$
X^{\prime}=\left\{(x, z) \in X \times \mathbf{A}^{1}: z^{m}+c_{1}(x) z^{m-1}+\cdots+c_{m}(x)=0\right\} .
$$

Show that $X^{\prime}$ is an irreducible algebraic variety and $\mathbf{C}\left(X^{\prime}\right)=\mathbf{C}(X)(f)$.
(iii) Show that $X^{\prime}(\mathbf{C})$ is connected.
(iv) Assume that the projection $p: X^{\prime}(\mathbf{C}) \rightarrow X(\mathbf{C})$ is unramified. Show that $m=1$ (assume that $m>1$ in which case $\varphi(X(\mathbf{C})) \subsetneq X^{\prime}(\mathbf{C})$; show that both $\varphi(X(\mathbf{C}))$ and $X^{\prime}(\mathbf{C}) \backslash \varphi(X(\mathbf{C}))$ are closed which is a contradiction with (iii)).
(v) Show how to reduce the proof of the theorem to (iv).

Theorem 2.4. Let $X$ and $Y$ be complete algebraic varieties. Any holomorphic map $f: X^{\text {an }} \rightarrow Y^{\text {an }}$ is of the form $g^{\text {an }}$ where $g: X \rightarrow Y$ is a morphism.

Remark 5. If $X$ is a nonsingular and quasi-projective as opposed to projective then the above is not true in general (e.g., think of the counterexample $\mathbf{C} \xrightarrow{\exp } \mathbf{C}$ ). Below, we discuss a theorem of Borel that tell us that any holomorphic map from $X$ to a locally symmetric variety that is a quotient of a Hermitian symmetric domain by an arithmetic torsion-free subgroup is necessarily regular.

## Exercise 5.

Finally, we are ready to prove on of the basic results on algebraization of smooth projective complex manifolds. Recall that a complex manifold is projective if it can be realized as a closed submanifold of the complex manifold $\mathbf{P}^{n}(\mathbf{C})$.

Theorem 2.5 (Chow). There is an equivalence of categories between complex projective smooth algebraic varieties and smooth complex projective manifolds given by $\left(X, \mathcal{O}_{X}\right) \mapsto\left(X^{\text {an }}, \mathcal{O}_{X, \text { an }}\right)$.

## Exercise 6.

Finally, we give an example of two non-isomorphic complex algebraic varieties are isomorphic as complex manifold (to indicate that the completeness condition is indeed necessary).
Exercise 7. Let $E \subset \mathbf{P}^{2}(\mathbf{C})$ be an elliptic curve and let $\mathcal{O}_{E}$ be the point at infinity. Consider the non-complete curve $C=E \backslash\left\{\mathcal{O}_{E}\right\}$ and let $P$ be any point on $C$. We define two varieties: $X$ defined to be the line bundle $X=E(D) \rightarrow C$ corresponding to the divisor $D=(P)$ and $Y=C \times \mathbf{A}^{1}$.
(i) Show that $X$ and $Y$ are non-isomorphic as complex algebraic varieties.
(ii) Show that $X^{\text {an }}$ and $Y^{\text {an }}$ are isomorphic as complex manifolds.
2.2.3 The classical modular curves $X_{0}(N)$. It is instructive to look at the case of compact Riemann surfaces. In this case, the functor $F$ yields a functor from the category of smooth projective curves over $\mathbf{C}$ to the category of compact Riemann surfaces and the latter is an equivalence of categories! We will now illustrate this with the example of the classical modular curves $X_{0}(N)$ for the congruence subgroup $\Gamma_{0}(N)=\left\{\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathbf{S L}_{2}(\mathbf{Z}), c \equiv 0 \bmod N\right\}$ of $\mathbf{S L}_{2}(\mathbf{Z})$.
Proposition 2.6. (i) The field $\mathbf{C}\left(X_{0}(N)\right)$ of modular functions is generated by $j(z)$ and $j(N z)$.
(ii) The minimal polynomia ${ }^{1} F(j, Y) \in \mathbf{C}(j)[Y]$ of $j(N z)$ over $\mathbf{C}(j)$ has degree $\mu=\left[\Gamma(1): \Gamma_{0}(N)\right]$. Moreover, $F(j, Y)$ is a polynomial in $j$ with integer coefficients that is symmetric whenever $N>1$. (iii) If $N=p$ then

$$
F(X, Y) \equiv\left(X^{p}-Y\right)\left(X-Y^{p}\right) \bmod p
$$

Now, suppose that $X$ is a Riemann surface (not necessarily compact) and suppose that the compactification holds, i.e., $X$ can be embedded as an open submanifold in a compact Riemann surface $X^{*}$ such that $X^{*} \backslash X$ is a finite set of points. Since the Zariski closed sets of $X^{*}$ (which is also algebraic by the above remark!) are precise the finite sets of points, $X$ is also a Zariski open subset of $X^{*}$ and hence, algebraic. In this case, we see that the compactification condition is also sufficient to guarantee algebraicity. Now, if a connected Riemann surface $M$ is algebraic then the maximum modulus principle applied to $M^{*}$ shows that every bounded holomorphic function on $M$ must be constant. The latter is a useful criterion for proving that certain complex manifolds are not algebraic. We use as an example the Poincaré upper-half plane $\mathcal{H}_{1}$ in the following exercise.

Exercise 8. Show that the complex manifold $X=\mathcal{H}_{1}$ does not arise from an algebraic variety (assume that the compactification condition holds; using the maximum modulus principle, show that a bounded function on a compact Riemann surface is necessarily constant; use then the fact that $\mathcal{H}_{1}$ is conformally equivalent to the open unit disk $\mathcal{D}_{1}$ to exhibit a non-constant and bounded function on $\mathcal{H}_{1}$ ).
2.2.4 More on algebraicity of the complex torus $\mathbf{C}^{g} / L$. Dimitar : Exhibit a complex torus that does not possess a Riemann form. It should be $L=\langle(1,0),(0,1),(i, 0),(\alpha, \beta)\rangle \subset \mathbf{C}^{2}$ where $\beta \in \mathbf{C}-\mathbf{R}$.

## 3. Theory of Compactification and Cohomology

The setting is as before: $D$ is a Hermitian symmetric domain corresponding to a reductive algebraic group $\mathbf{G} / \mathbf{Q}$ (that is, $\left.\operatorname{Hol}(D)^{+} \cong \mathbf{G}(\mathbf{R})^{+}\right)$and let $\Gamma \subset \mathbf{G}(\mathbf{R})$ be an arithmetic subgroup. Consider

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the locally symmetric space $X=\Gamma \backslash D$. We saw that if $\Gamma$ is torsion free then $X$ is smooth. Typically, it is non-compact.

Exercise 9. Show that there exists a reductive algebraic group $\mathbf{G} / \mathbf{R}$ such that $\mathbf{G}$ connected for the Zariski topology (i.e., as an algebraic variety over $\mathbf{R}$ ), but $\mathbf{G}(\mathbf{R})$ is disconnected. (Hint: consider $\mathbf{G}=\mathbf{G}_{m, \mathbf{R}}$.)

### 3.1 Preliminaries

Suppose that $\mathbf{G} / \mathbf{R}$ is a connected reductive algebraic group and consider $\mathbf{G}(\mathbf{R})^{+}$. Let $\mathfrak{k}=\operatorname{Lie}(K) \subset$ $\mathfrak{g}$ and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. We have a Cartan decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{p}$. The complexification $\mathfrak{p}_{\mathbf{C}}$ decomposes as $\mathfrak{p}_{\mathbf{C}}=\mathfrak{p}_{+} \oplus \mathfrak{p}_{-}$into eigenspaces for the action of $\pm i$. The subalgebras $\pm \mathfrak{p}_{ \pm}$are commutative since ( $\mathfrak{k}, \mathfrak{p}$ ) is a Cartan pair.

Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra.

### 3.2 Harish-Chandra embedding

Let $P_{ \pm}=\exp \left(\mathfrak{p}_{ \pm}\right) \subset \mathbf{G}(\mathbf{C})$. Then $K_{\mathbf{C}} \cdot P_{-} \subset \mathbf{G}(\mathbf{C})$ is a parabolic subgroup whose unipotent radical is $P_{-}$. A theorem of Borel and Harish-Chandra AMRT10, Thm.III.2.1] says that the multiplication map

$$
P_{+} \cdot K_{\mathbf{C}} \cdot P_{-} \rightarrow \mathbf{G}_{\mathbf{C}}
$$

is injective and its image contains $\mathbf{G}(\mathbf{R})$. Moreover, $\mathbf{G}(\mathbf{R}) \cap K_{\mathbf{C}} \cdot P_{-}=K$, so we get an embedding

$$
D \cong \mathbf{G}(\mathbf{R}) / K \rightarrow P_{+} \cdot K_{\mathbf{C}} \cdot P_{-} / K_{\mathbf{C}} \cdot P_{-} \cong P_{+} \stackrel{\exp }{\leftrightarrows} \mathfrak{p}_{+}
$$

(We are using that $\mathfrak{p}_{+}$is identified with $P_{+}$via the exponential map exp: $\mathfrak{p}_{+} \rightarrow P_{+}$). This gives us a canonical embedding (the Harish-Chandra embedding) $D \rightarrow \mathfrak{p}_{+}$. We can use this embedding to define the boundary components of $D$.

### 3.3 Siegel sets

To define Siegel sets, we start with a connected reductive group G/Q and a Hermitian symmetric domain $D=\mathbf{G}(\mathbf{R})^{+} / K$ where $K \subset \mathbf{G}(\mathbf{R})^{+}$is a maximal compact subgroup. Suppose that $\mathbf{P} \subset \mathbf{G}$ is a minimal $\mathbf{Q}$-parabolic subgroup.

Given a base point $p \in D$, there is a unique torus $\mathbf{A} \subset \mathbf{P}$ that is conjugate to the maximal Q-split torus of $\mathbf{P}$ such that $\operatorname{Lie} \mathbf{A}(\mathbf{R}) \perp \operatorname{Lie}\left(\operatorname{Stab}_{\mathbf{G}(\mathbf{R})^{+}}(p)\right)$.

Let $\Delta \subset X^{*}(\mathbf{A})=\operatorname{Hom}\left(\mathbf{A}, \mathbf{G}_{m}\right)$ denote the simple positive roots Dimitar : Doesn't this require a choice of a Borel subalgebra. . One way to define Siegel sets is to consider

$$
A^{+}=\{g \in A: \beta(g) \geqslant 1, \quad \forall \beta \in \Delta\}
$$

We define Siegel sets associated to the minimal $\mathbf{Q}$-parabolic $\mathbf{P}$ as

$$
\mathfrak{S}_{\omega}=\omega \cdot A^{+} \cdot p
$$

where $\omega \subset \mathbf{P}(\mathbf{R})^{+}$is a compact subset.
Remark 6. This is not the original definition of Siegel sets in the paper of Baily-Borel [BB66]. From the point of view of reduction theory, any system of sets $\left\{X_{\alpha}\right\}$ cofinal with respect to the sets $\left\{\mathfrak{S}_{\omega}\right\}$ will achieve the same properties. Here, cofinal means that

- For every $\alpha$, there exists a compact subset $\omega \subset \mathbf{G}(\mathbf{R})^{+}$such that $X_{\alpha} \subset \mathfrak{S}_{\omega}$.
- For every $\omega \subset \mathbf{G}(\mathbf{R})^{+}$, there exists an index $\alpha$ such that $\mathfrak{S}_{\omega} X_{\alpha}$.

The important point about Siegel sets is that they play an important role in the reduction theory.
3.3.5 Reduction theory. More precisely, suppose that $\Gamma \subset \mathbf{G}(\mathbf{Q})$ is an arithmetic subgroup. Then

- There exists a compact subset $\omega \subset \mathbf{P}(\mathbf{R})^{+}$and a finite set $F \subset \mathbf{G}(\mathbf{Q})$ such that

$$
D=\Gamma \cdot F \cdot \mathfrak{S}_{\omega} .
$$

The set $\Omega=F \cdot \mathfrak{S}_{\omega}$ is then called a fundamental set.

- For all compact $\omega$ and all $g_{1}, g_{2} \in \mathbf{G}(\mathbf{Q})$, the set

$$
\left\{\gamma \in \Gamma: g_{1} \mathfrak{S}_{\omega} \cap \gamma g_{2} \mathfrak{S}_{\omega} \neq 0\right\}
$$

is finite.

### 3.4 Rational boundary components and partial compactification

Dimitar : Define supporting hyperplanes and discuss the example of the action of $\mathbf{G L}_{n}(\mathbf{R})$ on $S_{n}(\mathbf{R})$.

## Dimitar : Define boundary components for a Hermitian symmetric domain.

### 3.5 Satake topology

Recall that $D$ embeds into $\mathfrak{p}_{+}$via the Harish-Chandra embedding. We will topologize the partial compactification $D^{*}$ (here, $D^{*}=\bar{D}^{\mathrm{BB}}$ is the union of $D$ and the rational boundary components) with a topology that is different from the topology induced from the vector space structure on $\mathfrak{p}_{+}$ that will make the quotient $\Gamma \backslash D^{*}$ Hausdorff (the same phenomenon that we saw for the Poincaré upper-half plane). Take any fundamental set $\Omega=F \cdot \mathfrak{S}_{\omega}$. For any $p \in D^{*}$, define a fundamental system of neighborhoods of p to be all sets $D^{*} \supset U \ni p$ such that

$$
\Gamma_{x} \cdot U=U, \quad \Gamma_{x}=\{\gamma \in \Gamma: \gamma x=x\}
$$

and such that whenever $\gamma \cdot p \in \bar{\Omega}, \gamma U \cap \bar{\Omega}$ is a neighborhood of $\gamma \cdot p$ for the topology induced from $\bar{D} \subset \mathfrak{p}^{+}$(here, $\bar{\Omega}$ is the closure of $\Omega$ in $\mathfrak{p}^{+}$for the usual topology). This topology is the unique topology on the partial compactification $\bar{D}^{\mathrm{BB}}$ satisfying the following properties:

- It induces the natural topology on $D$ (i.e., the topology coming from the topology on $\mathfrak{p}_{+}$via the Harish-Chandra embedding).
- The group $\mathbf{G}(\mathbf{Q})$ acts continuously on $\bar{D}^{\mathrm{BB}}$.
- If $x, x^{\prime} \in D$ such that $x^{\prime} \notin \Gamma x$ for an arithmetic subgroup $\Gamma$ then there exists neighborhoods $U \ni x$ and $U^{\prime} \ni x^{\prime}$ such that $\Gamma U \cap U^{\prime}=\emptyset$
- If $\Gamma$ is an arithmetic subgroup then for each point $x \in \bar{D}^{\mathrm{BB}}$ there is a fundamental system of neighborhoods $\{U\}$ of $x$ such that $\gamma \cdot U=U$ if $\gamma x=x$ and $\gamma U \cap U=\emptyset$ if $\gamma x \neq x$.


### 3.6 Baily-Borel-Satake compactification

To explain this, consider a baby case, namely, $\mathbf{G}=\mathbf{S L}_{2}, \Gamma \subset \mathbf{S L}_{2}(\mathbf{R})$ and $X=\Gamma \backslash \mathfrak{h}$. The quotient $X$ is not compact and to compactify it, we need to add the cusp at infinity, i.e., $X^{*}=X \cup\{i \infty\}$. Alternatively, we would like to view $X^{*}$ as a quotient of a partial compactifiction of the upper-half plane. More precisely, if we define ${\overline{\mathcal{H}_{1}}}^{\mathrm{BB}}=\mathfrak{h} \cup \mathbf{P}^{1}(\mathbf{Q})$ then $X^{*} \cong$

### 3.7 The theorem of Baily-Borel

Theorem 3.1 (Baily-Borel). Let $D$ be a Hermitian symmetric domain and let $\Gamma \subset \operatorname{Hol}(D)^{+}$be a torsion-free arithmetic subgroup. The quotient $D(\Gamma)=\Gamma \backslash D$ has a canonical realization as a Zariski open subset of a projective algebraic variety $D(\Gamma)^{*}$.

## Shimura Varieties

Remark 7. Sometimes, the compactification $D(\Gamma)^{*}$ is called the minimal compactification for the following reason (universal property): if $D(\Gamma)^{\dagger}$ is any other compactification such that $D(\Gamma)^{\dagger} \backslash D(\Gamma)$ has normal crossings then there exists a unique regular map $\varphi: D(\Gamma)^{\dagger} \rightarrow D(\Gamma)^{*}$ such that $\iota^{*}=\varphi \circ \iota^{\dagger}$ where $\iota^{*}$ and $\iota^{\dagger}$ are the inclusion morphisms for the two compactifications.

### 3.8 Borel's theorem

Theorem 3.2 (Borel). Let $X$ be a quasi-projective variety over $\mathbf{C}$ and suppose that $\Gamma \subset \operatorname{Hol}\left(X^{\mathrm{an}}\right)^{+}$ is arithmetic and torsion-free. Every holomorphic map of complex manifolds $X^{\text {an }} \rightarrow D(\Gamma)(\mathbf{C})$ is regular algebraic.

The proof makes use in an essential way of the following lemma proved originally by Borel:
Lemma 3.3. Every holomorphic map $\left(\mathcal{D}_{1}^{\times}\right)^{r} \times \mathcal{D}_{1}^{s} \hookrightarrow D(\Gamma)$ extends to a holomorphic map $\mathcal{D}_{1}^{r+s} \hookrightarrow$ $D(\Gamma)^{*}$, where $D(\Gamma)^{*}$ denotes the Baily-Borel compactification.

Remark 8. The lemma is a generalization of the big Picard theorem that says that a holomorphic function $\mathcal{D}_{1}^{\times} \rightarrow \mathbf{P}^{1}(\mathbf{C}) \backslash\{0,1, \infty\}$ extends to a holomorphic function $\mathcal{D}_{1} \rightarrow \mathbf{P}^{1}(\mathbf{C})$.
(Proof of Theorem 3.2). Let $f: X^{\text {an }} \rightarrow D(\Gamma)$ be a holomorphic map of complex manifolds. Using Hironaka's theorem on the resolution of singularities, there exists a non-singular projective variety $X^{*}$ and an embedding $X \hookrightarrow X^{*}$ such that the complement $X^{*} \backslash X$ is a divisor with normal crossings. The latter, when interpreted in terms of the complex topology, means that the embedding $X \hookrightarrow X^{*}$ locally looks like $\left(\mathcal{D}_{1}^{\times}\right)^{r} \times \mathcal{D}_{1}^{s} \hookrightarrow \mathcal{D}_{1}^{r+s}$ and hence, $f$ extends to a holomorphic map $f:\left(X^{*}\right)^{\text {an }} \rightarrow$ $D(\Gamma)^{*}$. By Chow's theorem (Theorem 2.5), it necessarily arises from a regular map $f: X^{*} \rightarrow D(\Gamma)^{*}$ (the latter considered as an algebraic variety, the Baily-Borel compactification of $D(\Gamma)$ ).
Remark 9. The condition that $\Gamma$ is torsion-free is needed: suppose that $\Gamma=\mathbf{S L}_{2}(\mathbf{Z})$ and $X=\mathbf{C} \subset$ $\mathbf{P}^{1}(\mathbf{C})$. The map $X \cong \mathbf{C} \xrightarrow{\text { exp }} \mathbf{C} \cong \Gamma \backslash \mathcal{H}_{1}$ is holomorphic, but is not algebraic.

## 4. Models over $\overline{\mathbf{Q}}$

### 4.1 Belyi covers

Here, we explain why modular curves have models over number fields from a more general point of view, namely, Belyi covers. A good reference is [Ser97, p.70-72]. The important point that this theorem shows is that any projective non-singular curve defined over a number field can arise as a quotient of the upper half-plane $\mathcal{H}_{1}$ modulo a finite-index subgroup $\Gamma \subset \mathbf{S L}_{2}(\mathbf{Z})$.

Theorem 4.1 (Belyi). Let $X$ be a non-singular projective curve. The following statements are equivalent:
(i) $X$ has a model over $\overline{\mathbf{Q}}$.
(ii) $X$ is isomorphic to $\Gamma \backslash \overline{\mathcal{H}}_{1}^{\mathrm{BB}}$ for some finite-index subgroup $\Gamma \subset \mathbf{S L}_{2}(\mathbf{Z})$.
(iii) There is a finite cover (called Belyi cover) $f: X \rightarrow \mathbf{P}^{1}$ unramified outside $\{0,1, \infty\}$.

Proof. To show $($ iiii $) \Rightarrow(i)$, we use a general argument descent argument (in fact, any cover of a variety defined over $\overline{\mathbf{Q}}$ has a model over $\overline{\mathbf{Q}})$.
$(i i) \Rightarrow(i i i)$ is not hard as well: if $\Gamma \subset \mathbf{S L}_{2}(\mathbf{Z})$ is a finite-index subgroup then

$$
\Gamma \backslash \mathcal{H}_{1} \rightarrow \mathbf{S L}_{2}(\mathbf{Z}) \backslash \mathcal{H}_{1}
$$

is a finite cover. But $\mathbf{S L}_{2}(\mathbf{Z}) \backslash \mathcal{H}_{1} \cong \mathbf{P}^{1}(\mathbf{C})-\{\infty\}$. Moreover, the above cover is ramified only at $j=0,1728$ and $\infty$. It is easy to check that one can move these three points to $\{0,1, \infty\}$ via an automorphism of $\mathbf{P}^{1}$.

## Shimura Varieties

$($ iii $) \Rightarrow(i i)$ : Recall from a previous discussion that $\Gamma(2) \backslash \mathcal{H}_{1} \simeq \mathbf{P}^{1}(\mathbf{C})-\{0,1, \infty\}$. Now, any finite, non-empty cover of $\mathbf{P}^{1}-\{0,1, \infty\}$ has the form $\Gamma \backslash \mathcal{H}_{1}$ for some $\Gamma \subset \Gamma(2)$. Since $\mathbf{S L}_{2}(\mathbf{Z}) / \Gamma(2)$ has order 12, it follows that $\Gamma \subset \mathbf{S L}_{2}(\mathbf{Z})$ is of finite index.

The implication that we will explain in more detail is $(i) \Rightarrow(i i i)$.
Let $f: X \rightarrow \mathbf{P}^{1}$ be a non-constant rational function. Suppose that $X$ is defined over $\overline{\mathbf{Q}}$ and choose $f$ defined over a number field $K$. The map $f$ is ramified at finitely many points $S$.

Theorem 4.2. There is a covering $\varphi: \mathbf{P}^{1} \rightarrow \mathbf{P}^{1}$ defined over $\overline{\mathbf{Q}}$ and ramified at only finitely many points $\{0,1, \infty\}$, such that $\varphi(S) \subset\{0,1, \infty\}$.

Before proving the theorem, here is a simpler exercise:
Exercise 10. Suppose that the points $S$ are all rational, i.e., $S \subset \mathbf{P}^{1}(\mathbf{Q})$.
(i) If $|S|=3$, show that there exists $\varphi: \mathbf{P}^{1} \rightarrow \mathbf{P}^{1}$ such that $\varphi(S)=\{0,1, \infty\}$.
(ii) If $|S| \geqslant 4$, show that there is a covering $\varphi: \mathbf{P}^{1} \rightarrow \mathbf{P}^{1}$ such that $\varphi(S) \cup S(\varphi)$ has at most $|S|-1$ points, where $S(\varphi)$ denotes the critical values of $\varphi$. (Hint: assuming that $\alpha \in \mathbf{P}^{1}(\mathbf{Q})$ is different from $\{0,1, \infty\}$, look for $\varphi$ of the form $z^{a}(z-1)^{b}$ for suitably chosen integers $a, b$ so that $\alpha$ turns out to be a critical point for $\varphi$ ).

## 5. Shimura's original approach

- [Shi94, §6.7] discusses models descending to $\mathbf{Q}$.


## 6. Examples

6.1 Modular curves: $D=\mathcal{H}_{1}$
6.2 Siegel modular varieties: $D=\mathcal{H}_{g}$

### 6.3 Hilbert-Bloomenthal varieties

### 6.4 Shimura curves

## References

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[^0]:    ${ }^{1}$ The transcendence degree of $\mathbf{C}\left(X_{0}(N)\right)$ is 1 , essentially by Theorem 2.2 .

