Stability of wave-plane solutions of the discrete complex Ginzburg-Landau equation

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The complex scaled Ginzburg-Landau equation :

$\partial_t A_j = A_j - (1+ic)|A_j|^2 A_j + \eta(1+ib)[A_{j+1} + A_{j-1} - 2A_j]$

Anzatz :

$A_j(t) = R\exp(ikj + i\Omega t)$

Inserting in the evolution equation and separating the real and imaginary parts :

$i\Omega = 1 - (1+ic)R^2 + \eta(1+ib)[e^{ik} + e^{-ik} - 2]$

$\Omega = -cR^2 + 2\eta b[\cos(k) - 1]$

$R^2 = 1 + 2\eta[\cos(k) - 1]$

Using half-angle trigonometric identity we find the dispertion relation :

$\Omega = -c - 2c\eta[\cos(k) - 1] + 2\eta b[\cos(k) - 1]$
$\Omega = 2\eta[1 - \cos(k)](c - b) - c$
$\Omega(k) = 4\eta\sin^2(k/2)(c - b) - c$

So for $(1 + 2\eta[\cos(k) - 1]) \geq 0$ we have a family of solutions :

$A_j(t) = R(k)\exp(ikj + i\Omega(k)t)$

Stability :

We go in a rotating frame, so we obtain a stationary solution $Z = R(k)\exp(ikj)$:

$A = \exp(i\Omega(k)t)Z,$

Gives the new evolution equation:

$\partial_t Z = Z[1 - i\Omega(k)] - (1+ic)|Z|^2 Z + \eta(1+ib)[Z_{j+1} + Z_{j-1} - 2Z_j]$

Then we get rid of the spatial dependence, so we obtain a homogeneous solution $Z = R$:

$Z \rightarrow Z\exp(ikj)$ (meaning $Z = Y\exp(ikj)$)

$\partial_t Z = Z[1 - i\Omega(k)] - (1+ic)|Z|^2 Z + \eta(1+ib)[e^{ik}Z_{j+1} + e^{-ik}Z_{j-1} - 2Z_j]$

Trying to write it like the continuous case :

$$-2 = -2[\cos(k) - \cos(k)] - 2 = -2([\cos(k) - \cos(k)] + 1)$$

$$(\cos(k) + i\sin(k))Z_{j+1} + (\cos(k) - i\sin(k))Z_{j-1} - 2([\cos(k) - \cos(k)] + 1)Z_j$$

$$\cos(k)[Z_{j+1} + Z_{j-1} - 2Z_j] + i\sin(k)[Z_{j+1} - Z_{j-1}] + 2[\cos(k) - 1]Z_j$$

$$\cos(k)\partial_{jj} + i\sin(k)\partial_j + 2[\cos(k) - 1]Z_j$$

Huge sucess !

$$\partial_t Z = Z[1 - i\Omega(k) + 2\eta(1+ib)(\cos(k) - 1)] - (1+ic)|Z|^2 Z + \eta(1+ib)[\cos(k)\partial_{jj} + i\sin(k)\partial_j]Z$$

We split the equation into real and imaginary parts : $Z = u + iv$

$$\partial_t u = f_1(u,v) + \eta(\partial_{jj}\cos(k)(u - bv) - \partial_j \sin(k)(bu + v))$$
$$\partial_t v = f_2(u,v) + \eta(\partial_{jj}\cos(k)(bu + v) + \partial_j \sin(k)(u - bv))$$

We linearize the equation around the solution, introducting $Q = (u,v)^T$:

$$\partial_t Q \approx \begin{bmatrix} -2(1 - 2\eta + 2\eta\cos(k)) & 0 \\ -2c(1 - 2\eta + 2\eta\cos(k)) & 0 \end{bmatrix} Q$$
$$+\eta \begin{bmatrix} -b\sin(k)\partial_j & -\sin(k)\partial_j \\ \sin(k)\partial_j & -b\sin(k)\partial_j \end{bmatrix} Q + \eta \begin{bmatrix} \cos(k)\partial_{jj} & -b\cos(k)\partial_{jj} \\ b\cos(k)\partial_{jj} & \cos(k)\partial_{jj} \end{bmatrix} Q$$

$$\partial_t Q \approx [\mathbf{J} + \mathbf{C}\partial_j + \mathbf{D}\partial_{\mathbf{jj}}]Q$$

Which is very similar to the continuous case.

We can probably rewrite the associated eigenvalue problem as a four dimensional discrete differential equation, but as the solution is homogeneous we will most likely have eigenfunctions like $\psi_j = \exp(iqj)$

$\partial_j \psi = 2i\sin(q)\psi$ and $\partial_{jj}\psi = 2(\cos(q) - 1)\psi$

Then :

$$[\mathbf{J} + 2i\sin(q)\mathbf{C} + 2(\cos(q) - 1)\mathbf{D}]\psi = \lambda\psi$$

Which can be easily solved numerically :

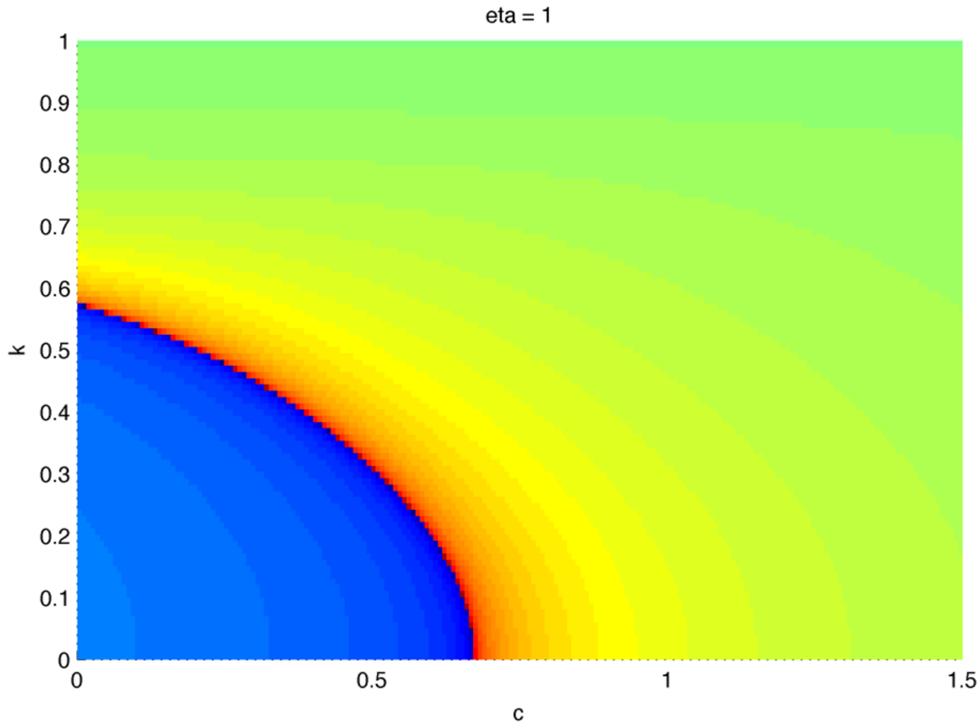

Figure 1: Maximum real part of the spectrum. The blue region is the stable region. $b = -1.5$

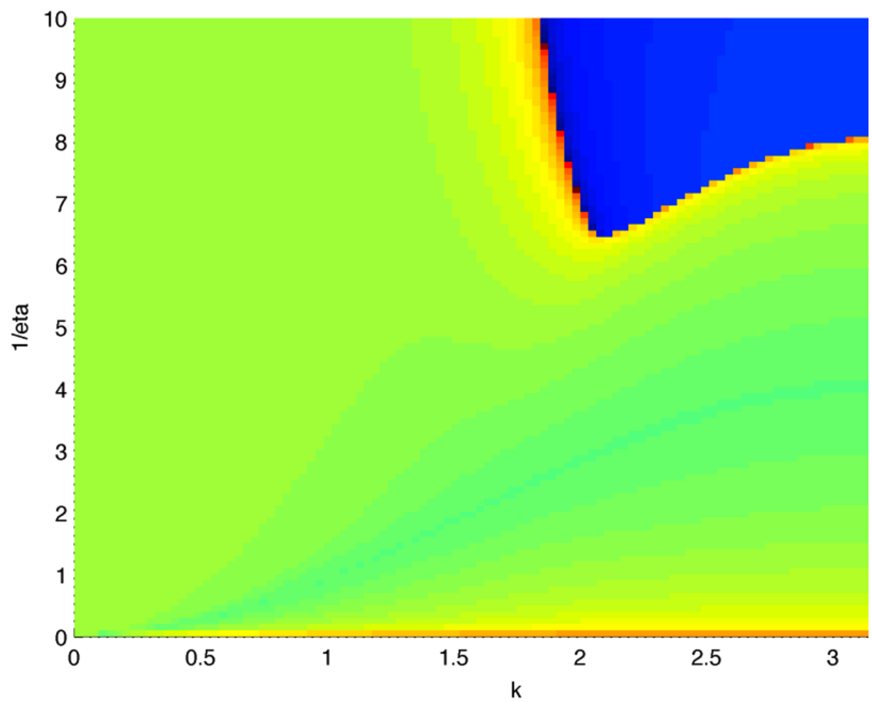

Figure 2: Maximum real part of the spectrum. The blue region is the stable region. $b = 3$ and $c = -1$.