8. On the non-existence of abelian, locally normal subgroups in $\mathcal{S}$

8.1. Why do we care?
We want to investigate t.d.l.c., compactly generated, topologically simple, non-discrete groups $G$ by looking at the induced action on the lattice $\mathcal{N}(G)$ of classes of locally normal (compact) subgroups modulo commensuration. It turns out, that in some cases, a different lattice, the so-called centralizer lattice, containing all classes in the image of the map

$$\perp : \mathcal{N}(G) \to \mathcal{N}(G), \quad [H] \mapsto [Q_G(H)]$$

is well-defined and has nicer properties than $\mathcal{N}(G)$. Remember, that the quasi-centralizer $QC_G(H)$ of some subgroup $H$ in a t.d.l.c group $G$ is defined by

$$QC_G(H) = \bigcup_{U \in \mathcal{B}(G)} C_G(H \cap U)$$

where $\mathcal{B}(G)$ denotes the basis of topology in $G$ consisting of all compact and open subgroups and $C_G(H)$ is the centralizer of $H$ in $G$.

The binary operations $\land_C$ and $\lor_C$ in the centralizer lattice are defined by $\land_C = \land$ and $\lor_C \beta = (\alpha \land \beta)^\perp$ with $\land$ being the join-operation from $\mathcal{N}(G)$ as defined in Section 4.

We have already proven in the seminars before, that a technical condition sufficing for the map $\perp$ to be well-defined is the so-called local C-stability, which for the sake of the reader, we will not define here.

In Section 6, we have seen, that local C-stability is equivalent to having the two following properties

- the quasi-center of $QZ(G)$ is trivial, that is: only the neutral element has an open centraliser; and
- there is no non-trivial, locally normal, amenable subgroup in $G$.

The first property has been shown to hold for all groups of class $\mathcal{S}$ (all t.d.l.c., compactly generated, top. simple, non-discrete groups) and the aim of this talk is, to show the non-existence of non-trivial, locally normal and amenable subgroups in $G$, for $G \in \mathcal{S}$. 

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8.2. Nilpotent groups and Fitting’s theorem.
Since the definitions have not been mentioned throughout the seminar, we recall, that a group $G$ is called **nilpotent**, if its lower central series terminates after finitely many steps in the trivial group. The length of the lower central series is defined to be the **nilpotency class**.

The **lower central series** $G = G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ consists of the subgroups $G_n < G$ defined recursively by $G_{n+1} = [G_n, G]$, where $[A, B]$ denotes the subgroup generated by all commutators of elements from $A$ and $B$.

Abelian (lower central series terminates after one step), or finite $p$-groups are examples of nilpotent groups.

We will need two properties of nilpotent groups.

**Lemma 8.1.** Nilpotent groups have a non-trivial center

**Proof.** The last non-trivial $G_n$ in the definition of the lower central series obviously lives in the center, since $[G_n, G] = \{1\}$. \hfill \Box

**Remark 8.2.** In fact, there is a equivalent definition of nilpotent groups stating that the upper central series terminates after finite steps in the whole group. With this definition, the above lemma is trivial.

**Theorem 8.3** (Fitting’s Theorem). If $M$ and $N$ are nilpotent normal subgroups of some group $G$, then the group $MN$ of products from elements in $M$ and $N$ is also nilpotent.

**Proof.** Idea: induction on the sum of the nilpotency classes of $M$ and $N$, the initial steps 0 and 1 being trivial.

- show, that $MN$ and $[M, N]$ are normal in $G$, when $M$ and $N$ are.
- show that $[M, N] \subset M \cap N$
- the centers of $Z(M)$ and $Z(N)$ are non-trivial and characteristic subgroups of $M$ and $N$ (i.e. fixed by all automorphisms), hence normal in $G$.
- by induction $MN/Z(M)$ and $MN/Z(N)$ are nilpotent
- this implies, that $MN/(Z(M) \cap Z(N))$ is nilpotent
- as $Z(M) \cap Z(N) < Z(MN)$, this implies $MN$ to be nilpotent
\hfill \Box
8.3. Non-existence of locally normal, abelian subgroups in $\mathcal{S}$.

To prove the non-existence of locally normal abelian subgroups, we need the following proposition

**Proposition 8.4.** Let $G \in \mathcal{S}$ and $\{e\} \neq L < G$ a compact, locally normal and abelian. Then, there is some $k \in \mathbb{N}$ and locally normal, compact subgroups $M_1, \ldots, M_k < G$ such that

1. every $M_i$ is conjugate to an open subgroup on $L$,
2. $M_i$ normalizes $M_j$ for all pairs $(i, j)$ and
3. the product group $M := M_1 \cdots M_k$ is commensurated and locally normal.

**Proof.** Let $V \in \mathcal{B}(G)$ be a subgroup of $N_G(L)$. Then $U := VL \subset N_G(L)$ is an open ($V$ being open), compact ($V$ and $L$ are compact) subgroup ($L$ being normal in $N_G(L)$) of $N_G(L)$ (in particular, it is open and compact in $G$) with $L \triangleleft U$.

Hence we get $U \cdot L \triangleright L \triangleright G$, as $G$ is topologically simple. Using, that $G$ is compactly generated, we see that $G$ is generated by $U$ and a finite number $L = L_1, \ldots, L_n$ of conjugates of $L$.

As the normalizer of $L$ is open, the space of conjugates of $L$ (with the quotient topology from $G$) is discrete and hence $U$ (being compact) acts with finite orbits.

Therefore, by adding finitely many conjugates, we get $G = \langle U, L_1, \ldots, L_k \rangle$ with the $L_i$ being conjugate to $L$ and the set $\{L_1, \ldots, L_k\}$ being closed under the action of $U$ by conjugation.

As conjugates of $L$, obviously all $L_i$ are locally normal and abelian. Define

$$V := \bigcap_{i=1}^{k} N_U(L_i \cap U) = U \cap \bigcap_{i=1}^{k} N_G(L_i)$$

As all $L_i$ are locally normal, $V$ is an open subgroup (both, in $U$ and $G$) and since $\{L_1, \ldots, L_k\}$ is closed under the conjugation with elements in $U$, we also get $V \triangleleft U$.

Now, we set $M_j := L_j \cap V$ and claim, that the proposition holds for those $M_j$.

The first property automatically holds and needs not proof.

For the second fact, one sees

$$M_j \subset N_G(L_i) \cap V \subset N_G(L_i \cap V) = N_G(M_i)$$

Now, as $V \triangleleft U$ and $\{L_1, \ldots, L_k\}$ is $U$-invariant, so is $\{M_1, \ldots, M_k\}$ forcing $M = M_1 \cdots M_k \triangleleft U$, which in turn implies, that $M$ is locally normal.

Finally, as $M$ and $L_j$ are compact and $M_j \subset L_j$ is open, $M_j$ has to be of finite index in $L_j$ and also $[L_j : M] < \infty$.

In particular, $L_j M$ commensurates $M$ ($j$ arbitrary). But this shows that $G$ commensurates $M$, as $M \triangleleft U$ and $G$ is generated by $U$ and $\{L_1, \ldots, L_k\}$. \qed
We can now prove the main theorem of this section

**Theorem 8.5.** Let $G \in \mathcal{F}$, then it does not contain any non-trivial, compact, locally normal abelian subgroup. In particular, groups in $\mathcal{F}$ are locally $C$-stable and they have a well-defined centralizer lattice.

**Proof.** We will use the following facts from the last sections as black boxes:

1. $QZ(G)$ contains all discrete subgroups of $G$,
2. $QZ(G)$ is trivial for $G \in \mathcal{F}$ and
3. for $G \in \mathcal{F}$, infinite, locally normal, compact and commensurated subgroups have trivial quasi-centralizer.

Let us assume for contradiction, that $H$ is a non-trivial, locally normal and abelian subgroup of $G$. Then, without loss of generality we may assume $H$ to be compact: in fact, either $H$ is discrete (and therefore trivial by the first two black boxes), or it intersects non-trivially with a group $K \in \mathcal{B}(G)$. The resulting intersection would then be a non-trivial, compact and abelian subgroup of $H$. Also, as an intersection of two locally normal subgroups, it will be locally normal.

In addition to this, by the first two black boxes $H$ cannot be finite - as finite groups are discrete.

Now, by the proposition, we have just proven, $G$ communsurates a locally normal subgroup $M = M_1 \cdots M_k$, where all $M_i$ are locally normal, nilpotent (even abelian, being conjugate to an open subgroup of $H$) and normalize each other.

Using Fitting's Theorem (all $M_i$ normalize each other!), we immediately see, that $M$ is nilpotent and thus has a non-trivial center $Z(M)$.

On the other hand, using the third black box, $M$ will have to have a trivial quasi-centralizer $QC_G(M)$. But $Z(M) \subset QC_G(M)$, which is a contradiction. □