WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

Draft by T. Dumont, Spring 2014, EPFL.

1. Outline of the seminar

Maxime Gheysens, the 17th of March 2014.

In this first lecture, we gave an overview of the state of the art and of what we would like to cover during this working seminar.

1.1. **Classification.** As mathematicians, we classify objects. To this purpose we try to establish a catalogue of simple objects among them and a reduction process of a general object to the simpler ones composing it.

Two trivial examples are vector spaces, classified by their fields and dimensions, and integers which are product of primes. A less trivial example is the family of finite groups. Finiteness guaranties the existence of maximal normal subgroups which yield (finite) simple quotients. Though the latter have been fully classified, it is not clear how the simple pieces assembles together.

We would like to have something similar for locally compact groups, but it is much harder. Usually, the existence of maximal proper normal subgroup is not guarantied, already in $\mathbf{R}$. The same is true for closed subgroups. Classification itself is hard since the class of locally compact groups contains all groups with the discrete topology which are known to be unclassifiable thanks to a result of Champetier.

Theorem 1 (Champetier's thesis, 1991)**.** *Let X be a standard Borel space. For any Borel map*

$$\varphi : \{\textit{Finitely generated groups}\} \longrightarrow X,$$

which is invariant (constant) on isomorphism classes, there exist two finitely generated non-isomorphic G_1, G_2 such that $\varphi(G_1) = \varphi(G_2)$.

Therefore we shall always exclude the discrete groups from this discouraging picture. Hopefully,

Theorem 2 (Caprace, Monod, 2011)**.** *Let G be a compactly generated locally compact group. Then exactly one of the following happens :*

- *G has a discrete infinite quotient.*
- *G has a cocompact normal subgroup, which is connected and solvable.*
- *G has a cocompact normal subgroups, which admits exactly n simple quotients for some integer $0 < n < \infty$. These quotients are non-discrete and non-compact.*

Remarks.

(i) Here and in what follows, a 'quotient' of a topological group means a quotient by a normal closed subgroup. Also 'simple' will always means 'topologically simple' except if explicitly stated otherwise.

(ii) Champetier's result shows that already compactly generated groups are difficult to tame.

Date: June 23, 2014.

1.2. **Consequence of Hilbert's fifth problem and simplicity.** Let G be a (Hausdorff) topological group, let G° denote the connected component of the identity. It is a closed normal subgroup of G, thus the sequence

$$0 \to G^\circ \to G \to G/G^\circ \to 0$$

is exact. The group G/G° is *totally disconnected*, often abbreviated *t.d.*, that is, every connected component is a singleton. The next theorem is a version of the solution to Hilbert's fifth problem.

Theorem 3 (Gleason, Montgomery-Zippin, Yamabe, ∼1950)**.** *Let G be a locally compact group. For every identity neighbourhood U, there is an open (hence closed) subgroup G' of G and a closed subgroup $K \lhd G'$ such that $K \subset U$ and G'/K is a Lie group.*

Corollary 4. *A connected locally compact simple group G is a Lie group.*

Remark. If G is simple, then either $G^\circ = G$ or $G^\circ = \{e\}$. In other words, either G is a Lie group or G is totally disconnected.

Corollary 5 (Meta-corollary)**.** *We need to understand the classe $\mathscr{S}_{\text{t.d.}}$ of totally disconnected, simple, compactly generated, locally compact, non-discrete groups.*

1.3. **Locally normal subgroups of simple locally compact groups.** A general approach to the study of $\mathscr{S}_{\text{t.d.}}$ was missing until Caprace, Reid, Willis started publishing a serie of three papers who give a local insight. Their key concept is the study of compact locally normal subgroups.

Definition 6. A subgroup $H < G$ is called *locally normal* if its normalizer $N_G(H)$ is open.

In their paper, locally normal subgroups are always assumed compact. We won't follow this convention. A normal subgroup is obviously locally normal and so is any open subgroup, because any subgroup containing an open subgroup is itself open.

Our main goal will be to understand the following statement and, if we are ambitious, its proof.

Theorem 7 (Caprace, Reid, Willis, 2013)**.** *For every $G \in \mathscr{S}_{\text{t.d.}}$, there is a compact totally disconnected G-space $\Omega = \Omega(G)$ such that either $\Omega = \{*\}$ or the following hold :*

- *Ω has no isolated point,*
- *the G-action on Ω is continuous and faithful,*
- *$G \curvearrowright \Omega$ is minimal, i.e. every G-orbit is dense,*
- *$G \curvearrowright \Omega$ is strongly proximal.*

The G-space Ω is called the *Stone-space* of G. Strong proximality means that for every probability measure μ on Ω, there is a sequence (g_n) in G such that $g_n\mu$ converges to a Dirac mass in the weak-*-topology. It is important to know that the situation $\Omega \neq \{*\}$ appears if and only if there is a pair (L, M) of non-trivial, locally normal, compact subgroups of G such that $[L, M] = \{e\}$.

Corollary 8. *If $\Omega \neq \{*\}$, then G is non-amenable.*

Proof. An invariant measure must be a Dirac mass by strong proximality, hence there is a fixed point. On the other hand, minimality insures that the orbits are dense which contradict $\Omega \neq \{*\}$ being Haussdorf. □

This leads us to think it is unlikely to have amenable groups in $\mathscr{S}_{\text{t.d.}}$. On the other hand recent results of Juschenko and Monod show that in the discrete case there exist finitely generate simple groups.