Topics in Number Theory: Part 1 (continued)
Functional equation for Dirichlet L-functions

Recall the integral representation of the completed Dirichlet L-function.

$$\pi^{-s/2}\Gamma(s/2)L(s,\chi) = \int_0^\infty y^{s/2}\frac{\theta_\chi(iy)}{2}\frac{dy}{y}.$$

Splitting the integral into two, based on the intervals $(1/N,\infty)$ and $(0,1/N)$, we note that the first integral gives an entire function. We wish to convert the second integral, using the functional equation, to obtain a broadly symmetric expression.

3.(a) Make a substitution in the second integral so that its limits of integration are the same as the first.

(b) (i) Replace χ by χ^{-1} in the functional equation in 2 (h) (ii) and substitute this into the integral.

(ii) Conclude that the completed L-function is entire.

Now, let us note that for primitive χ

$$| < \chi, \psi > | = \sqrt{N},$$

and we define

$$\epsilon(\chi) = \frac{\sqrt{N}}{< \chi^{-1}, \psi >}.$$

(c) (i) Show that $|\epsilon(\chi)| = 1$.

(ii) Relate $\epsilon(\chi)$ to $\epsilon(\chi^{-1})$, using the fact that χ is even.

(d) (i) Substitute in your equation from (b) (i) for the Gauss sum and multiply through by $N^{s/2}$.

(ii) Obtain the functional equation for $L(s,\chi)$: factor out $\epsilon(\chi)$ from the integral, and make use of the symmetry of the resulting integral.

Odd case

The odd case will have some similarities and differences compared to the even case. We will go through this case more quickly.

Here, the theta series used in the even case will not work. Instead we use

$$\widetilde{\theta}_\chi(iy) = \sum_{n\in\mathbb{Z}} \chi(n) n \sqrt{y} e^{-\pi n^2 y}.$$

4. (a) Show that this gives an integral representation of the completed L-function.

(b) (i) Break up the theta series into a double-sum (the outer over $\mathbb{Z}/N\mathbb{Z}$ and the inner over $\mathbb{Z}$), as before.

(ii) Find the Fourier transform of the expression in the inside sum. You can make use of the fact that $xe^{-\pi x^2}$ is an eigenfunction of the Fourier transform, with eigenvalue $-i$.

(iii) Apply Poisson summation.

We've already shown that for primitive χ, the inner sum is zero when $(l, N) > 1$. We proceed as in the even case to obtain

$$\widetilde{\theta}_\chi(iy) = \frac{-i < \chi, \psi >}{N\sqrt{y}} \cdot \widetilde{\theta}_{\chi^{-1}}\left(\frac{i}{N^2 y}\right),$$

for the functional equation.

(c) (i) Now split the integral expression for the completed Dirichlet L-function into two pieces with respect to the intervals $(1/N, \infty)$ and $(0, 1/N)$. As before, make a substitution in the second integral so that its limits of integration are the same as the first.

(ii) Use the functional equation for the theta series to make a substitution in the integral.

We now define

$$\epsilon(\chi) = \frac{-i\sqrt{N}}{< \chi^{-1}, \psi >}.$$

(d) (i) Show that $\epsilon(\chi) \cdot \epsilon(\chi^{-1}) = 1$ (recall that the Gauss sum has absolute value $\sqrt{N}$ and show that $< \chi^{-1}, \psi >= -\overline{< \chi, \psi >}$).

(ii) As before, use this to find the functional equation.