WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

3. Van Dantzig's theorem and around

Maxime Gheysens, the 14th of April 2014

The goal of this lecture is to understand and prove the following fundamental result about totally disconnected locally compact groups.

Theorem 3.1 (van Dantzig, 1932)**.** *Let G be a totally disconnected locally compact group. Then $\mathcal{B}(G)$, the set of all compact open subgroups of G, is a basis of identity neighbourhoods.*

Remarks.

- In particular, totally disconnected locally compact groups do have small subgroups.
- Observe the striking contrast with connected groups, which admits only one open subgroup, themselves.
- By dropping the local compactness assumption, we should get examples of totally disconnected (complete) groups without small subgroups (Nicolas).

Before proving this theorem, we will begin by studying two examples.

3.1. **Examples.**

3.1.1. *The p-adic numbers.* Let $G = \mathbf{Q}_p$ be the additive group of p-adic numbers. The dilated subgroups of the p-adic integers, $p^n\mathbf{Z}_p$, are compact and open. Indeed, we only need to check it for $\mathbf{Z}_p$, as the multiplication is a homeomorphism, and $\mathbf{Z}_p$ is both the open ball around zero of radius p (hence open) and the projective limit of $\mathbf{Z}/p^n\mathbf{Z}$ (hence compact).

Moreover, let U be any neighbourhood of zero. Then it must contain an open ball $\mathbf{B}(0, p^{-n})$ for n big enough. But $p^{n+1}\mathbf{Z}_p \subseteq \mathbf{B}(0, p^{-n})$. Therefore, the set $\{p^n\mathbf{Z}_p \mid n \geqslant 0\}$ is both a subset of $\mathcal{B}(G)$ and a basis of identity neighbourhoods.

3.1.2. *Group of tree automorphisms.* Let $3 \leqslant d < \infty$ and $T = T_d$ be the regular d-tree. Consider its full automorphism group $G = \mathrm{Aut}(T)$, endowed with the compact-open topology. The latter is defined by the subbasis of sets of the form $U_{K,V} = \{g \in G \mid g(K) \subseteq V\}$, for K ranging among the compacts subsets of T and V among the open ones.

Fact. G is a locally compact group. This is true, more generally, for the full isometry group of any proper metric space, endowed with the compact-open topology.

Moreover, G is totally disconnected. In order to see this, consider, for any $\xi \in \partial T$, the orbital map $G \to \partial T\colon g \mapsto g\xi$. This is a continuous map onto a totally disconnected space, hence it has to be constant on the connected components of G.

But, for any $g \neq e$, we can find a $\xi \in \partial T$ such that $g\xi \neq \xi$, showing that g is not in the connected component of the identity.

Define now, for any finite subtree F of T, G_F to be the pointwise stabilizer of F. These obviously form a basis of identity neighbourhoods for the compact-open topology. Of course, G_F is the (finite) intersection of the fixators $\mathrm{Fix}_G(v)$ of the vertices v of F, therefore we only need to check that the latter are compact and open.

For a vertex v, let φ_v be the orbital map $G \to T\colon g \mapsto gv$. As vertices can only be sent to vertices, the fixator $\mathrm{Fix}_G(v)$ is the preimage by φ_v of the open ball around v of radius $\frac{1}{2}$ and is thus open.

Here are three methods to show compactness :

- Use the Arzelà–Ascoli theorem (all isometries are equicontinuous ; pointwise compactness is due to v being fixed, hence the orbit of any vertex is bounded);
- First prove sequential compactness by a diagonal argument (take any sequence (g_n) in $\mathrm{Fix}_G(v)$, then extract a subsequence converging on the closed ball of radius 1, from which you can extract a subsequence converging on the closed ball of radius 2, and so on), then prove that $\mathrm{Fix}_G(v)$ is second countable;
- Use the preceding strategy to any *net* (g_α) in $\mathrm{Fix}_G(v)$.

3.2. **Proof of van Dantzig's theorem.** The proof splits in two lemmas: the first one is about the topology of totally disconnected locally compact *spaces* and the second one will deduce from it some algebraic content for topological groups.

Lemma 3.2. *Let X be a totally disconnected locally compact space. Then the set of* clopen *subsets of X is a basis for the topology.*

Proof. Firstly, we may assume, without loss of generality, that X is compact. Indeed, suppose the lemma proved for totally disconnected *compact* spaces and let x be a point in a totally disconnected locally compact space X. We need to find a basis of neighbourhoods for x that are closed and open *in* X. Let Y be a compact neighbourhood of x. We may find a set $Z \subset Y$, open in X, containing x and such that the intersection of Z with $\partial Y = \overline{Y} \cap \overline{(X \setminus Y)}$ is empty. Indeed, for any point in ∂Y, we can find Z_y and Z'_y, two open disjoint neighbourhoods of x and y, respectively. By compactness of ∂Y, we can extract a finite subcover $\left\{Z'_{y_i}\right\}_{i=1}^n$ and the set $Z = \bigcap_{i=1}^n Z_{y_i}$ will then be the one we were looking for. Hence any set clopen in Y and contained in Z will still be clopen in X. Thus, the basis of neighbourhoods of x made of clopen sets of Y that are in Z will still be a basis of clopen neighbourhoods in X. Therefore we will now assume X to be compact.

Secondly, it is enough to show that clopen sets separate points. Indeed, if it is the case, consider any point x in X and an open neighbourhood U of x. For any $y \in X \setminus U$, we can find two disjoint clopen sets C_y and K_y, containing x and y, respectively. The complement $X \setminus U$ is closed, hence compact: we can extract a finite subcover $\{K_{y_i}\}_{i=1}^n$. Define now C to be $\bigcap_{i=1}^n C_{y_i}$: it is a clopen set, containing x and disjoint from $X \setminus U$, hence included in U.

After these two reductions, we now have to prove that clopen sets separate points in any totally disconnected compact space X. Let $x \in X$ and $K = \bigcap_C C$, where the intersection is taken over all the clopen sets containing x. We need to show that

K is reduced to the point x; for this, we will show that K is connected[1]. Assume, for the sake of contradiction, that K is the disjoint union of two closed sets K_1 and K_2, we may assume $x \in K_1$. The K_i are closed in K, but the latter is itself closed in X, hence they are also closed in X. A compact space being normal, we can find two disjoint open sets U_1 and U_2 containing K_1 and K_2, respectively.

Consider now the border $\partial U_2 = \overline{U_2} \cap \overline{X \setminus U_2} = \overline{U_2} \cap (X \setminus U_2)$ (as U_2 is open). Obviously, it is disjoint from K_2, since $K_2 \subseteq U_2$. Moreover, $K_1 \subseteq U_1 \subseteq X \setminus U_2$ but, as U_1 is open, it is even included in the interior of $X \setminus U_2$, i.e. in $X \setminus \overline{U_2}$, which shows that ∂U_2 is also disjoint from K_1. Therefore, ∂U_2 is disjoint from K. By definition of K, this means that, for any $y \in \partial U_2$, there is a clopen set containing x and avoiding y. The complements of these sets are clopen sets that covers ∂U_2; by compactness we can extract a finite subcover of these: we then have a clopen set L containing x and disjoint from ∂U_2.

Finally, consider the closed set $C = L \setminus U_2$, which contains x. It is also a closed set, since $L \setminus U_2 = L \setminus \overline{U_2}$ (as L is disjoint from ∂U_2). Therefore, C is a clopen set, containing x and disjoint from K_2: this is a contradiction with the definition of K. □

Of course, thanks to the local compactness, we may restrict the basis to the compact open subsets of X. Therefore, van Dantzig's theorem will follow from the next lemma.

Lemma 3.3. *Let G be a Hausdorff topological group. If U is a compact open subset of G containing the identity, then it contains a compact open* subgroup.

Proof. For any $x \in U$, the map $G \times G \to G\colon (y, y') \mapsto xyy'$ is continuous: as U is open, there is an open identity neighbourhood V_x such that $xV_x^2 \subseteq U$ (and of course $V_x \subseteq V_x^2$ as $e \in V_x$). By compactness, we can extract a finite subcover $\{V_{x_i}\}_{i=1}^n$ of U. Define $V = \bigcap_{i=1}^n V_{x_i}$, we then have:

$$UV \subseteq \left(\bigcup_{i=1}^n x_i V_{x_i}\right) V \subseteq \bigcup_{i=1}^n x_i V_{x_i}^2 \subseteq U.$$

Finally, set $W = U \cap U^{-1} \cap V \cap V^{-1}$ It is easy to check, by induction :

$$\begin{aligned} W &\subseteq U, \\ W^2 &= WW \subseteq UV \subseteq U, \\ W^{n+1} &= W^n W \subseteq UV \subseteq U \qquad \text{for any } n \geqslant 1, \\ W^{-1} &= W. \end{aligned}$$

Therefore, the subgroup Q generated by W, which is equal to $\bigcup_{n \geqslant 1} W^n$, is both open and included in U. As an open subgroup, it is closed, hence compact since U is compact, and this ends the proof.

□

3.3. **Application. Profinite groups.**

Proposition 3.4 (Characterization of profinite groups)**.** *Let G be a topological group. Then the following are equivalent.*

[1] Observe it is the only place in this talk where we use total disconnectedness!

(1) *G is a closed subgroup of a product of finite groups (i.e. is a projective limite of finite groups, hence the name* profinite groups*).*
(2) *G is a totally disconnected* compact *group.*

Proof. Obviously, profinite groups are compact (by Tychonoff's theorem) and totally disconnected. Conversely, let G be a totally disconnected compact group. Let Q be a (compact) open subgroup of G. The normalizer of Q, having Q as a subgroup, is also open. But an open subgroup of a compact group has to be of finite index and the finite index of $\mathcal{N}_G(Q)$ exactly means that Q has only a finite number of conjugates: therefore the intersection $\tilde{Q}$ of all the conjugates of Q is again an open finite-index subgroup.

Let now φ be the morphism from G to the product of all the quotients of G by $\tilde{Q}$, for Q ranging among the open subgroups of G. These quotients are finite since $\tilde{Q}$ has finite index in G. The kernel of φ is the intersection $\bigcap \tilde{Q}$ (for Q ranging among the open subgroups), which is inside $\bigcap Q$; the latter being trivial by van Dantzig's theorem, φ is injective. As the projections are continuous, so is φ. Finally, G is compact, hence so is $\varphi(G)$, which is thus closed. □

Remark. The proof has also shown the following interesting lemma : if a compact group contains an open subgroup, it also contains a normal one. In particular, for profinite groups, the compact open *normal* subgroups form a basis of identity neighbourhood.

3.4. **Converses to van Dantzig's theorem.** Let us end this talk by quoting two results in the neighbourhood of van Dantzig's theorem. First, van Dantzig's theorem actually characterizes total disconnectedness among locally compact groups.

Proposition 3.5. *Let G be a locally compact group. If the open subgroups form a basis of identity neighbourhoods, then G is totally disconnected.*

This is an obvious consequence of the following lemma.

Lemma 3.6. *Let G be a locally compact group. The connected component of the identity is the intersection of the open subgroups.*

Proof. Let C be the intersection of all the open subgroups of G. As an open subgroup is automatically closed, it must contain the connected component of the identity, hence $G^\circ \leqslant C$. Now consider the projection π of G onto G/G°. As the projection is continuous, the preimage of an open subgroup of G/G° is still an open subgroup of G. Therefore, by van Dantzig's theorem,

$$C \leqslant \bigcap_{\substack{H \leqslant G/G^\circ \\ H \text{ open}}} \pi^{-1}(H) = \pi^{-1}(\{e\}) = G^\circ.$$

Hence $C = G^\circ$. □

The final remark of the last section suggests to investigate compact open normal subgroups. This is done by the following characterization (Corollary 4.1 of [CM11]).

Theorem 3.7 (Caprace–Monod, 2011)**.** *Let G be a compactly generated locally compact group. The following are equivalent.*

(1) *G is residually discrete.*
(2) *The compact open normal subgroups form a basis of identity neighbourhoods.*

Addendum. Groups versus spaces

Whereas the behaviour of topological spaces can be quite wild, the addition of a group structure rigidifies the category, as shown by the following two consequences of Lemma 3.6.

A.1. **Quotients of totally disconnected spaces.** By Lemma 3.6, the quotient of a totally disconnected group is still totally disconnected (*"Once G° is killed, it is killed forever."*). Indeed, if N is a closed normal subgroup of G, then the open subgroups of G/N are in bijection with the subgroups of G of the form VN, where V is an open subgroup of G. Compare this with the fact that *any* compact metric space is a continuous image of the Cantor set (cf. [Wil70, Theorem 30.7]).

A.2. **Intersection of clopen sets.** Lemma 3.6 shows in particular that the intersection of clopen sets containing a fixed point x in a locally compact group is the connected component of this point. This is not the case anymore for spaces. For instance, consider the subspace following subspace of $\mathbf{R}^2$:

$$X = I_1 \cup I_2 \cup \bigcup_{n \geqslant 1} \left\{ (\frac{1}{n}, y) \;\middle|\; -1 \leqslant y \leqslant 1 \right\},$$

where $I_1 = \{(0, y) \mid 0 < y \leqslant 1\}$ and $I_2 = \{(0, y) \mid -1 \leqslant y < 0\}$. Then the connected component of the point $x = (0, \frac{1}{2})$ in X is I_1 whereas the intersection of the clopen sets containing x is $I_1 \cup I_2$.

References

[CM11] P.-E. Caprace & N. Monod – "Decomposing locally compact groups into simple pieces", *Math. Proc. Cambridge Philos. Soc.* **150** (2011), no. 1, p. 97–128.

[Wil70] S. Willard – *General Topology*, Addison–Wesley, Reading, MA, 1970.