WORKING SEMINAR ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

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8. On the non-existence of abelian, locally normal subgroups in $\mathscr{S}$

8.1. **Why do we care?**

We want to investigate t.d.l.c., compactly generated, topologically simple, non-discrete groups G by looking at the induced action on the lattice $\mathscr{N}(G)$ of classes of locally normal (compact) subgroups modulo commensuration. It turns out, that in some cases, a different lattice, the so-called centralizer lattice, containing all classes in the image of the map

$$\perp : \mathscr{N}(G) \to \mathscr{N}(G),\ [H] \mapsto [Q_G(H)]$$

is well-defined and has nicer properties than $\mathscr{N}(G)$.
Remember, that the **quasi-centralizer** $QC_G(H)$ of some subgroup H in a t.d.l.c group G is defined by

$$QC_G(H) = \bigcup_{U \in \mathscr{B}(G)} C_G(H \cap U)$$

where $\mathscr{B}(G)$ denotes the basis of topology in G consisting of all compact and open subgroups and $C_G(H)$ is the centralizer of H in G.

The binary operations $\wedge_C$ and $\vee_C$ in the centralizer lattice are defined by $\wedge_C = \wedge$ and $\alpha \vee_C \beta = \left(\alpha^\perp \wedge \beta^\perp\right)^\perp$ with $\wedge$ being the join-operation from $\mathscr{N}(G)$ as defined in Section 4.

We have already proven in the seminars before, that a technical condition sufficing for the map $\perp$ to be well-defined is the so-called local C-stability, which for the sake of the reader, we will not define here.

In Section 6, we have seen, that local C-stability is equivalent to having the two following properties

- the quasi-center of $QZ(G)$ is trivial, that is: only the neutral element has an open centraliser; and
- there is no non-trivial, locally normal, amenable subgroup in G.

The first property has been shown to hold for all groups of class $\mathscr{S}$ (all t.d.l.c., compactly generated, top. simple, non-discrete groups) and the aim of this talk is, to show the non-existence of non-trivial, locally normal and amenable subgroups in G, for $G \in \mathscr{S}$.

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8.2. **Nilpotent groups and Fitting's theorem.**
Since the definitions have not been mentioned throughout the seminar, we recall, that a group G is called **nilpotent**, if its lower central series terminates after finitely many steps in the trivial group. The length of the lower central series is defined to be the **nilpotency class**.

The **lower central series** $G = G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n \trianglerighteq \cdots$ consists of the subgroups $G_n < G$ defined recursively by $G_{n+1} = [G_n, G]$, where $[A, B]$ denotes the subgroup generated by all commutators of elements from A and B.

Abelian (lower central series terminates after one step), or finite p-groups are examples of nilpotent groups.

We will need two properties of nilpotent groups.

Lemma 8.1. *Nilpotent groups have a non-trivial center*

Proof. The last non-trivial G_n in the definition of the lower central series obviously lives in the center, since $[G_n, G] = \{1\}$. □

Remark 8.2. In fact, there is a equivalent definition of nilpotent groups stating that the upper central series terminates after finite steps in the whole group. With this definition, the above lemma is trivial.

Theorem 8.3 (Fitting's Theorem)**.** *If M and N are nilpotent normal subgroups of some group G, then the group MN of products from elements in M and N is also nilpotent.*

Proof. Idea: induction on the sum of the nilpotency classes of M and N, the initial steps 0 and 1 being trivial.

- show, that MN and $[M, N]$ are normal in G, when M and N are.
- show that $[M, N] \subset M \cap N$
- the centers of $Z(M)$ and $Z(N)$ are non-trivial and characteristic subgroups of M and N (i.e. fixed by all automorphisms), hence normal in G.
- by induction $MN/Z(M)$ and $MN/Z(N)$ are nilpotent
- this implies, that $MN/(Z(M) \cap Z(N))$ is nilpotent
- as $Z(M) \cap Z(N) < Z(MN)$, this implies MN to be nilpotent

□

8.3. **Non-existence of locally normal, abelian subgroups in $\mathscr{S}$.**
To prove the non-existence of locally normal abelian subgroups, we need the following proposition

Proposition 8.4. *Let $G \in \mathscr{S}$ and $\{e\} \neq L < G$ a compact, locally normal and abelian. Then, there is some $k \in \mathbb{N}$ and locally normal, compact subgroups $M_1, .., M_k < G$ such that*

1. *every M_i is conjugate to an open subgroup on L,*
2. *M_i normalizes M_j for all pairs (i, j) and*
3. *the product group $M := M_1 \cdots M_k$ is commensurated and locally normal.*

Proof. Let $V \in \mathscr{B}(G)$ be a subgroup of $N_G(L)$. Then $U := VL \subset N_G(L)$ is an open (V being open), compact (V and L are compact) subgroup (L being normal in $N_G(L)$) of $N_G(L)$ (in particular, it is open and compact in G) with $L \trianglelefteq U$.
Hence we get $U\cdot \ll L \gg \supset \overline{\ll L \gg} = G$, as G is topologically simple. Using, that G is compactly generated, we see that G is generated by U and a finite number $L = L_1, .., L_n$ of conjugates of L.

As the normalizer of L is open, the space of conjugates of L (with the quotient topology from G) is discrete and hence U (being compact) acts with finite orbits.

Therefore, by adding finitely many conjugates, we get $G =< U, L_1, .., L_k >$ with the L_i being conjugate to L and the set $\{L_1, .., L_k\}$ being closed under the action of U by conjugation.

As conjugates of L, obviously all L_i are locally normal and abelian. Define

$$V := \bigcap_{i=1}^{k} N_U(L_i \cap U) = U \cap \bigcap_{i=1}^{k} N_G(L_i)$$

As all L_i are locally normal, V is and open subgroup (both, in U and G) and since $\{L_1, .., L_k\}$ is closed under the conjugation with elements in U, we also get $V \trianglelefteq U$.

Now, we set $M_j := L_j \cap V$ and claim, that the proposition holds for those M_j.

The first property automatically holds and needs not proof.

For the second fact, one sees

$$M_j \subset N_G(L_i) \cap V \subset N_G(L_i \cap V) = N_G(M_i)$$

Now, as $V \trianglelefteq U$ and $\{L_1, .., L_k\}$ is U-invariant, so is $\{M_1, .., M_k\}$ forcing $M = M_1 \cdots M_k \trianglelefteq U$, which in turn implies, that M is locally normal.

Finally, as M and L_j are compact and $M_j \subset L_j$ is open, M_j has to be of finite index in L_j and also $[L_j M : M] < \infty$.

In particular, $L_j M$ commensurates M (j arbitrary). But this shows that G commensurates M, as $M \trianglelefteq U$ and G is generated by U and $\{L_1, .., L_k\}$. □

We can now prove the main theorem of this section

Theorem 8.5. *Let $G \in \mathscr{S}$, then it does not contain any non-trivial, compact, locally normal abelian subgroup.*
In particular, groups in $\mathscr{S}$ are locally C-stable and they have a well-defined centralizer lattice.

Proof. We will use the following facts from the last sections as black boxes:

(1) $QZ(G)$ contains all discrete subgroups of G,
(2) $QZ(G)$ is trivial for $G \in \mathscr{S}$ and
(3) for $G \in \mathscr{S}$, infinite, locally normal, compact and commensurated subgroups have trivial quasi- centralizer.

Let us assume for contradiction, that H is a non-trivial, locally normal and abelian subgroup of G. Then, without loss of generality we may assume H to be compact: in fact, either H is discrete (and therefore trivial by the first two black boxes), or it intersects non-trivially with a group $K \in \mathscr{B}(G)$. The resulting intersection would then be a non-trivial, compact and abelian subgroup of H. Also, as an intersection of two locally normal subgroups, it will be locally normal.

In addition to this, by the first two black boxes H cannot be finite - as finite groups are discrete.

Now, by the proposition, we have just proven, G communsurates a locally normal subgroup $M = M_1 \cdots M_k$, where all M_i are locally normal, nilpotent (even abelian, being conjugate to an open subgroup of H) and normalize each other.

Using Fitting's Theorem (all M_i normalize each other!), we immediately see, that M is nilpotent and thus has a non-trivial center $Z(M)$.

On the other hand, using the third black box, M will have to have a trivial quasi-centralizer $QC_G(M)$. But $Z(M) \subset QC_G(M)$, which is a contradiction. □