WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

4. Locally normal subgroups, local equivalence and the structure lattice

Waltraud Lederle (ETH Zürich), the 28th of April 2014
Notes written by Maxime Gheysens, with the collaboration of Thibaut Dumont

This talk covers Section 2 of [CRW13].

4.1. **Locally normal subgroups.** Let G be a totally disconnected locally compact group. Recall that $\mathcal{B}(G)$, the set of compact open (in short, *compen*[1]) subgroups of G, is a basis of identity neighbourhoods.

Definition 4.1. A subgroup H of G is called *locally normal* if its normalizer $\mathcal{N}_G(H)$ is open[2], i.e. contains an element of $\mathcal{B}(G)$.

Obvious examples are normal subgroups, open subgroups and normal subgroups of open subgroups. On the other hand, let H be the (cyclic) subgroup generated in $G = \mathrm{Aut}(T_d)$ by a translation g along a line L. As, for any finite subtree F of T_d, there is an automorphism $h \in \mathrm{Fix}(F)$ which does not preserve the line L (hence the conjugate of g by this h will be a translation along another line), we see that H cannot be normalized by any $\mathrm{Fix}(F)$, hence it is not locally normal.

Lemma 4.1. *Let H and L be locally normal subgroups of G.*

1. *The intersection $H \cap L$ is locally normal.*
2. *If H is compact, every open subgroup of H is locally normal.*
3. *For every open subgroup U of G, the centralizer $\mathcal{C}_U(H)$ is locally normal.*
4. *There is an open subgroup U such that H is a normal subgroup of U (if H is compact, U can be chosen to be compen).*

Proof.

1. The intersection $H \cap L$ is obviously normalized by $\mathcal{N}_G(H) \cap \mathcal{N}_G(L)$, which is open.
2. If $K \leqslant H$ is open with H compact, then we may write $K = H \cap U$ for some open subgroup U, hence it is locally normal by the previous point (cf. Proposition A.4 below to see why we can choose U to be a group).
3. Let us show that $\mathcal{N}_U(H) = U \cap \mathcal{N}_G(H)$, which is an open subgroup, normalizes $\mathcal{C}_U(H)$. Let $g \in \mathcal{N}_U(H)$, $c \in \mathcal{C}_U(H)$, and $h \in H$; we need to show that gcg^{-1} commutes with h. As g normalizes H, there is an $h' \in H$ such that $hg = gh'$, hence also $g^{-1}h = h'g^{-1}$. Therefore,
$$gcg^{-1}h = gch'g^{-1} = gh'cg^{-1} = hgcg^{-1},$$
as wanted.

[1]Nicolas and Henrik suggest rather *opact*.
[2]Recall that [CRW13] requires moreover the compactness of H in the definition.

(4) Let U be an open compact subgroup of $\mathcal{N}_G(H)$. Then H is a normal subgroup of HU (which is indeed a group since U normalizes H). Moreover, if H is compact, then so is HU (by the continuity of the group multiplication). □

4.2. **Basics about order lattices.**

Definition 4.2. A *lattice* is either (equivalently):

- a partially ordered set such that every pair of elements has a unique infimum and supremum;
- a set with two operations, the *meet* $\wedge$ and the *join* $\vee$, that are commutative and associative and satisfy the "absorption laws" ($a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ for any a and b).

These two definitions are indeed equivalent: from a partially ordered set, we may define the join (respectively the meet) of a and b to be the supremum (respectively the infimum) of $\{a, b\}$; conversely, from the two operations $\wedge$ and $\vee$, we may define the partial order $a \leqslant b$ by $a = a \wedge b$ (equivalently, thanks to the absorption laws, $b = a \vee b$).

Examples.

- The set of natural numbers $\mathbf{N}$, endowed with the order given by divisibility (the join is then the least common multiple and the meet is the greatest common divisor), is a lattice.
- For any set X, the power set $(\mathcal{P}(X), \subseteq, \cap, \cup)$ is a lattice.
- For any group, the set of subgroups is a lattice for the inclusion (the join is then the intersection and the meet of H_1 and H_2 is the subgroup generated by $H_1 \cup H_2$). From this, the set of normal subgroup is a sublattice.

Definition 4.3. A lattice is called *modular* if, for any $x \leqslant b$, we have $x \vee (a \wedge b) = (x \vee a) \wedge b$, for any a.

Exercise 4.1. If needed, solutions for this exercise can be found in [Bir79, I.7] or in [Jac85, Chapter 8].

(1) Show that any totally ordered set is a modular lattice.
(2) Show that, in any lattice, if $x \leqslant b$, then we have $x \vee (a \wedge b) \leqslant (x \vee a) \wedge b$, for any a (hence the modular identity could be replaced by the corresponding $\geqslant$ inequality).
(3) Deduce from the two preceding points that a lattice is modular if and only if it does not contain the lattice N_5 shown in Figure 3 as a sublattice.
(4) Deduce from the preceding characterization that the power set lattice and the lattice of natural numbers with the order given by divisibility are modular lattices[3], whereas the subgroup lattice of the dihedral group of order 8 is *not* modular.
(5) Show yet that the lattice of normal subgroups of a group *is* modular[4].

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[3]Alternatively, we can also show that modularity of $(\mathbf{N}, |)$ follows from the modularity of $\mathbf{N}$ for the usual (total) order.

[4]The key point for this result is the fact that the subgroup generated by two normal subgroups L and N can be explicitly written as LN. Therefore, the modular identity would be true for any subgroups L, M, N such that $L \leqslant N$ and $LM = ML$.

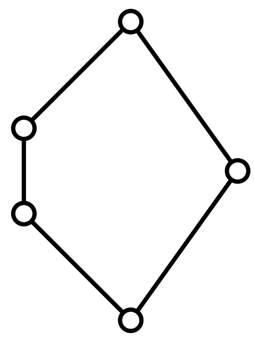

FIGURE 1. Hasse diagram of N_5.

4.3. **Local equivalence.**

Definition 4.4. Two subgroups H_1 and H_2 of a totally disconnected locally compact group are called *locally equivalent* if there is some $U \in \mathcal{B}(G)$ such that $H_1 \cap U = H_2 \cap U$.

Remark. Two subgroups H_1 and H_2 are locally equivalent if and only if their intersection $H_1 \cap H_2$ is open in both H_1 and H_2. Indeed, if $H_1 \cap H_2 = H_1 \cap U_1 = H_2 \cap U_2$ for some open sets U_1 and U_2, then, for any compen subgroup $V \subseteq U_1 \cap U_2$, we would have $H_1 \cap V = H_2 \cap V$. Conversely, if $H_1 \cap U = H_2 \cap U$ for some $U \in \mathcal{B}(G)$, then $H_1 \cap U = H_2 \cap U \leqslant H_1 \cap H_2$, hence $H_1 \cap H_2$ is open in both H_1 and H_2. In particular, this shows that local equivalence does not depend on a choice of a particular supergroup (i.e. two subgroups H_1 and H_2 of $K \leqslant G$ are locally equivalent in G if and only if they are locally equivalent in K).

Local equivalence is clearly an equivalence relation on the set of subgroups of G. We will denote by $[H]$ the class of H for this relation. We may order the set of local equivalence classes by defining $[K] \leqslant [H]$ by $[H \cap K] = [K]$ (i.e. $H \cap K \cap U = K \cap U$ for some $U \in \mathcal{B}(G)$). As, by definition, H is always locally equivalent to $H \cap U$ for any $U \in \mathcal{B}(G)$, we may also say that $[K] \leqslant [H]$ if and only if there are representatives K' and H' of $[K]$ and $[H]$ such that $K' \leqslant H'$.

Examples.

- All relatively open subgroups of a subgroup $H \leqslant G$ are locally equivalent to each other.
- Two compact subgroups are locally equivalent if and only if they are commensurate[5]. This is due to the fact that, in totally disconnected compact groups, a subgroup is open if and only if it is closed and has finite index.
- Local normality is not preserved by local equivalence. For instance, in the automorphism group of a tree, the subgroup of translations along a fixed line is discrete, hence locally equivalent to the locally normal trivial subgroup $\{e\}$, but not locally normal, as shown above.

[5]Recall that two subgroups are commensurate if their intersection has finite index in both subgroups.

4.4. **The structure lattice.**

Definition 4.5. The *structure lattice* of a totally disconnected locally compact group G, written $\mathcal{LN}(G)$, is the set of local equivalence classes that admit a locally normal representative, endowed with the two following operations:

- $[H_1] \wedge [H_2] = [H_1 \cap H_2]$ (by Lemma 4.1, the intersection of two locally normal groups is still locally normal);
- $[H_1] \vee [H_2] = [(H_1 \cap U)(H_2 \cap U)]$ for some compen subgroup $U \leqslant \mathcal{N}_G(H1) \cap \mathcal{N}_G(H_2)$.

Remarks.

- This definition is independent of the choice of representatives and of the choice of U.
- Unless otherwise stated, when dealing with representatives of classes of the structure lattice, we will alway choose a locally normal one.
- As H is locally equivalent to $H \cap U$ for any $U \in \mathcal{B}(G)$, and since the latter is locally normal if H is so, we may even choose a compact locally normal representative.
- The structure lattice is obviously a local invariant, in the sense that $\mathcal{LN}(G)$ is isomorphic to $\mathcal{LN}(U)$ for any open subgroup U of G.
- Any automorphism of G will preserve the lattice operations $\wedge$ and $\vee$. In particular, $\mathcal{LN}(G)$ is endowed with an action induced by conjugation.

The structure lattice always contains at least two trivial elements: the class of discrete subgroups (that we will denote by 0) and the class of compen subgroups (that we will denote by ∞), which are obviously respectively a global minimum and a global maximum (hence $\mathcal{LN}(G)$ is a bounded lattice).

Lemma 4.2. *Let G be a totally disconnected locally compact group.*

1. *Let X be a finite set of $\mathcal{LN}(G)$ and $\mathcal{L}$ the sublattice generated by X. Then there exist $U \in \mathcal{B}(G)$ and a choice Y of (compact locally normal) representatives of elements of X such that every $y \in Y$ is a normal subgroup of U and, moreover, $\mathcal{L}$ is isomorphic to the lattice of (normal) subgroups of U generated by Y, modulo commensuration.*
2. *The structure lattice satisfies all lattice identities that are satisfied by all normal subgroup lattices of compen subgroups of G. In particular, the structure lattice is modular.*

Proof. Let $X = \{[K_1], \dots, [K_n]\}$, with the K_i's compact and locally normal and let U be a compen subgroup of $\bigcap \mathcal{N}_G(K_i)$. Then $Y = \{L_1, \dots, L_n\}$, where $L_i = K_i \cap U$ is a set of compact locally normal representatives for X consisting of normal subgroups of U. For them, the join operation is simply given by $[L_i] \vee [L_j] = [L_i L_j]$. Thus the map $L \mapsto [L]$ is a surjection from the lattice generated by Y onto the lattice $\mathcal{L}$. Two subgroups have the same image for this map if and only if they have an open subgroup in common, i.e. they are commensurate. This proves the first part of the Lemma, and the second one is an obvious corollary. □

4.4.1. *The p-adic numbers.* The field of p-adic numbers $\mathbf{Q}_p$ has $\mathbf{Z}_p$ as an open subgroup, hence they have the same structure lattice. But all non-trivial closed subgroups of $\mathbf{Z}_p$ are of the form $p^k\mathbf{Z}_p$ ($k \in \mathbf{N}$), hence open. Therefore, the structure lattice is trivial: $\mathcal{LN}(\mathbf{Q}_p) = \mathcal{LN}(\mathbf{Z}_p) = \{0, \infty\}$.

We could generalize this example with the following definition.

Definition 4.6. A group is called *hereditarily just infinite* if all its open subgroups are just infinite[6].

Let G be a totally disconnected locally compact group with a hereditarily just infinite open subgroup U, as $\mathbf{Q}_p$ or $\mathrm{PSL}_n(\mathbf{Q}_p)$. Then, for any locally normal subgroup K, the group $K \cap U$, which is locally equivalent to K, is a normal subgroup of an open subgroup of U, hence it is either finite or open, i.e. locally equivalent to 0 or ∞.

Remark. This example also shows that the structure lattice can be strictly smaller than the subgroup lattice modulo local equivalence, i.e. a subgroup can be never locally equivalent to any locally normal subgroup. For instance, this is the case whenever the group G has a trivial structure lattice and a (closed) subgroup which is neither discrete or open.

4.4.2. *Group of tree automorphisms.* Let G be the full automorphism group of a d-regular tree T. For any finite subtree F, the complement $T \setminus F$ has n connected components $T_1, \dots, T_n$. Let K_i be the pointwise stabilizer of the complement of T_i (this is also called the *rigid stabilizer* of the root v of T_i, relatively to F). Then the fixator $\mathrm{Fix}(F)$ is obviously isomorphic to the direct product of K_i. For each subset ℓ of $\{1, \dots, n\}$, the subgroup $H_\ell = \prod_{i \in \ell} K_i$ is a normal subgroup of $\mathrm{Fix}(F)$, hence a compact locally normal subgroup of G. Moreover, each K_i is obviously infinite, hence if $\ell \neq \ell'$ are two distinct subsets of $\{1, \dots, n\}$, the subgroups H_ℓ and $H_{\ell'}$ are not commensurated. Therefore, F gives rise to 2^n distinct elements of the structure lattice. By letting F to be bigger and bigger, we see that $\mathcal{LN}(G)$ is infinite.

4.4.3. *The p-adic vector space.* Let $G = \mathbf{Q}_p^n$ with $n \geqslant 2$. Then all non-trivial compact subgroups are isomorphic to $\mathbf{Z}_p^m$ for some $1 \leqslant m \leqslant n$ (be careful, this does not necessarily mean that these subgroups are canonically embedded, we just have an abstract isomorphism). Two of them are commensurate if and only if they span the same $\mathbf{Q}_p$-vector subspace. Therefore, the structure lattice of $\mathbf{Q}_p^n$ is isomorphic to its vector subspace lattice, hence is uncountable.

Addendum. Relatively open subgroups in totally disconnected locally compact groups

The proof of Lemma 4.1(2) required some information about the algebraic structure of relatively open subgroups, which is given in the Proposition A.4 below.

Lemma A.3. *Let G be a compact totally disconnected group. Any (closed) subgroup K of G is the intersection of the open subgroups U of G containing K.*

Proof. Let C be the intersection of the open subgroups U of G containing K; obviously, $K \leqslant C$. Conversely, let $x \in C$. For any open normal subgroup W of G, the group WK is open and contains K, hence $x \in WK$. Therefore, there is a $k_W \in K$ such that $Wx = Wk_W$. Now, the net $\{k_W\}$ converges to x as W decreases (since open normal subgroups form a basis of identity neighbourhoods as we have proved in the third talk). But K is closed, hence $x \in K$, as we had to show. □

[6]Recall that a group is *just infinite* if all its non-trivial quotients are finite.

Proposition A.4. *Let K be a compact subgroup of a totally disconnected locally compact group G. Any open subgroup of K is the intersection of K with a compen* subgroup *of G.*

Proof. Let L be an open subgroup of K. It is also a closed subgroup of K, hence of G. For any compen subgroup U of G containing L, consider the chain $L \leqslant K \cap U \leqslant K$ made of open subgroups of compact ones. Hence each has finite index in the following one. By choosing any U such that the index of $K \cap U$ in K is maximal, we thus have $L = K \cap U$ (otherwise the maximality would contradict the previous lemma), as required. □

Remark. Compactness is needed in the Proposition A.4. Indeed, let $G = \mathrm{SL}_3(\mathbf{Q}_p)$ and H be any infinite discrete subgroup of G. Choose now a proper infinite subgroup K of H. Though open, it cannot be written as the intersection of H with an open subgroup of G. Indeed, open proper subgroups of G are compact[7] (hence K would be finite).

However, it is not clear whether Lemma 4.1(2) is still true without compactness.

References

[Bir79] Garrett Birkhoff, *Lattice theory*, third ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I., 1979. MR 598630 (82a:06001)

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[Jac85] Nathan Jacobson, *Basic algebra. I*, second ed., W. H. Freeman and Company, New York, 1985. MR 780184 (86d:00001)

[7]This follows from the Howe–Moore property applied on the quasi-regular left representation of G on $\ell^2(G/U)$, where U is any open subgroup: the stabilizer of δ_U has to be compact or G.