WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

6. Global and local C-stabilities

Maxime Gheysens, the 19th of May 2014

This talk covers Sections 3.3 and 3.4 of [CRW13].

Our local goal is to be able to define a centraliser lattice whose objects are robust under the choice of a representative in some local equivalence classes. This requires both studying "quasification" of centres and centralisers (which was done by Thibault in the previous talk) and introducing some technical conditions under which the so-called centraliser lattice will be well defined, which is the core of this talk.

6.1. **Memories from STD IV and V.** For the sake of the reader, we collect here two definitions and four lemmas proved in the last two talks. As always, G is a totally disconnected locally compact group.

The *quasicentre* of G, written $\mathrm{QZ}(G)$, is the set of all elements whose centraliser is open (equivalently, that commute with some open set of G). The *quasicentraliser* of some subgroup H inside another one K, written $\mathrm{QC}_K(H)$, is defined by

$$\mathrm{QC}_K(H) = \bigcup_{V \in \mathcal{B}(G)} \mathrm{C}_K(H \cap V).$$

It is obviously an invariant of the local equivalence class of H.

Lemma 6.1. *Let G be a totally disconnected locally compact group and H be a subgroup of G.*

1. *If H is locally normal, then so is every centraliser of the form $\mathrm{C}_U(H)$ for $U \in \mathcal{B}(G)$.*
2. *The quasicentre $\mathrm{QZ}(G)$ is a (non necessarily closed) characteristic subgroup of G, containing any discrete locally normal subgroup of G.*
3. *If the quasicentre of H is trivial, then $\mathrm{QC}_G(H) \cap \mathcal{N}_G(H) = \mathrm{C}_G(H)$.*
4. *If H is locally normal and its quasicentre $\mathrm{QZ}(H)$ is discrete and torsion-free, then so is $H \cap \overline{\mathrm{QC}_G(H)}$ and, moreover, $[\mathrm{QC}_G(H)] = [\mathrm{C}_G(H \cap U)]$ for any $U \in \mathcal{B}(G)$.*

Note that the last statement is the kind of robustness we are looking for.

6.2. **C-stable subgroups.** Here begin the technicalities.

Definition 6.1. A closed subgroup H of G is said to be *C-stable* in G if the intersection $\mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H))$ is discrete and torsion-free.

Before going further, let us state a black box, whose proof will be postponed to the Addendum below.

Black Box 1. *If H is compact, then* $\mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H)) = \mathrm{QC}_G(H\,\mathrm{C}_G(H))$.

Lemma 6.2. *Let G be a totally disconnected locally compact group and H be a closed C-stable subgroup of G.*

(1) *$H \cap U$ is C-stable for any open subgroup U of G.*
(2) *$\mathrm{QZ}(H)$ is discrete and torsion-free.*
(3) *If H is discrete, then it is torsion-free.*

Proof. (1) U is a closed subgroup, hence so is $H \cap U$. The quasicentraliser $\mathrm{QC}_G(H)$ is the same as $\mathrm{QC}_G(H \cap U)$, by definition. Finally, the centraliser $C_G(H \cap U)$ contains $\mathrm{C}_G(H)$, hence $\mathrm{QC}_G(\mathrm{C}_G(H \cap U))$ is contained in $\mathrm{QC}_G(\mathrm{C}_G(H))$. Therefore, the subgroup we are interested in, $\mathrm{QC}_G(H \cap U) \cap \mathrm{QC}_G(\mathrm{C}_G(H \cap U)))$, is a subgroup of $\mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H))$, which is discrete and torsion-free by hypothesis.

(2) We have $\mathrm{QZ}(H) = \mathrm{QC}_G(H)$, the latter being obviously a subgroup of $\mathrm{QC}_G(H)$. Moreover, we have the following inclusions:

$$\mathrm{QZ}(H) \leqslant H \leqslant \mathrm{C}_G(\mathrm{C}_G(H)) \leqslant \mathrm{QC}_G(\mathrm{C}_G(H)).$$

Hence $\mathrm{QZ}(H)$ is a subgroup of the discrete and torsion-free $\mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H))$.

(3) If H is discrete, then it coincides with its quasicentre, and the previous point concludes.

□

Corollary 6.3. *If H is locally normal and C-stable, then* $[\mathrm{C}_U(H \cap U)] = [\mathrm{QC}_G(H)]$ *for any* $U \in \mathcal{B}(G)$. *In particular,* $[\mathrm{C}_U(H \cap U)]$ *depends only on the local equivalence class of* H.

6.3. **Locally C-stable groups.** The last corollary motivates the following definition.

Definition 6.2. A totally disconnected locally compact group G is called *locally C-stable* if its quasicentre is trivial and all its compact locally normal subgroups are C-stable in G.

Remark. The maybe obscure condition of the triviality of the quasicentre will appear to be both useful in the proofs below and automatically satisfied for groups in the class $\mathscr{S}_{\mathrm{t.d.}}$.

This definition wouldn't be of any interest without the following characterisation.

Proposition 6.4. *A totally disconnected locally compact group is locally C-stable if and only if its quasicentre is trivial and it has no nontrivial* abelian *locally normal subgroup.*

Moreover, in this case, every compact locally normal subgroup has a trivial quasicentre.

Once again, we will need a black box whose proof is postponed to the Addendum.

Black Box 2. *Let G be a totally disconnected locally compact group with a trivial quasicentre and L be a compact locally normal subgroup of G. If $\mathrm{C}_U(L)$ is trivial for some open subgroup U, then the whole centraliser $\mathrm{C}_G(L)$ is also trivial.*

Proof of Proposition 6.4. Let us start with the easy $\Rightarrow$ direction. Let L be a nontrivial abelian locally normal subgroup of G. As $\mathrm{QZ}(G) = 1$, L cannot be discrete nor finite ; by taking a nontrivial intersection with some compen subgroup, we may assume L is compact. But L, being abelian, is a subgroup of $\mathrm{QC}_G(L) \cap \mathrm{QC}_G(\mathrm{C}_G(L))$. Therefore L is a compact locally normal subgroup which is not C-stable, hence G is not locally C-stable.

The harder $\Leftarrow$ direction will be split into two steps.

- **Step I:** Every compact locally normal subgroup has a trivial quasicentre.
- **Step II:** For any compact locally normal subgroup L, $\mathrm{QC}_G(L\,\mathrm{C}_G(L))$ is trivial.

The Black Box 1 will then conclude the proof.

Step I. Let $L \leqslant G$ be compact and locally normal and choose a $U \in \mathcal{B}(G)$ containing L as a normal subgroup. Let g be an element of the quasicentre $\mathrm{QZ}(L)$. By definition, $g \in \mathrm{C}_L(M)$ for some open subgroup M of L. Being the intersection of L and $\mathrm{C}_U(M)$, the centraliser $\mathrm{C}_L(M)$ is a locally normal. As G has no nontrivial discrete locally normal subgroup (since its quasicentre is trivial), $\mathrm{C}_L(M)$ is therefore either trivial or infinite. In the latter case, $\mathrm{C}_L(M)$ has to intersect M nontrivially, because M has finite index in the compact group L. This implies that $Z(M) = \mathrm{C}_M(M) = M \cap \mathrm{C}_U(M)$ is a nontrivial abelian locally normal subgroup of G, which is impossible by hypothesis. Hence $\mathrm{C}_L(M)$ is trivial, i.e. $g = 1$ and the quasicentre $\mathrm{QZ}(L)$ is also trivial.

Step II. As above, let $L \leqslant G$ be compact and locally normal and choose a $U \in \mathcal{B}(G)$ containing L as a normal subgroup. We have

$$\begin{aligned}
\mathrm{QC}_G(L\,\mathrm{C}_G(L)) &= \mathrm{QC}_G(L\,\mathrm{C}_G(L) \cap U) && \text{(since } U \text{ is open)}\\
&= \mathrm{QC}_G(L\,\mathrm{C}_U(L)) && \text{(because } L \text{ is normal in } U\text{).}
\end{aligned}$$

It is therefore equivalent to show that $\mathrm{QC}_G(K)$ is trivial, with $K = L\,\mathrm{C}_U(L)$.

As the open subgroup U normalises K, K is locally normal. Moreover, $\mathrm{C}_U(K)$ is a subgroup of $\mathrm{C}_U(L) \cap \mathrm{C}_U(\mathrm{C}_U(L))$, which is an abelian locally normal subgroup, hence trivial. By Black Box 2. this implies that $\mathrm{C}_G(K)$ is trivial.

By Lemma 6.13, this show that $\mathrm{QC}_G(K) \cap \mathcal{N}_G(K)$ is trivial. As $\mathcal{N}_G(K)$ is open, the quasicentraliser $\mathrm{QC}_G(K)$ is thus a discrete subgroup. Moreover, it is easy to check that the quasicentraliser of a subgroup H is normalised by the normaliser of H; therefore $\mathrm{QC}_G(K)$ is locally normal. Finally, this shows that $\mathrm{QC}_G(K)$ is a subgroup of $\mathrm{QZ}(G)$, which is trivial and so ends the proof. □

Addendum. Proofs of black boxes

Black Box 1. *If H is compact, then* $\mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H)) = \mathrm{QC}_G(H\,\mathrm{C}_G(H))$.

Proof. Note first that $H\,\mathrm{C}_G(H)$ is closed, as H is compact (use nets). Let $x \in \mathrm{QC}_G(H\,\mathrm{C}_G(H))$. As x centralises an open subgroup of $H\,\mathrm{C}_G(H)$, it also centralises an open subgroup of any subgroup of $H\,\mathrm{C}_G(H)$; in particular x is in $\mathrm{QC}_G(H)$ and in $\mathrm{QC}_G(\mathrm{C}_G(H))$.

For the reverse inclusion, let $x \in \mathrm{QC}_G(H) \cap \mathrm{QC}_G(\mathrm{C}_G(H))$ and $U \in \mathcal{B}(G)$ such that H is a subgroup of U. There are K and V, open subgroups of H and U, respectively, such that x centralises K and $\mathrm{C}_V(H)$. So x centralises the compact group $K\,\mathrm{C}_V(H)$. But the latter has finite index in $H\,\mathrm{C}_U(H)$, hence is open in it.

Finally, as $H\,\mathrm{C}_U(H) = (H\,\mathrm{C}_G(H)) \cap U$, we have shown that x centralises an open subgroup of $H\,\mathrm{C}_G(H)$. $\square$

Black Box 2. *Let G be a totally disconnected locally compact group with a trivial quasicentre and L be a compact locally normal subgroup of G. If $\mathrm{C}_U(L)$ is trivial for some open subgroup U, then the whole centraliser $\mathrm{C}_G(L)$ is also trivial.*

This will first require two lemmas.

Lemma A.5. *Let G be a group, U and V be subgroup of G and W be a subgroup of $U \cap V$. Suppose $\theta : U \to V$ is an abstract isomorphism fixing pointwise W. If $\mathrm{C}_G(W \cap g^{-1}Wg)$ is trivial for any $g \in U$, then $U = V$ and $\theta_{|U} = \mathrm{id}_U$.*

Proof. Let $g \in U$ and $x \in W \cap g^{-1}Wg$. Then
$$\underbrace{gxg^{-1}}_{\in W} = \theta(gxg^{-1}) = \theta(g)\underbrace{\theta(x)}_{\in W}\theta(g^{-1}) = \theta(g)x\theta(g^{-1}),$$
which shows that $g^{-1}\theta(g)$ is in $\mathrm{C}_G(W \cap g^{-1}Wg)$, hence $g = \theta(g)$. $\square$

The second lemma is a variation of the previous one.

Lemma A.6. *Let G, L and U be as in the hypotheses of Black Box 2, with $\mathrm{C}_U(L)$ trivial. Let H and K be two open subgroups of G both containing L. Suppose $\theta : H \to K$ is an isomorphism of topological groups fixing pointwise L. Then $H = K$ and $\theta_{|H} = \mathrm{id}_H$.*

Proof. Without loss of generality, U is a subgroup of $\mathcal{N}_G(L)$. Choose $W \in \mathcal{B}(G)$ such that $W\theta(W)$ is a subgroup of U. Let $g \in W$ and $x \in L$. The same algebraic trick as above shows that $g^{-1}\theta(g) \in \mathrm{C}_U(L) = 1$, therefore $\theta_{|W} = \mathrm{id}_W$. It suffices now to apply the previous lemma, since $\mathrm{C}_G(W \cap g^{-1}Wg)$ is, by definition, inside the quasicentre (as $W \cap g^{-1}Wg$ is open), which is trivial. $\square$

Proof of Black Box 2. Apply the previous lemma to $H = K = G$ and $\theta = \theta_g \colon G \to G \colon x \mapsto gxg^{-1}$ for any $g \in \mathrm{C}_G(L)$. This obviously fixes pointwise L and hence is trivial on the whole G. Therefore $\mathrm{C}_G(L)$ is contained in the centre of G, which is itself in the quasicentre, which is trivial. $\square$

References

[CRW13] Pierre-Emmanuel Caprace, Colin Reid, and George Willis, *Locally normal subgroups of totally disconnected groups. Part I: General theory*, preprint (2013).