WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

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2. Hilbert's fifth problem and its solutions

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"What is the physicist's definition of a group?
A Lie group without the manifold structure."
— An anonymous math*overflow*.net user.

In this second lecture, we present Hilbert's fifth problem, its interpretations and solutions. This talk was largely inspired by the book of Tao. Also Montgomery and Zippin wrote a nice book on the topic.

2.1. **Hilbert's fifth problem.** In the beginning of the 20th century, Hilbert addressed a list of important open problems; the fifth one asks to characterize Lie groups among the topological groups. The problem was stated in a general form which lead to several interpretations of it. The first results were obtained by von Neumann and Pontryagin for compact groups and abelian groups respectively. Later, in 1950, Gleason, then Montgomery and Zippin, obtained a complete solution to one interpretation of Hilbert's fifth problem. Three years later, Yamabe gave what is considered as the final answered to the topic.

In this section, we present the problem asked by Hilbert in 1900. Then we expose the solution of Gleason and Montgomery-Zippin. The second part treats Yamabe's improvement of these results and we finally give a list of examples. In a futur presentation we could discuss the work of von Neumann and Pontryagin, as well as some representation theory. This should close the topic of Lie groups and we shall focus more on totally disconnected groups afterward.

Definition 1 (Topological structures)**.**

(i) A topological space X is called *locally Euclidean* it is locally homeomorphic to $\mathbf{R}^n$, i.e. every point $x \in X$ has an open neighbourhood homeomorphic to an open subset of some space $\mathbf{R}^n$.
(ii) An *n-dimensional* (topological) *manifold* is a Hausdorff locally Euclidean topological space such that all homeomorphisms in definition (i) are with respect to the same $\mathbf{R}^n$.
(iii) A group G together with a topology is called a *topological group* if the multiplication and inverse maps are continuous.
(iv) A *locally compact group* is a Hausdorff topological group which is a locally compact space, i.e. every point has a neighbourhood basis consisting of compact neighbourhoods.
(v) A *locally Euclidean group* is a Hausdorff topological group which is a locally Euclidean space.

We don't assume manifolds to be second countable as it is common in the literature. Here is some straightforward observations and well known results :

- A locally Euclidean space is first countable and locally compact.
- A Hausdorff topological group is first countable if and only if it is metrizable. Therefore locally Euclidean groups are metrizable.

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- In a topological group, all translations are homeomorphisms, hence Brouwer's theorem on the invariance of the dimension implies that a locally Euclidean group is an n-dimensional manifold for a fixed $n \in \mathbf{N}$. All discrete groups are 0-dimensional manifolds; they are exactly the 0-dimensional Lie groups.

Definition 2 (Smooth structures)**.** Let M be an n-dimensional manifold.

(i) A *smooth atlas* is a set of local homeomorphisms $(\phi_\alpha)_{\alpha \in A}$ from open subsets U_α, covering M, into $\mathbf{R}^n$, and such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth whenever they are well defined. Two atlases are *compatible* if their union is still a smooth atlas. An equivalence class of atlas for the compatibility relation is called a *smooth structure* on M.
(ii) An *n-dimensional smooth manifold* is a n-dimensional manifold together with a smooth structure. It is clear how to define *smooth maps* between smooth manifolds and finite products of manifolds.
(iii) A *Lie group* is a locally Euclidean group endowed with a smooth structure such that the multiplication and inverse maps are smooth.

The most commonly accepted interpretation of Hilbert's fifth problem is to determine whether the smoothness assumption is redondant in the definition of a Lie group.

Question (Hilbert's 5th Problem)**.** *Are locally Euclidean topological groups in fact Lie groups ?*

In other words, is there a smooth structure on a locally Euclidean topological group making the group operations smooth? The philosophy is that a weak regularity condition such as continuity and a group-like structure should necessarily come from a stronger regularity, namely smoothness.

The next proposition and corollary show that if a smooth structure exists it is unique, hence the problem is well defined. It is a first instance of the aforementioned philosophy.

Proposition 3. *Let G, H be Lie groups and $\Phi : G \to H$ be a continuous homomorphism. Then, Φ is a smooth map.*

Applying this proposition to the identity map of a topological group, we immediately get the following corollary.

Corollary 4. *There is at most one smooth structure on a topological group making it a Lie group.*

Lemma 5 (The exponential map of a Lie group)**.** *Let G be a Lie group and denote $\mathfrak{g}$ its Lie algebra, i.e. the tangent space at the identity. There is a smooth map $\exp : \mathfrak{g} \to G$ satisfying :*

(i) $\exp(sX)\exp(tX) = \exp((s+t)X)$*, for all $s, t \in \mathbf{R}$ and $X \in \mathfrak{g}$,*
(ii) $\exp(X+Y) = \lim_n(\exp(X/n)\exp(Y/n))^n$*, for all $X, Y \in \mathfrak{g}$,*
(iii) $\exp$ *is a local diffeomorphism at 0, i.e. there are $U \subset \mathfrak{g}$ and $V \subset G$ open neighbourhoods of 0 and 1 respectively such that $\exp|_U : U \to V$ is a diffeomorphism.*

Definition 6 (One-parameter subgroup)**.** Let G be a topological group. A *one-parameter subgroup ϕ* of G is a continuous group homomorphism $\phi : \mathbf{R} \to G$.

If G is a Lie group, the exponential map induces an identification between the set of one-parameter subgroups and $\mathfrak{g}$ given by the formula

$$\phi(t) = \exp(tX).$$

Proof of Proposition 3. Smoothness needs only be verified in a neighbourhood of the identity. The previous identification is now the key. The continuous homomorphism Φ sends one-parameter subgroups of G to one-parameter subgroups of H. In fact, it defines a linear, hence smooth, map between the Lie algebras. Composing with respective exponential maps on sufficiently small neighbourhoods gives the conclusion. □

Definition 7 (NSS group)**.** A topological group G has *no small subgroup*, abbreviated NSS, if there is a identity neighbourhood U containing no subgroup of G other than $\{e\}$.

The study of NSS groups was motivated by the fact that Lie groups are NSS. Once more, the exponential map plays a crucial role and we see that regularity at the identity is key in these developments.

Proposition 8. *Let G be a Lie groups, then G has NSS.*

Proof. Let $U \subset \mathfrak{g}$ be a small neighbourhood of 0 such that the restriction of the exponential map to it is a diffeomorphism onto its image. We may assume that U is a ball with respect to some norm on $\mathfrak{g}$. The structure of additive group of $\mathfrak{g}$ implies that given any non-zero vector $X \in \mathfrak{g}$, there is an integer n such that nX escapes U, i.e. $nX \notin U$. Consequently, $\exp(U)$ can not contain a subgroup of G. □

Theorem 9 (Gleason, ∼1950)**.** *Let G be a NSS locally Euclidean group. Then G is a Lie group.*

This profound result was a major breakthrough and relaunched investigations on Hilbert's fifth problem. Indeed, soon afterward Montgomery and Zippin proved that the NSS assumption was not necessary for locally Euclidean groups, yielding a positive answer to the question asked by Hilbert nearly fifthy years ago.

Theorem 10 (Montgomery, Zippin, ∼1950)**.** *Let G be a locally Euclidean group, then G has NSS.*

Corollary 11 (Hilbert's fifth problem)**.** *Let G be a locally Euclidean group, then G is a Lie group.*

2.2. **Approximation by Lie groups.** The answer to the first formulation of Hilbert's fifth problem was very satisfactory, but one can ask what about more general locally compact group. How *close* are they to Lie groups? We present now the results given by Yamabe soon after the works of Gleason, Montgomery and Zippin. It turns out many locally compact groups can be approximated by Lie groups. To this purpose we introduce the notion of projective limit.

Definition 12.

(i) Let $\mathcal{G} = (G_i)_{i\in I}$ be a family of topological groups indexed on a partially ordered set $(I, <)$ together with a family of continuous homomorphisms $\Phi = (\varphi_{ij} : G_j \to G_i)$, one for each pair $i < j$. We say that $(\mathcal{G}, \Phi)$ is a *projective system* if $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ whenever $i < j < k$.
(ii) The *projective limit* of a projective system $(\mathcal{G}, \Phi)$ is defined by

$$\varprojlim G_i := \{(g_i)_{i\in I} \mid \varphi_{ij}(g_j) = g_i \text{ for all } i < j\} \subset \prod_{i\in I} G_i.$$

The product is endowed with the product topology, so that the projective limit with the subspace topology is a closed subgroup of the product. Therefore if the groups G_i are locally compact so is $\varprojlim G_i$. Each projection map $\pi_j : \varprojlim G_i \to G_j$ is a continuous homomorphism and, for every pair $i < j$, we have by definition $\pi_i = \varphi_{ij} \circ \pi_j$. The projective limit $\varprojlim G_i$ and the maps $(\pi_i)_{i\in I}$ are universal with respect to the latter property.

If G is the projective limit of a family $(G_i)_{i\in I}$ of Lie groups, it may not be a Lie group. However, he can be *approximated by Lie groups* in some sense. The kernel K_i of the projection $\pi_i : G \to G_i$ is a closed normal subgroup of G. Since the map

$$\overline{\pi}_i : G/K_i \to G_i$$

is continuous and injective (but not necessarily surjective), a theorem of von Neumann implies that the quotient G/K_i is a Lie group. Moreover for each pair $i < j$, we have $K_j \subset K_i$, thus

$$G/K_i \hookrightarrow G/K_j.$$

"As the kernels K_i shrink to the identity, the Lie groups G/K_i grow to approximate G."

Theorem 13 (Gleason, Yamabe, 1953)**.** *Let G be a locally compact group. Then for every identity neighbourhood U, there is an open subgroup $G' < G$ and a compact subgroup $K \lhd G'$ such that $K \subset U$ and G'/K is a Lie group.*

In the connected case, one can always take $G' = G$, hence we obtain the following.

Corollary 14 (Connected Case)**.** *Let G be a connect locally compact group, then G is a projective limit of Lie groups,*

$$G = \varprojlim G/K,$$

where the projective system is indexed on the set of compact normal subgroups K of G. Furthermore, if G has NSS then G is a Lie group.

Corollary 15. *A locally compact group G has NSS if and only if G is a Lie group.*

Proof. A locally compact group containing an open subgroup that is a Lie group is itself a Lie group. □

In the next talk, we will prove van Dantzig's theorem on totally disconnected locally compact groups and see how it can be used to improve Theorem 13.

2.3. **Examples of Lie groups.** The groups $\mathbf{R}^n$, $\mathbf{R}^n/\mathbf{Z}^n$, $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$, with their usual topology, are Lie groups. E. Cartan proved that a (topologically) closed subgroup of a Lie group is a Lie group; adding $\mathrm{SL}_n(\mathbf{C})$, $\mathrm{U}_n(\mathbf{C})$, $\mathrm{SU}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{O}_n(\mathbf{R})$, $\mathrm{SO}_n(\mathbf{R})$ to the list. Also the Heisenberg group and one of its quotient, the Heinsenberg-Weil group, are both Lie groups :

$$\mathrm{H}_3 = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ & 1 & \mathbf{R} \\ & & 1 \end{pmatrix}, \qquad \mathrm{H} = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R}/\mathbf{Z} \\ & 1 & \mathbf{R} \\ & & 1 \end{pmatrix}.$$

However, H is not a matrix group. In other words, there is no non-trivial continuous injective homomorphism $\mathrm{H} \to \mathrm{GL}_n(\mathbf{C})$, (see p. 9 of Tao's book).

2.4. **Examples of locally compact groups without Lie structure.** The topological groups in the list below are locally compact but not Lie. We shall also see how they fit in Gleason-Yamabe's theorem.

(i) The infinite torus $G = (\mathbf{R}/\mathbf{Z})^{\mathbf{N}}$ with the product topology is connected. Any neighbourhood of the identity contains a subgroup of the form :

$$G_n = \prod_{i=1}^{n} \{1\} \times \prod_{n<i} \mathbf{R}/\mathbf{Z},$$

for some integer n large enough, i.e. G is not NSS. The quotients $G/G_n \cong (\mathbf{R}/\mathbf{Z})^n$ are the finite dimensional torus and G is the projective limit of them with respect to the inclusion maps.

(ii) At the opposite of connected groups, the p-adic integers $\mathbf{Z}_p$, the p-adic field $\mathbf{Q}_p$ and $\mathrm{GL}_n(\mathbf{Q}_p)$ are non-discrete locally compact totally disconnected groups. The compact open subgroups $p^n\mathbf{Z}_p$, for $n \in \mathbf{N}$, form a basis of neighbourhood of the identity in $\mathbf{Q}_p$ and $\mathbf{Z}_p$. These groups are not NSS and they are projective limits of discrete groups :

$$\mathbf{Q}_p \cong \varprojlim \mathbf{Q}_p/p^n\mathbf{Z}_p, \qquad \mathbf{Z}_p \cong \varprojlim \mathbf{Z}_p/p^n\mathbf{Z}_p \cong \varprojlim \mathbf{Z}/p^n\mathbf{Z}.$$

(iii) The two previous example were genuine projective limits of Lie groups. The last example is not and the open subgroup G' in Gleason-Yamabe's theorem has to be a proper subgroup of G. Consider the continuous automorphism of the additive group $\mathbf{Q}_p$ defined by $x \mapsto px$ and form the semi-direct product $G = \mathbf{Q}_p \rtimes \mathbf{Z}$. The underlying topological space is the product $\mathbf{Q}_p \times \mathbf{Z}$ and the group operation is

$$(x, n)(y, m) = (x + p^n y, n + m),$$

for all $x, y \in \mathbf{Q}_p$ and $n, m \in \mathbf{Z}$. The group G is not a Lie group since $p^n\mathbf{Z}_p \times \{0\}$, for $n \in \mathbf{N}$, are arbitrary small subgroups. They are all contained in the open identity neighbourhood $U = \mathbf{Z}_p \times \{0\}$. The latter does not contain any non-trivial subgroup that is normal in G. Indeed, the conjugation by the element $(0, 1)$ acts like the multiplication by p in the $\mathbf{Q}_p$ factor. However, $G' = \mathbf{Q}_p \times \{0\}$ is an open subgroup which is a projective limit of Lie groups.