STABILITY OF WAVE-PLANE SOLUTIONS OF THE COMPLEX GINZBURG-LANDAU EQUATION

The complex scaled Ginzburg-Landau equation :

$$\partial_t A = A + (1+ib)\partial_{xx}A - (1+ic)|A|^2 A$$

Anzatz :

$$A(x,t) = R\exp(ikx + i\Omega t)$$

Inserting in the evolution equation and separating the real and imaginary parts :

$$i\Omega = 1 - k^2(1+ib) - (1+ic)R^2$$

$$\Omega = -bk^2 - cR^2$$

$$R = \sqrt{1-k^2}$$

So for $k^2 < 1$ we have a family of solutions :

$$A(x,t) = \sqrt{1-k^2}\exp[ikx - i(bk^2 + c(1-k^2))t] = \sqrt{1-k^2}\exp[ikx - i\Omega(k)t]$$

With the dispersion relation :

$$\Omega(k) = bk^2 + c(1-k^2) = k^2(b-c) + c$$

Stability :

We go in a rotating frame, so we obtain a stationary solution $Z = \sqrt{1-k^2}\exp[ikx]$:

$$A = \exp(-i\Omega(k)t)Z,$$

Gives the new evolution equation:

$$-i\Omega(k)Z + \partial_t Z = Z + (1+ib)\partial_{xx}Z - (1+ic)|Z|^2 Z$$

$$\partial_t Z = [1 + i\Omega(k)]Z + (1+ib)\partial_{xx}Z - (1+ic)|Z|^2 Z$$

Then we get rid of the spatial dependence, so we obtain a homogeneous solution $Z = \sqrt{1-k^2}$:

$$Z \to Z\exp(ikx)$$

$$\partial_t Z = [1 - k^2(1+ib) + i\Omega(k)]Z + (1+ib)[\partial_{xx}Z + 2ik\partial_x Z] - (1+ic)|Z|^2 Z$$

$$\partial_t Z = (1-k^2)(1+ic)Z + (1+ib)[\partial_{xx}Z + 2ik\partial_x Z] - (1+ic)|Z|^2 Z$$

We split the equation into real and imaginary parts : $Z = u + iv$:

$$\partial_t u = (1-k^2)(u - cv) + \partial_{xx}u - 2k\partial_x v - b\partial_{xx}v - 2bk\partial_x u - (v - cu)(u^2+v^2)$$
$$\partial_t v = (1-k^2)(v + cu) + \partial_{xx}v + 2k\partial_x u + b\partial_{xx}u - 2bk\partial_x v - (u + cv)(u^2+v^2)$$

$$\partial_t u = (1 - k^2 - u^2 - v^2)(u - cv) + \partial_{xx}u - 2k\partial_x v - b\partial_{xx}v - 2bk\partial_x u$$

$\partial_t v = (1 - k^2 - u^2 - v^2)(v + cu) + \partial_{xx} v + 2k\partial_x u + b\partial_{xx} u - 2bk\partial_x v$

We linearize the equation, introduction $Q = (u, v)^T$:

$$\partial_t Q \approx \begin{bmatrix} 1 - k^2 - 3u^2 - v^2 + 2cuv & -2uv + c(-1 + k^2 + u^2 + 3v^2) \\ -2uv - c(-1 + k^2 + 3u^2 + v^2) & 1 - k^2 - u^2 - 2cuv - 3v^2 \end{bmatrix} Q$$

$$+ \begin{bmatrix} -2bk\partial_x & -2k\partial_x \\ 2k\partial_x & -2bk\partial_x \end{bmatrix} Q + \begin{bmatrix} \partial_{xx} & -b\partial_{xx} \\ b\partial_{xx} & \partial_{xx} \end{bmatrix} Q$$

$$\partial_t Q \approx [\mathbf{J} + \mathbf{C}\partial_x + \mathbf{D}\partial_{\mathbf{xx}}]Q$$

It is fortunate that the Jacobian evaluated at $Z = \sqrt{1 - k^2}$ simplifies to :

$$\mathrm{J} = \begin{bmatrix} -2(1 - k^2) & 0 \\ -2c(1 - k^2) & 0 \end{bmatrix}$$

We can write the associated eigenvalue problem as a four dimensional ODE :

$$\partial_x \psi = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}^{-1}(\mathbf{J} - \lambda \mathbf{I}) & -\mathbf{D}^{-1}\mathbf{C} \end{pmatrix} \psi$$

$$\partial_x \psi = \mathcal{M}_\lambda \psi$$

As the the solutions are homogeneous, $\mathcal{M}_\lambda$ do not depend on space. So if $\mathcal{M}_\lambda$ is hyperbolic then there is no bounded solutions. There exists bounded solutions only if the eigenvalues of $\mathcal{M}_\lambda$ are purely imaginary so the eigenfunctions are :

$$\psi = \psi_o \exp(i\omega x)$$

with :

$$[\mathbf{J} + i\omega \mathbf{C} - \omega^2 \mathbf{D}]\psi = \lambda \psi$$

Which can be easily solved numerically :

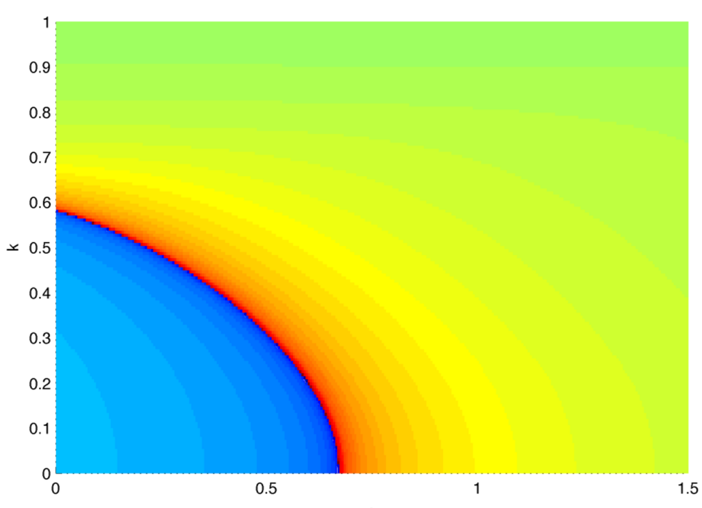

FIGURE 0.1. Maximum real part of the spectrum. The blue region is the stable region. $b = -1.5$