Continuous complex model with diffusion

Model :

$$z_t(x,t) = (\mu + i\omega)z - z|z|^2 + e^{i\theta}A(x,t)$$
$$A_t = \alpha z(x,t) - \gamma A + DA_{xx}$$

Anzatz :

$$z(x,t) = R(x)e^{i\Omega t + i\phi(x)}$$

$$A(x,t) = e^{i\ \Omega\ t}\frac{\alpha}{2D^{1/2}\sqrt{(i\Omega+\gamma)}}\int R(\tau)e^{i\ \phi(\tau)}e^{-\sqrt{(i\Omega+\gamma)/D}\ |x-\tau|}\ d\tau$$

Let's assume $R(x) = R$ and $\phi(x) = \beta x$:

$$A(x,t) = e^{i\ \Omega\ t}\frac{\alpha R}{2D^{1/2}\sqrt{(i\Omega+\gamma)}}\int e^{i\beta\tau}e^{-\sqrt{(i\Omega+\gamma)/D}\ |x-\tau|}\ d\tau$$

$$\int e^{i\beta\tau}e^{-\sqrt{(i\Omega+\gamma)/D}\ |x-\tau|}d\tau = \int_{-\infty}^{x} e^{i\beta\tau}e^{-\sqrt{(i\Omega+\gamma)/D}(x-\tau)}d\tau + \int_{x}^{\infty} e^{i\beta\tau}e^{-\sqrt{(i\Omega+\gamma)/D}(\tau-x)}d\tau$$

$$= \int_{-\infty}^{x} e^{\tau(i\beta+\sqrt{(i\Omega+\gamma)/D})}e^{-x\sqrt{(i\Omega+\gamma)/D}}d\tau + \int_{x}^{\infty} e^{\tau(i\beta-\sqrt{(i\Omega+\gamma)/D})}e^{x\sqrt{(i\Omega+\gamma)/D}}d\tau$$

$$= \frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}}e^{\tau(i\beta+\sqrt{(i\Omega+\gamma)/D})}e^{-x\sqrt{(i\Omega+\gamma)/D}}\Big|_{-\infty}^{x} + \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}}e^{\tau(i\beta-\sqrt{(i\Omega+\gamma)/D})}e^{x\sqrt{(i\Omega+\gamma)/D}}\Big|_{x}^{\infty}$$

$$= \frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}}e^{x(i\beta+\sqrt{(i\Omega+\gamma)/D})}e^{-x\sqrt{(i\Omega+\gamma)/D}} - \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}}e^{x(i\beta-\sqrt{(i\Omega+\gamma)/D})}e^{x\sqrt{(i\Omega+\gamma)/D}}$$

$$= \frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}}e^{xi\beta} - \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}}e^{xi\beta} = [\frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}} - \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}}]e^{xi\beta}$$

$$(\frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}} - \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}} = \frac{i\beta-\sqrt{(i\Omega+\gamma)/D}}{-(i\Omega+\gamma)/D-\beta^2} - \frac{i\beta+\sqrt{(i\Omega+\gamma)/D}}{-(i\Omega+\gamma)/D-\beta^2} = \frac{-2\sqrt{(i\Omega+\gamma)/D}}{-\beta^2-(i\Omega+\gamma)/D} = \frac{2D\sqrt{(i\Omega+\gamma)/D}}{\beta^2 D+i\Omega+\gamma})$$

So :

$$\boxed{\begin{aligned} A(x,t) &= e^{i\Omega t}\frac{\alpha R}{2D^{1/2}\sqrt{(i\Omega+\gamma)}}[\frac{1}{i\beta+\sqrt{(i\Omega+\gamma)/D}} - \frac{1}{i\beta-\sqrt{(i\Omega+\gamma)/D}}]e^{i\beta x} = e^{i\Omega t}\frac{\alpha R}{2D^{1/2}\sqrt{(i\Omega+\gamma)}}[\frac{2D\sqrt{(i\Omega+\gamma)/D}}{\beta^2 D+i\Omega+\gamma}]e^{i\beta x} \\ &= e^{i\Omega t}\frac{\alpha R}{\beta^2 D+i\Omega+\gamma}e^{i\beta x} = \frac{\alpha R}{\sqrt{(\beta^2 D+\gamma)^2+\Omega^2}}e^{i\Omega t+i\arctan[\Omega/(\beta^2 D+\gamma)]+i\beta x} = z(x,t)\frac{\alpha}{\sqrt{(\beta^2 D+\gamma)^2+\Omega^2}}e^{i\arctan[\Omega/(\beta^2 D+\gamma)]}\end{aligned}}$$

Verification :

$$A_t = \alpha R e^{i\Omega t+i\beta x} - \gamma A + DA_{xx}$$

$A_t = i\Omega A$ and $A_{xx} = -\beta^2 A$

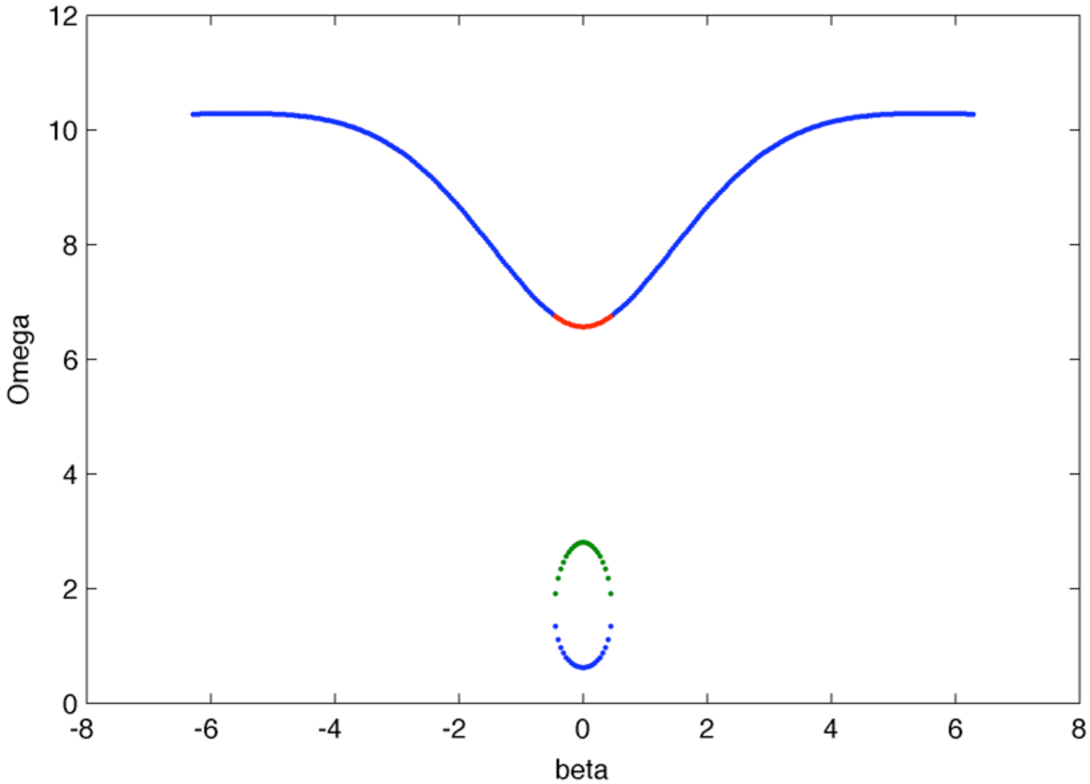
$$i\Omega\frac{1}{\beta^2 D+i\Omega+\gamma} = 1 - \gamma\frac{1}{\beta^2 D+i\Omega+\gamma} - \beta^2 D\frac{1}{\beta^2 D+i\Omega+\gamma}$$

$(i\Omega + \gamma + \beta^2 D)\frac{1}{\beta^2 D+i\Omega+\gamma} = 1$ **Ok!**

Self-consistent equation :

$\Omega = \omega(x) + Im\left(\frac{1}{z(x,t)}e^{i\theta}A(x,t)\right) = \omega + Im\left(\frac{\alpha}{\sqrt{(\beta^2 D+\gamma)^2+\Omega^2}}e^{i\arctan[\Omega/(\beta^2 D+\gamma)]+i\theta}\right)$

$$\omega - \Omega + \frac{\alpha}{\sqrt{(\beta^2 D+\gamma)^2+\Omega^2}}\sin(\arctan[\Omega/(\beta^2 D+\gamma)] + \theta) = 0$$



If $\theta = 0$ we can write it as a cubic equation :

$\sin(\arctan[\Omega/(\beta^2 D+\gamma)]) = \frac{\Omega}{\sqrt{\Omega^2+(\beta^2 D+\gamma)^2}}$

$\omega - \Omega + \frac{\alpha\Omega}{\Omega^2+(\beta^2 D+\gamma)^2} = 0$

$-\Omega^3 + \Omega^2\omega - \Omega[(\beta^2 D+\gamma)^2 - \alpha] + \omega(\beta^2 D+\gamma)^2 = 0$

Stability :

1) Need a stationnary solution
2) Can we get an homogeneous solution ?
3) Symmetries ?

$z(x,t) = Re^{i\Omega t + i\beta x}$

$A(x,t) = \frac{\alpha}{\beta^2 D + i\Omega + \gamma} z(x,t) = \frac{\alpha}{\sqrt{(\beta^2 D+\gamma)^2+\Omega^2}} e^{i\arctan[\Omega/(\beta^2 D+\gamma)]} z(x,t)$

Change of variable :

$$\boxed{y(x,t) = z(x,t)e^{-i\Omega t - i\beta x} = R}$$

$$\boxed{B(x,t) = A(x,t)e^{-i\Omega t - i\beta x} = \frac{\alpha R}{\beta^2 D + i\Omega + \gamma} = B^*}$$

$z_t = (\mu + i\omega(x))z - z|z|^2 + e^{i\theta}A(x,t)$
$A_t = \alpha z(x,t) - \gamma A + DA_{xx}$

$y_t = z_t e^{-i\Omega t - i\beta x} - i\Omega y$ or $(y_t + i\Omega y)e^{i\Omega t + i\beta x} = z_t$

$B_t = A_t e^{-i\Omega t - i\beta x} - i\Omega B$ or $(B_t + i\Omega B)e^{i\Omega t + i\beta x} = A_t$

$B_x = A_x e^{-i\Omega t - i\beta x} - i\beta A e^{-i\Omega t - i\beta x}$, so $B_x + i\beta B = A_x e^{-i\Omega t - i\beta x}$
$B_{xx} = A_{xx} e^{-i\Omega t - i\beta x} - 2i\beta A_x e^{-i\Omega t - i\beta x} + \beta^2 B$
$B_{xx} = A_{xx} e^{-i\Omega t - i\beta x} - 2i\beta(B_x + i\beta B) + \beta^2 B$
$B_{xx} + 2i\beta B_x - 3\beta^2 B = A_{xx} e^{-i\Omega t - i\beta x}$

Then :

$$\boxed{\begin{aligned} y_t &= (\mu + i(\omega - \Omega))y - y|y|^2 + e^{i\theta}B(x,t) \\ B_t &= \alpha y(x,t) - (\gamma + i\Omega + 3\,\beta^2 D)B + D(B_{xx} + 2i\beta B_x) \end{aligned}}$$

Let's define $X = (y, B)^T$ then we study perturbation like : $X + (c_1, c_2)^T e^{\lambda t} e^{ikx}$ around the stationnary solution :

Let's write the jacobian of the real system $y = a + ib$ and $B = c + id$, $X = (a, b, c, d)^T$

$$X_t \approx \begin{pmatrix} \mu - 3a^2 - b^2 & \Omega - \omega - 2ab & \cos(\theta) & -\sin(\theta) \\ \omega - \Omega - 2ab & \mu - a^2 - 3b^2 & \sin(\theta) & \cos(\theta) \\ \alpha & 0 & -(\gamma + 3\beta D) & \Omega \\ 0 & \alpha & -\Omega & -(\gamma + 3\beta D) \end{pmatrix} X + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D\partial_{xx} & -2D\beta\partial_x \\ 0 & 0 & 2D\beta\partial_x & D\partial_{xx} \end{pmatrix} X$$

$= (\mathbf{J} + \mathbf{D})X = \mathbf{M}X$

Looking for function s.t. $\mathbf{M}\psi(x) = \lambda\psi(x)$

Since the solution is homogeneous the eigenfunctions should be : $(c_1, c_2, c_3, c_4)^T e^{ikx}$

If so :

$$D\psi = \begin{pmatrix} 0 \\ 0 \\ -Dk^2 c_3 - i2D\beta k c_4 \\ i2D\beta k c_3 - Dk^2 c_4 \end{pmatrix} e^{ikx}$$

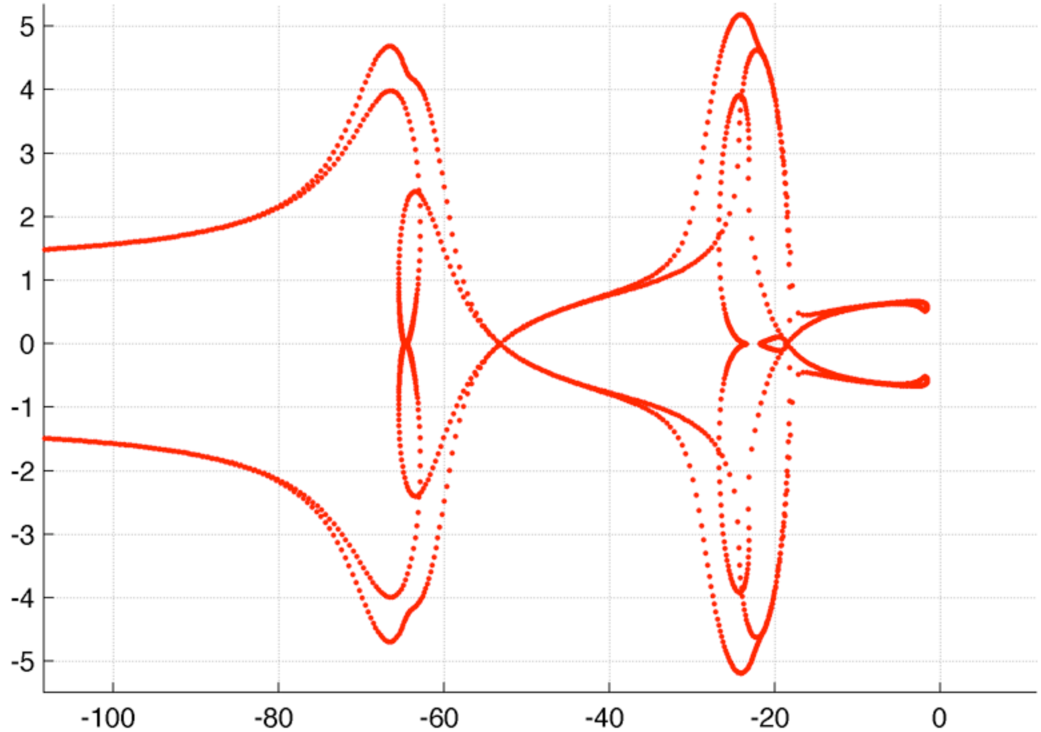
The final eigenvalue problem can be written as (we can divide by e^{ikx}) :

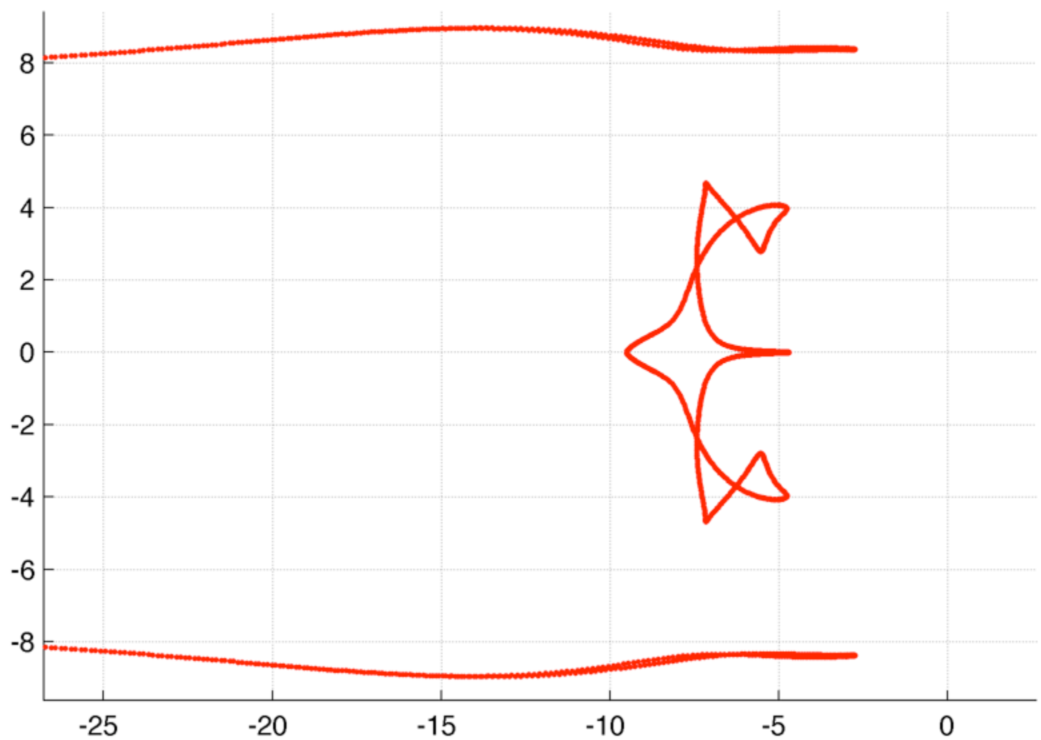
$$\begin{pmatrix} \mu - 3a^2 - b^2 & \Omega - \omega - 2ab & \cos(\theta) & -\sin(\theta) \\ \omega - \Omega - 2ab & \mu - a^2 - 3b^2 & \sin(\theta) & \cos(\theta) \\ \alpha & 0 & -(\gamma + 3\beta D) - Dk^2 & \Omega - i2D\beta k \\ 0 & \alpha & -\Omega + i2D\beta k & -(\gamma + 3\beta D) - Dk^2 \end{pmatrix} \psi = \lambda\psi$$

Need to evaluate it at the stationnary solution :

$$\begin{pmatrix} \mu - 3R^2 & \Omega - \omega & \cos(\theta) & -\sin(\theta) \\ \omega - \Omega & \mu - R^2 & \sin(\theta) & \cos(\theta) \\ \alpha & 0 & -(\gamma + 3\beta D) - Dk^2 & \Omega - i2D\beta k \\ 0 & \alpha & -\Omega + i2D\beta k & -(\gamma + 3\beta D) - Dk^2 \end{pmatrix} \psi = \lambda\psi$$

Some spectra:





No zero eigenvalue ?