Topics in Number Theory: Part 1

1. Find the Fourier transform of the normalised Gaussian $e^{-\pi x^2}$.

2. Functional equation for L-functions of even Dirichlet characters:

Let χ be a primitive Dirichlet character mod $N > 1$. It has L-function

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

The idea is to prove the functional equation for this L-function, using a similar idea to that for the Riemann zeta function. Namely, we will define a suitable theta series, with functional equation obtained through Poisson summation, exhibit an integral representation of the L-function in terms of this theta series, and use this to prove the functional equation for the L-function.

a) Consider

$$\theta_\chi(iy) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 y}.$$

What can we say about this function if χ is odd (i.e., $\chi(-1) = -1$)? Does the same problem occur if χ is even (i.e., $\chi(-1) = +1$)?

We will here on fix χ to be *even*.

b) (i) Prove that

$$\int_0^\infty y^{s/2} \frac{\theta_\chi(iy)}{2} \frac{dy}{y}$$

is an integral representation of (the completed) Dirichlet L-function.

(ii) Why might one write $\frac{dy}{y}$ separately, rather than absorb the y into the term at the front of the integral?

c) We now wish to obtain a functional equation for θ_χ. Can we apply Poisson summation immediately? If not, why not?

d) Decompose the theta series into a double sum, making use of the fact that we can express integers n as $\ell N + b$, where $l \in \mathbb{Z}$, $b \in \{0, 1, \ldots, N-1\}$.

e) Hopefully now you are in a position to consider the Fourier transform of

$$f(x) = \exp(-\pi(xN+b)^2 y).$$

What is it?

f) Apply Poisson summation to the theta series. Express it as a double sum, where the inner sum is over integers modulo N, and the outer sum is over $\mathbb{Z}$.

We would like to replace b by $b\ell^{-1}$ mod N, but ℓ may not be invertible mod N. Instead, we will first prove that

$$\sum_{b \bmod N} \chi(b)\mathrm{e}(\ell b/N) = 0$$

when χ is primitive and ℓ is not invertible mod N.

g) (i) Let $\gcd(\ell, N) = N/m > 1$, and replace b by $b(1+xm)$ for any x mod N/m. Show that

$$\sum_{b \bmod N} \chi(b)\mathrm{e}(\ell b/N) = \chi(1+xm) \sum_{b \bmod N/m} \sum_{\beta \bmod m} \chi(b)\mathrm{e}(\ell\beta/N).$$

(ii) Explain why this sum is zero, given that χ is primitive.

h) (i) Now make the change of variables, as suggested above (i.e., replace b by $b\ell^{-1}$ mod N). The inner sum should become a Gauss sum $< \chi, \psi >$, which does not depend on ℓ, where

$$< \chi, \psi > = \sum_{b \bmod N} \chi(b)\overline{\psi}(b)$$

and $\psi(b) = \mathrm{e}(-b/N)$.

(ii) Obtain the functional equation

$$\theta_\chi(iy) = \frac{< \chi, \psi >}{N\sqrt{y}} \theta_{\chi^{-1}}(i/N^2 y).$$

- The functional equation here relates θ_χ to $\theta_{\chi^{-1}}$. Why does this not contradict with what we saw for the Riemann zeta function?

- Note that the flip in the argument is now $y \mapsto 1/N^2 y$, not $y \mapsto 1/y$ as before.